Direct Numerical Solution of the Steady 1D Compressible Euler Equations for Transonic Flow Profiles with Shocks

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1. Introduction

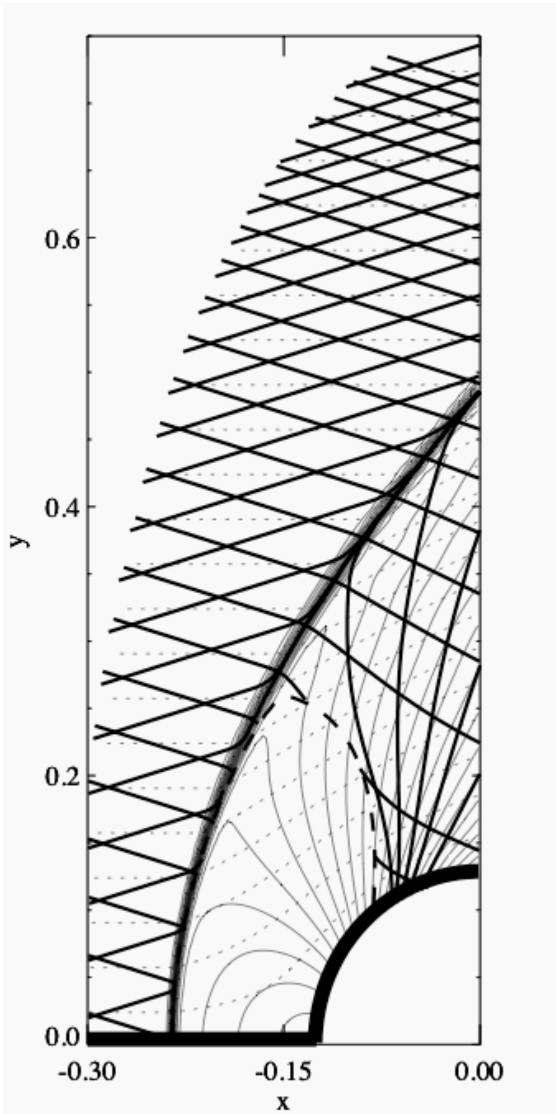
- consider stationary solutions of hyperbolic conservation law

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{F}(U) = 0$$

- in particular, compressible Euler equations

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho\,\vec{v} \\ \frac{p}{\gamma-1} + \frac{\rho\,v^2}{2} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\,\vec{v} \\ \rho\,\vec{v}\vec{v} + I\,p \\ (\frac{\gamma\,p}{\gamma-1} + \frac{\rho\,v^2}{2})\,\vec{v} \end{bmatrix} = 0$$

Transonic steady Euler flows



$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{F}(U) = 0$$

$$\nabla \cdot \vec{F}(U) = 0$$

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Standard approach for steady flow simulation

- time marching (often implicit)

$$\frac{U^{n+1} - U^n}{\Delta t} + \nabla \cdot \vec{F}(U^{n+1}) = 0$$

- Newton: linearize $\vec{F}$
- Krylov: iterative solution of linear system in every Newton step
- Schwarz: parallel (domain decompositioning), or multigrid

$\Rightarrow$ NKS methodology for steady flows

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Main advantages of NKS

- use the hyperbolic BCs for steady problem
- 'physical' way to find suitable initial conditions for the Newton method in every timestep
- it works! (in the sense that it allows one to converge to a solution, in many cases, with some trial-and-error)

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Disadvantages of NKS

- number of Newton iterations required for convergence grows as a function of resolution
- number of Krylov iterations required for convergence of the linear system in each Newton step grows as a function of resolution
- grid sequencing/nested iteration: often does not work as well as it could (need many Newton iterations on each level)
- robustness, hard to find general strategy to increase timestep

$\Longrightarrow$ NKS methodology not very scalable, and expensive

Why not solve the steady equations directly?

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{F}(U) = 0 \qquad \nabla \cdot \vec{F}(U) = 0$$

- too hard!
- let's try anyway:
  - maybe we can understand why it is difficult
  - maybe we can find a method that is more efficient than implicit time marching
- start in 1D

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2. 1D model problems

- radial outflow from extrasolar planet

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho\, r^2 \\ \rho\, u\, r^2 \\ (\frac{p}{\gamma-1} + \frac{\rho\, u^2}{2})\, r^2 \end{bmatrix} + \frac{\partial}{\partial r}\begin{bmatrix} \rho\, u\, r^2 \\ \rho\, u^2\, r^2 + p\, r^2 \\ (\frac{\gamma\, p}{\gamma-1} + \frac{\rho\, u^2}{2})\, u\, r^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\rho G M + 2\, p\, r \\ -\rho G M\, u + q_{heat}\, r^2 \end{bmatrix}$$

