LSFEM for Scalar Hyperbolic Conservation Laws: Convergence and Numerical Conservation

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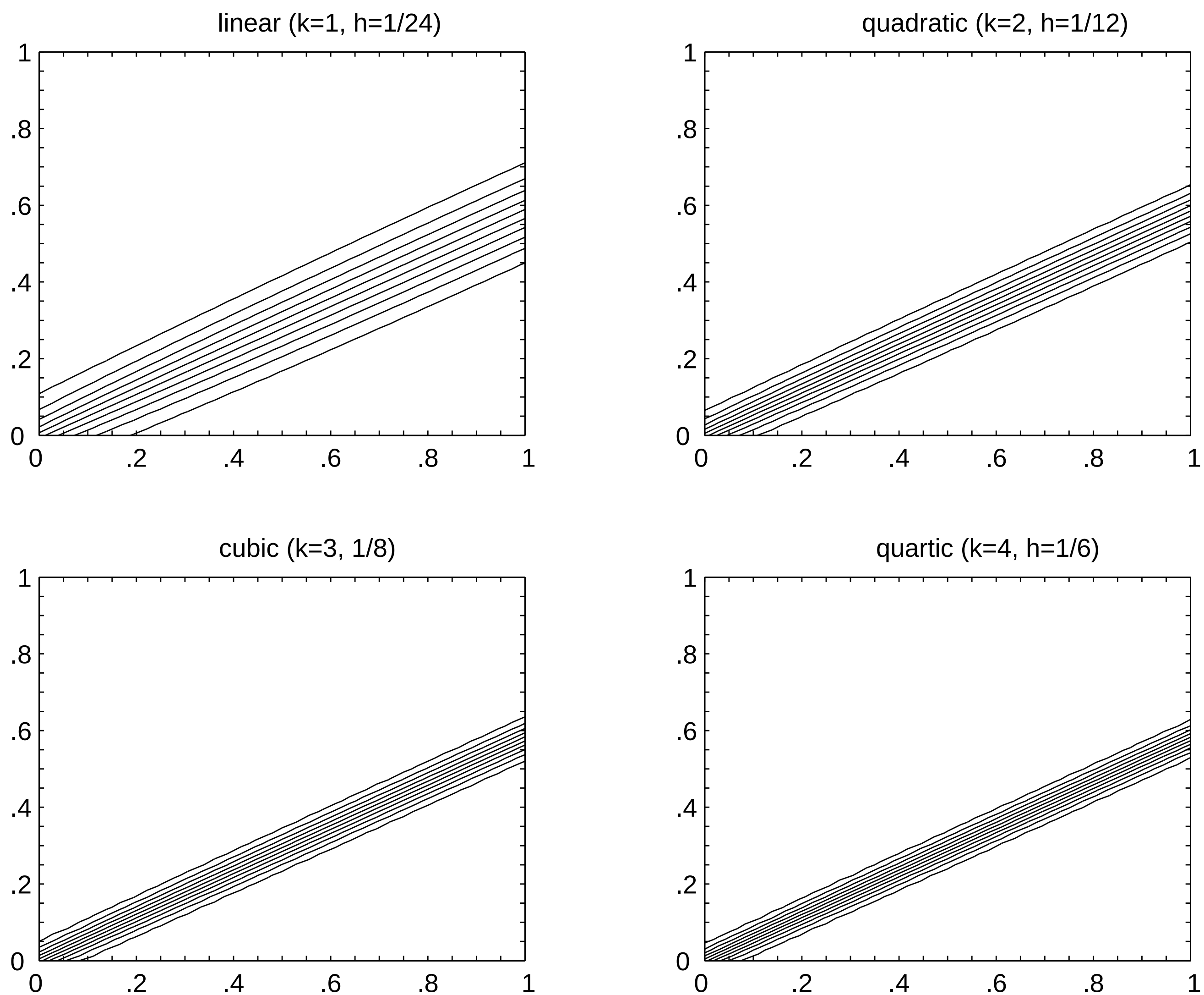
- ‘Least–Squares Finite Element Methods and Algebraic Multigrid Solvers for Linear Hyperbolic PDEs’, SIAM J. Sci. Comput. 26, 31-54, 2004.