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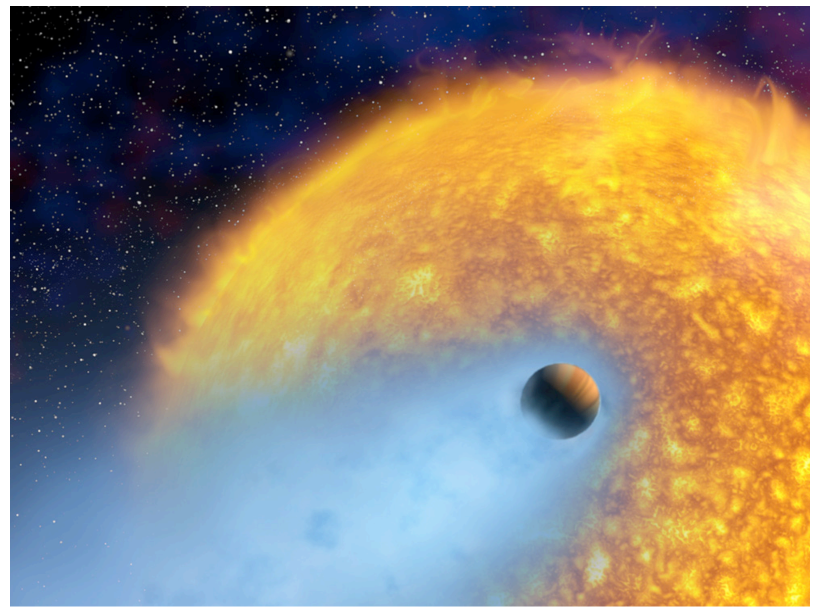
Radial outflow from exoplanet

- http://exoplanet.eu
- 236 extrasolar planets known, as of May 2007
- 24 multiple planet systems
- many exoplanets are gas giants (“hot Jupiters”)
- many orbit very close to star (~0.05 AU)
- hypothesis: strong irradiation leads to supersonic hydrogen escape



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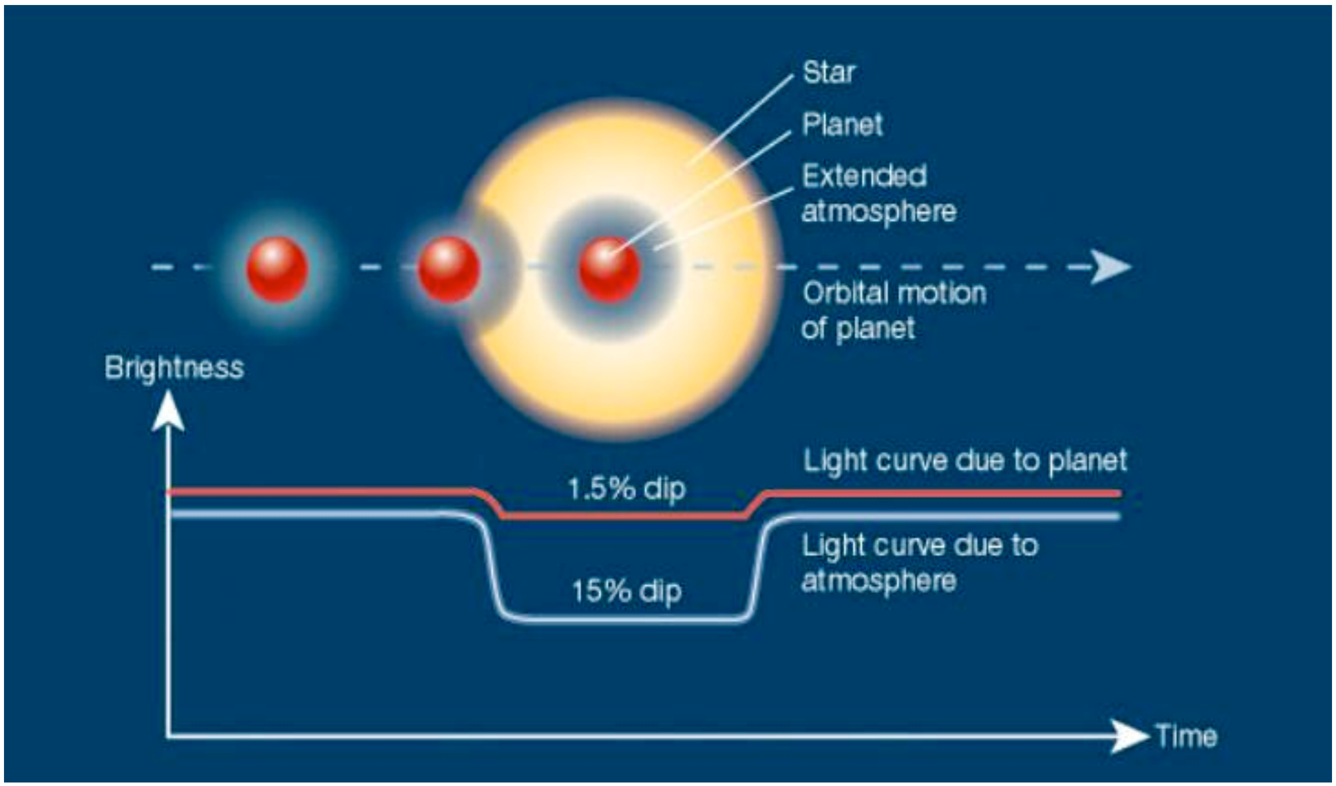
Transiting exoplanet



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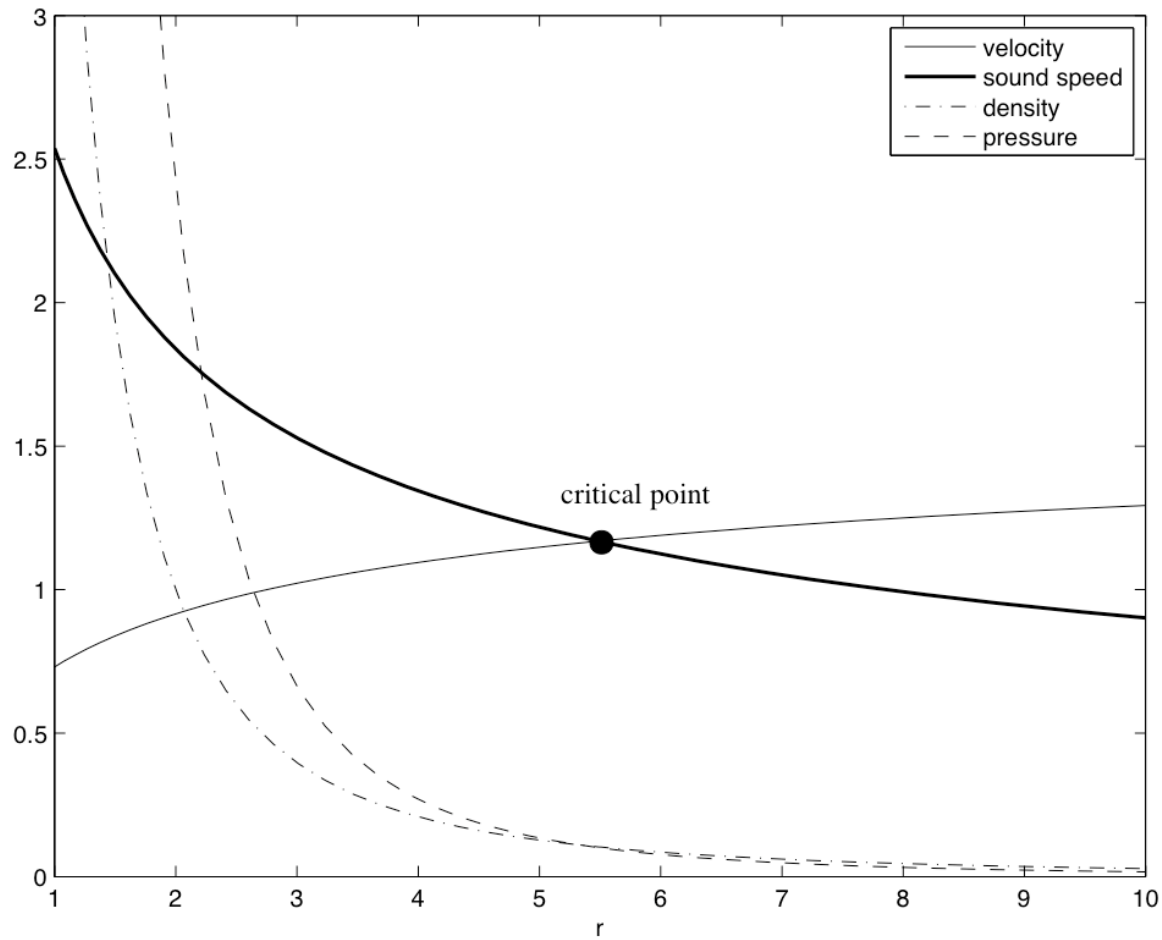
Transonic radial outflow solution of Euler equations of gas dynamics



subsonic ⇒|⇐ supersonic

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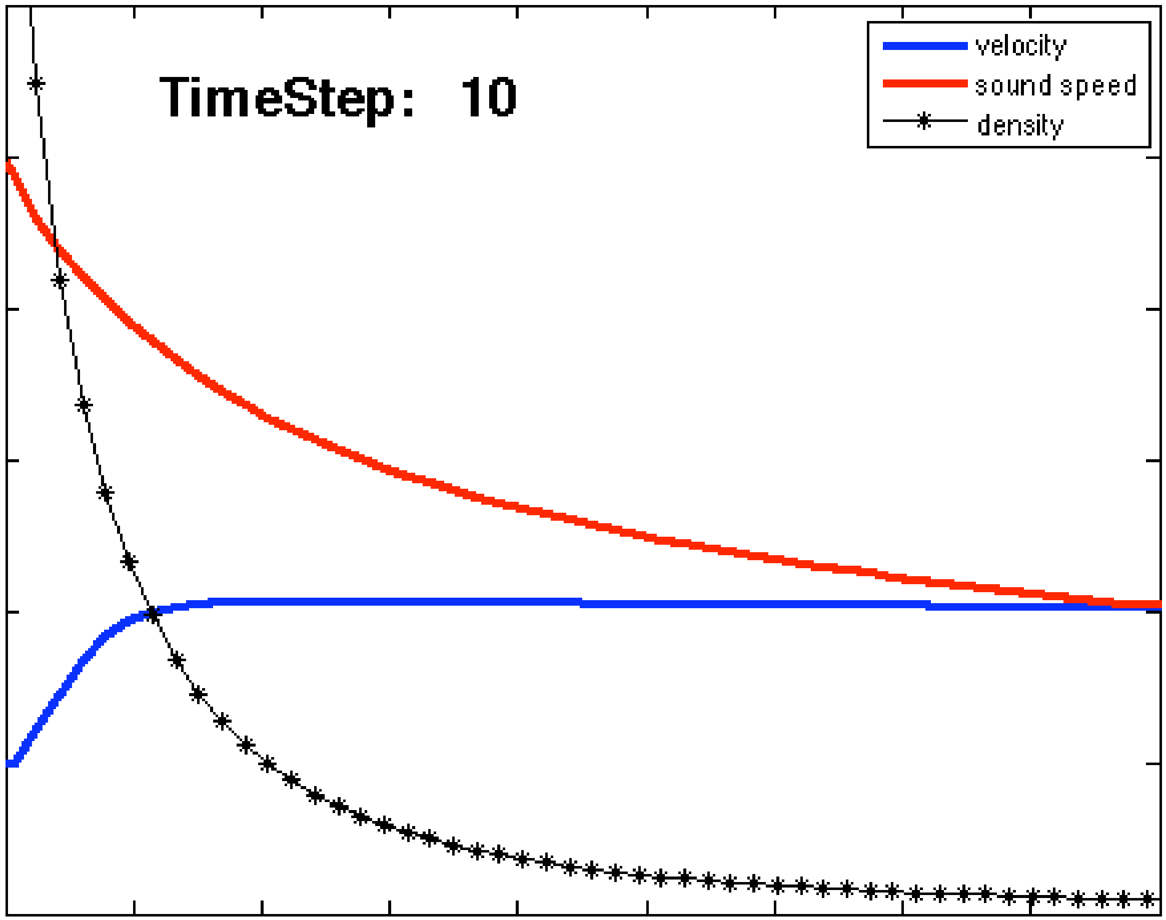
Use time marching method (explicit)



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Simplified 1D problem: radial isothermal Euler

- 2 equations (ODEs), 2 unknowns ($u, \ \rho$)

$$\frac{d}{dr}(\rho u r^2) = 0$$

$$\frac{du}{dr} = \frac{2\,u\,c^2\,(r-r_c)}{r^2\,(u^2-c^2)}$$

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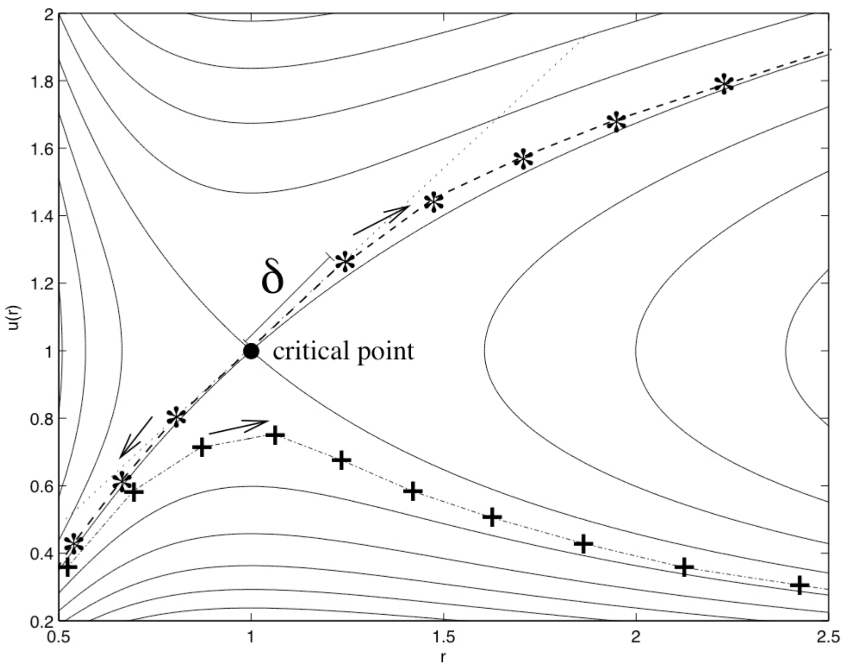
Solving the steady ODE system is hard...

- solving ODE from the left does not work...
- also: 2 equations, 2 unknowns, but only 1 BC needed! (along with transonic solution requirement)
- but... integrating outward from the critical point does work!!!



$$\frac{d}{dr}(\rho u r^2) = 0$$

$$\frac{du}{dr} = \frac{2\,u\,c^2\,(r-r_c)}{r^2\,(u^2-c^2)}$$

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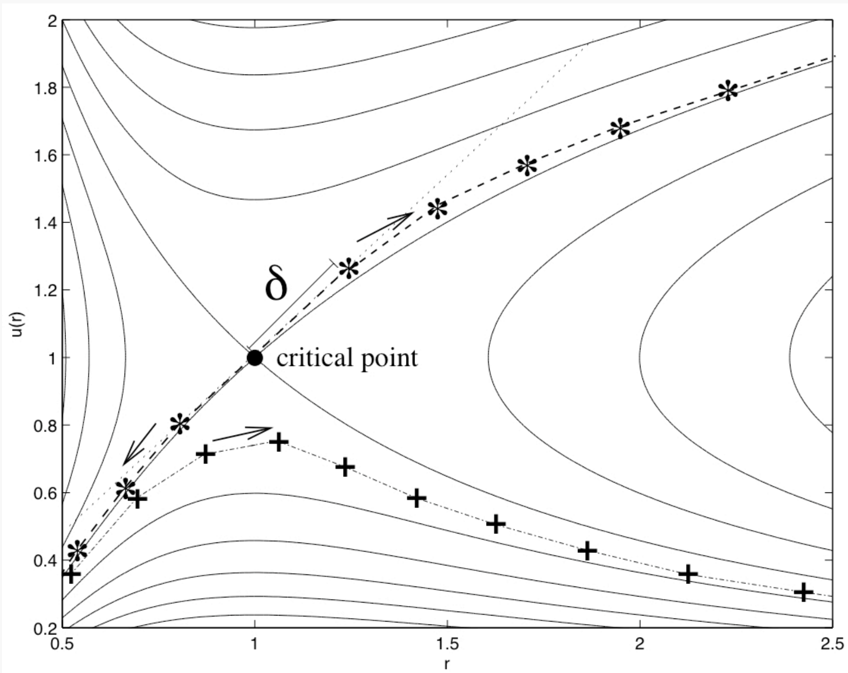
3. Newton Critical Point (NCP) method for steady transonic Euler flows

- First component of NCP: integrate outward from critical point, using dynamical systems formulation



$$\frac{du}{dr} = \frac{2\,u\,c^2\,(r-r_c)}{r^2\,(u^2-c^2)}$$

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First component of NCP

$$\frac{du}{dr} = \frac{2\,u\,c^2\,(r - r_c)}{r^2\,(u^2 - c^2)}$$

1. Write as dynamical system...

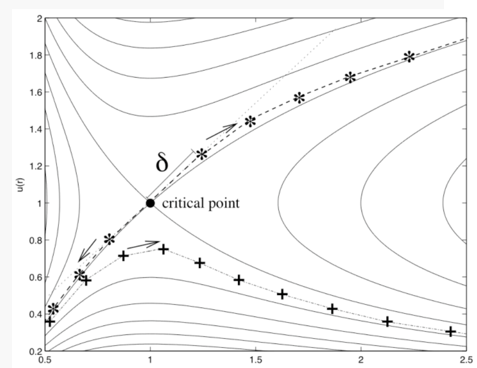
$$\frac{du(s)}{ds} = -2\,u\,c^2\left(r - \frac{GM}{2c^2}\right)$$
$$\frac{dr(s)}{ds} = -r^2\,(u^2 - c^2)$$

$$\frac{dV}{ds} = G(V)$$

2. find critical point: $G(V) = 0$
3. linearize about critical point

$$\left.\frac{\partial G}{\partial V}\right|_{V_{crit}} = \begin{bmatrix} 0 & 2\,c^3 \\ \frac{(GM)^2}{2\,c^3} & 0 \end{bmatrix}$$

4. integrate outward from critical point



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For the Full Euler Equations

$$\frac{d}{dr}\begin{bmatrix} \rho\, u\, r^2 \\ \rho\, u^2\, r^2 + p\, r^2 \\ \left(\frac{\gamma\, p}{\gamma-1} + \frac{\rho\, u^2}{2}\right) u\, r^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\rho G M + 2\, p\, r \\ -\rho G M\, u + q_{heat}\, r^2 \end{bmatrix}$$

- 3 equations, 3 unknowns, but only 2 inflow BC
- problem: there are many possible critical points! (two-parameter family)

Full Euler dynamical system

$$
\begin{aligned}
\frac{dF}{ds} &= 0, \\
\frac{du}{ds} &= 2\,u\,c^2\,(r - \frac{GM}{2c^2}) - (\gamma-1)\,q_{heat}\frac{r^4\,u}{F}, \\
\frac{dr}{ds} &= r^2\,(u^2 - c^2), \\
\frac{dT}{ds} &= (\gamma-1)\,T\,(GM - 2\,u^2\,r) - \\
&\quad (\gamma-1)\,q_{heat}\frac{r^4}{F}\,(T-u^2).
\end{aligned}
$$

$$
\Longrightarrow \qquad
\begin{aligned}
T_{crit} &= \frac{GM}{2\,\gamma\,r_{crit}} + (\gamma-1)\,\frac{q_{heat}\,r_{crit}^3}{2\,\gamma\,F_{crit}}, \\
u_{crit} &= \sqrt{\gamma\,T_{crit}}.
\end{aligned}
$$

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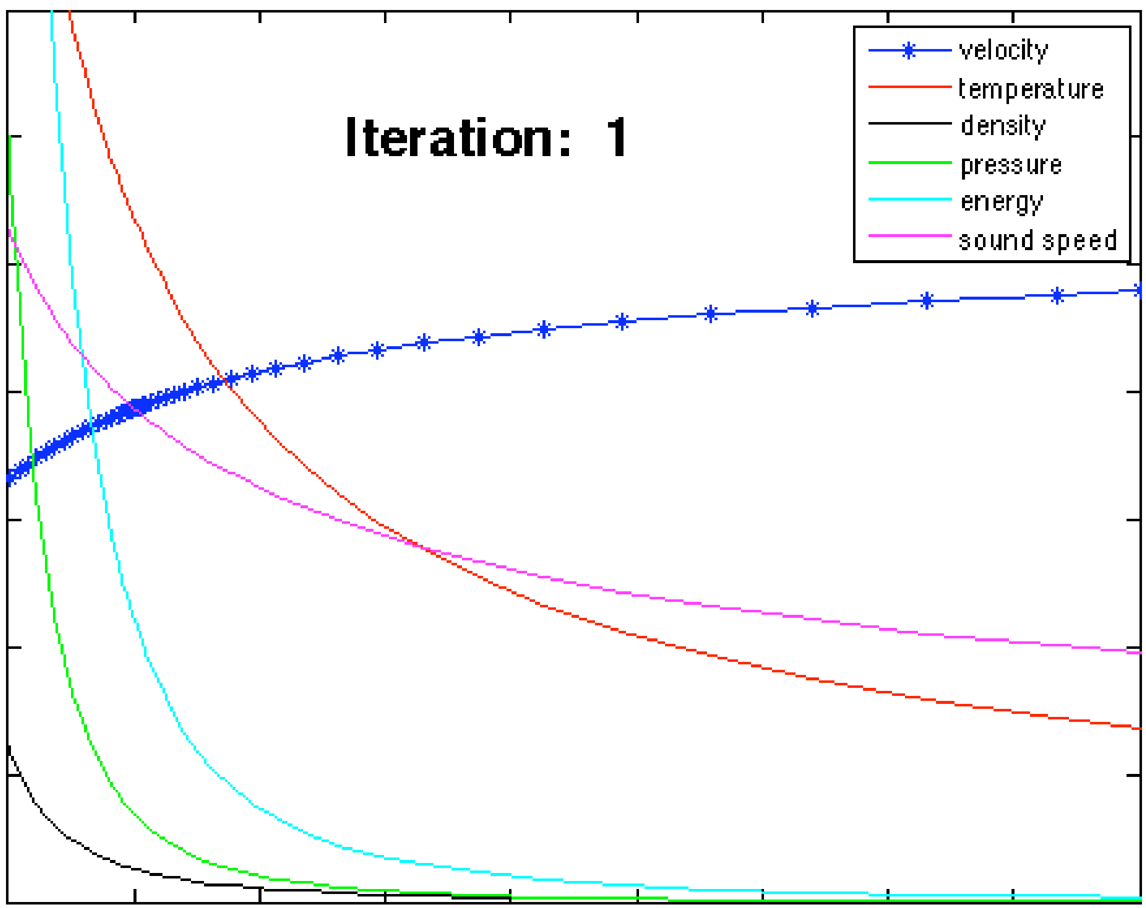
Second component of NCP: use Newton method to match critical point with BCs

guess initial critical point

1. use adaptive ODE integrator to find trajectory
2. modify guess for critical point depending on deviation from desired inflow boundary conditions (2x2 Newton method)
3. repeat



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Quadratic Newton covergence

Newton step k	error $\|B^{(k)} - B^*\|_2$
1	4.41106268600662
2	2.28831581534917
3	1.43924405447424
4	0.10259052732943
5	0.00125578478131
6	0.00000037420499

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NCP method for 1D steady flows

- it is possible to solve steady equations directly, if one uses critical point and dynamical systems knowledge

- (Newton) iteration is still needed

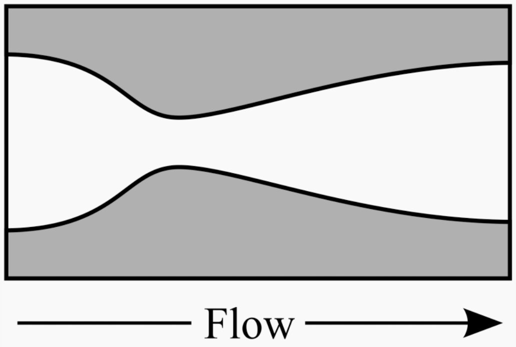
- NCP Newton method solves a 2x2 nonlinear system

(adaptive integration of trajectories is explicit)

- much more efficient than solving a 1500x1500 nonlinear system, and more well-posed

4. Extension to problems with shocks

- consider quasi-1D nozzle flow



$$\frac{\partial}{\partial t}\begin{bmatrix} \rho\, A \\ \rho\, u\, A \\ \left(\frac{p}{\gamma-1}+\frac{\rho\, u^2}{2}\right)\, A \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho\, u\, A \\ \rho\, u^2\, A + p\, A \\ \left(\frac{\gamma\, p}{\gamma-1}+\frac{\rho\, u^2}{2}\right)\, u\, A \end{bmatrix} = \begin{bmatrix} 0 \\ p\,\frac{dA}{dx} \\ 0 \end{bmatrix}.$$

$$\Longrightarrow \qquad \begin{aligned} u_{crit} &= \sqrt{\gamma\, T_{crit}} = c_{crit}, \\ \frac{dA}{dx}(x_{crit}) &= 0. \end{aligned}$$

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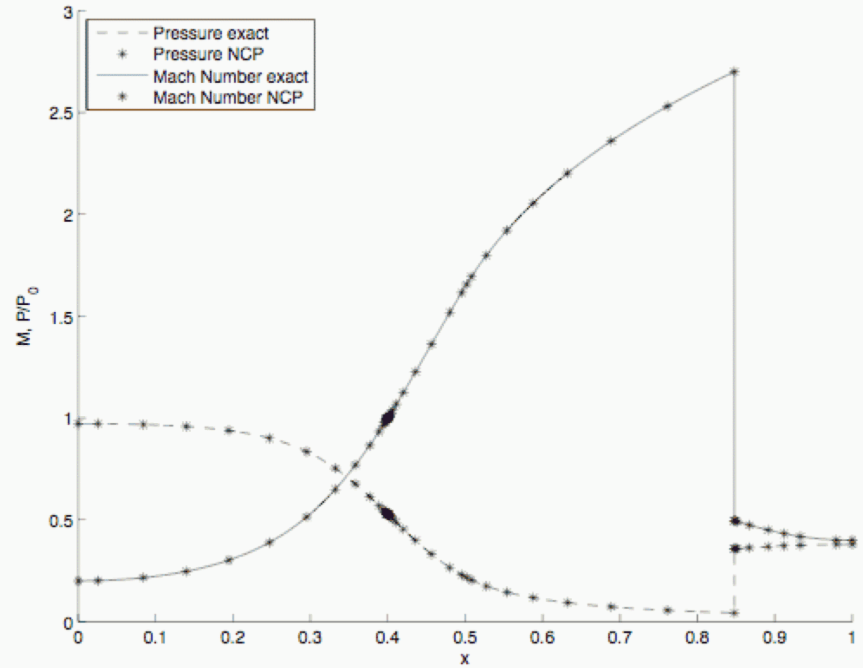
NCP method for nozzle flow with shock

- subsonic in: 2 BC
- subsonic out: 1 BC

- NCP from critical point to match 2 inflow BC

- Newton method to match shock location to outflow BC (using Rankine-Hugoniot relations, 1 free parameter)



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5. Extension to problems with heat conduction

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho\, r^2 \\ \rho\, u\, r^2 \\ \left(\frac{p}{\gamma-1}+\frac{\rho\, u^2}{2}\right)\, r^2 \end{bmatrix} + \frac{\partial}{\partial r}\begin{bmatrix} \rho\, u\, r^2 \\ \rho\, u^2\, r^2 + p\, r^2 \\ \left(\frac{\gamma\, p}{\gamma-1}+\frac{\rho\, u^2}{2}\right)\, u\, r^2 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ -\rho\, G M + 2\, p\, r \\ -\rho\, G M\, u + q_{heat}\, r^2 + \frac{\partial}{\partial r}\left(\kappa\, r^2\, \frac{\partial T}{\partial r}\right) \end{bmatrix}$$

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Dynamical system for Euler with heat conduction

$$\phi = \kappa\, r^2 \tfrac{dT}{dr}$$

$$
\begin{aligned}
\frac{dr}{ds} &= -r^2(u^2-c^2)(u^2-T), \\
\frac{dF}{ds} &= 0, \\
\frac{du}{ds} &= -2uc^2\left(r-\frac{GM}{2c^2}\right)(u^2-T)+\frac{\phi u(u^2-c^2)}{\kappa} \\
&\quad -(\gamma-1)uT(GM-2u^2r), \\
\frac{dT}{ds} &= \frac{-\phi(u^2-c^2)(u^2-T)}{\kappa}, \\
\frac{d\phi}{ds} &= \frac{-\phi F(u^2-c^2)^2}{(\gamma-1)\kappa}+FT(GM-2u^2r)(u^2-c^2) \\
&\quad +q_{heat}r^4(u^2-c^2)(u^2-T).
\end{aligned}
$$

Two types of critical points!

- sonic critical point (node):

$$u_{crit} = \sqrt{\gamma T_{crit}} = c_{crit}$$

- thermal critical point (saddle point):

$$\begin{aligned} u_{crit} &= \sqrt{T_{crit}} = c_{crit}/\sqrt{\gamma}, \\ \frac{\phi_{crit}}{\kappa} + GM - 2u_{crit}^2 r_{crit} &= 0. \end{aligned}$$

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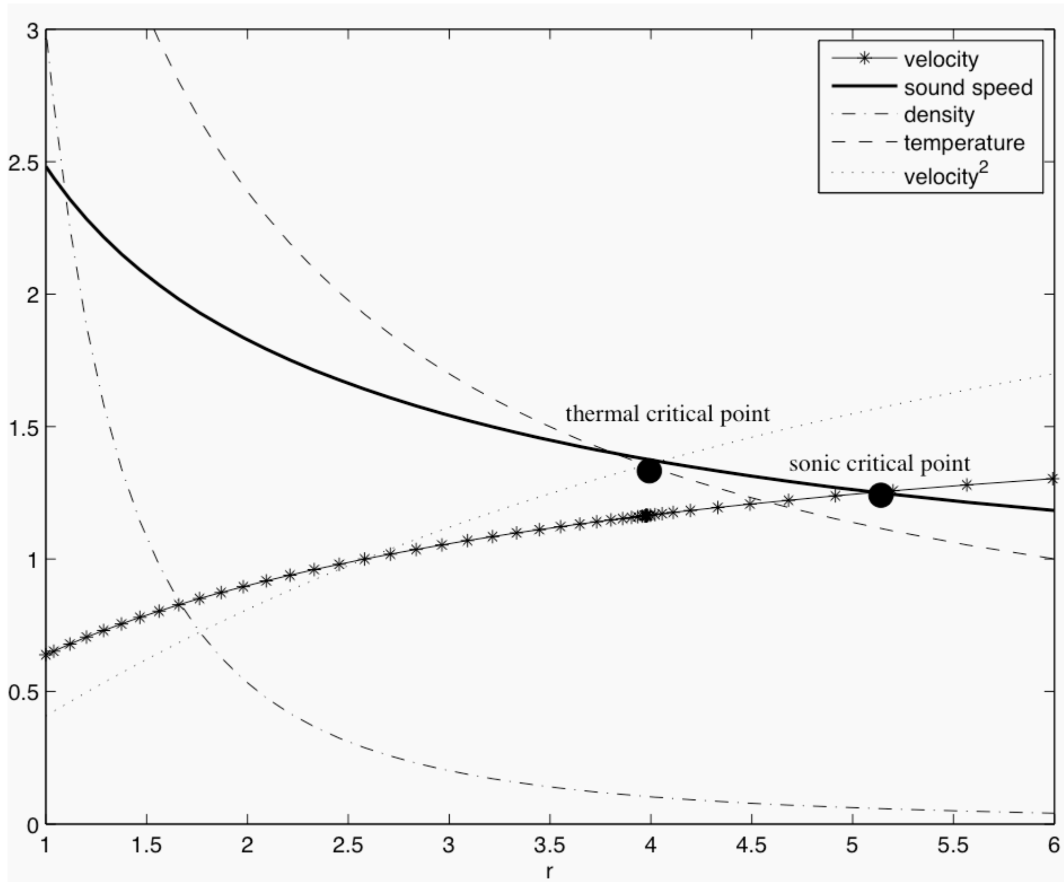
Transonic flow with heat conduction

- subsonic inflow: 3 BC (ρ, p, ϕ)
- supersonic outflow: 0 BC
- 3-parameter family of thermal critical points
- NCP matches thermal critical point with 3 inflow BC



6. Conclusions

- solving steady Euler equations directly is superior to time-marching methods for 1D transonic flows
- NCP uses
  - adaptive itegration outward from critical point
  - dynamical system formulation
  - 2x2 Newton method to match critical point with BC

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Conclusions

- 2D: work in progress
  - integrate separately in different domains of the flow, ‘outward’ from critical curves
  - match conditions at critical curves with BCs using Newton method
  - issues:
    - change of topology
    - solve PDE in different regions
    - cost
  - potential advantages are significant: problem more well-posed
    - fixed number of Newton steps, linear iterations (scalable)
    - better grid sequencing (nested iteration)

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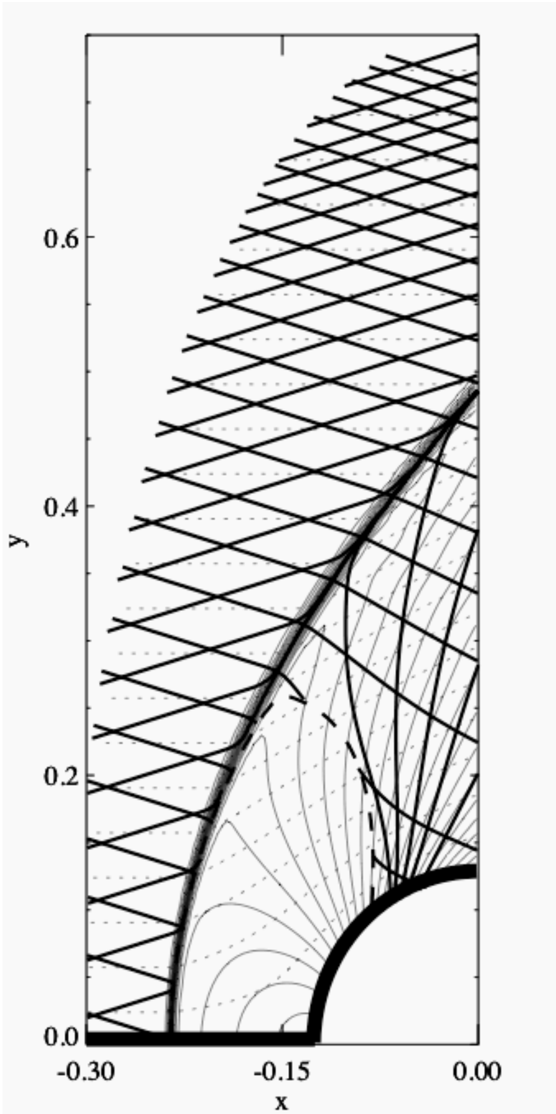
Transonic steady Euler flows



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