- ‘Numerical Conservation Properties of H(div)-Conforming Least-Squares Finite Element Methods for the Burgers Equation’, SIAM J. Sci. Comput. 26, 1573-1597, 2005.

Outline

- (1) LSFEM for the linear advection equation
- (2) Standard LSFEM for the Burgers equation
- (3) $H(div)$-conforming LSFEM for Burgers
- (4) Potential $H(div)$-conforming LSFEM for Burgers
- Numerical results – convergence study
- Numerical conservation – Weak conservation proofs
- Conclusions

(1) Linear advection: $\partial_t u + a\,\partial_x u = 0$



Linear advection: LSFEM

- existence and uniqueness, convergence of discrete problem proved

- continuous and discontinuous elements

- high order elements

(2) Nonlinear hyperbolic conservation law

$$\begin{aligned} \nabla \cdot \vec{f}(u) = 0 \quad & \Omega \\ u = g \quad & \Gamma_I \end{aligned}$$

- $\Omega \subset \mathbb{R}^2$ Γ_I inflow boundary
- space-time domains: $\nabla = (\partial_x, \partial_t)$
- inviscid Burgers equation: $\vec{f}(u) = (u^2/2, u)$

Nonlinear Hyperbolic Conservation Laws

$$\nabla \cdot \vec{f}(u) = 0 \quad \Omega$$
$$u = g \quad \Gamma_I$$

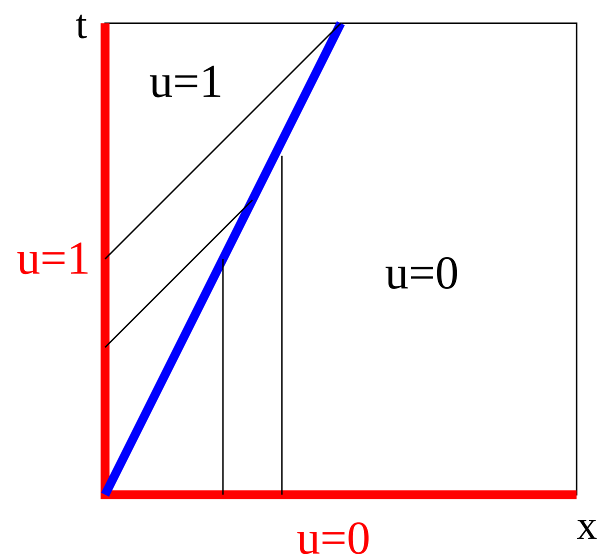
inviscid Burgers equation:

$$\vec{f}(u) = (u^2/2, u)$$

Rankine-Hugoniot relation:

$$[\ \vec{f}(u)\ ] \cdot \vec{n} = 0$$

$$\Rightarrow \vec{f}(u) \in H(div)$$



$$H(div) = \{(v_1, v_2) \in (L_2)^2 \mid \|\nabla \cdot (v_1, v_2)\|_0^2 < \infty\}$$

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Nonlinear hyperbolic conservation law

$$\begin{aligned} \nabla \cdot \vec{f}(u) = 0 & \quad \Omega \\ u = g & \quad \Gamma_I \end{aligned}$$

- weak solutions:
  $$-\left\langle \vec{f}(u), \nabla\phi \right\rangle_{0,\Omega} + \left\langle \vec{n} \cdot \vec{f}(g), \phi \right\rangle_{0,\Gamma_I} = 0 \quad \forall\, \phi \,\in\, C^1_{\Gamma_O}(\overline{\Omega})$$
- restrict to piecewise C^1 functions with jump discontinuities

$$\Rightarrow \quad u \in H^{1/2-\epsilon}(\Omega) \quad \forall\, \epsilon > 0$$

$\Rightarrow$ *THEOREM:* $\vec{f}(u) \in H(div, \Omega)$

Standard LSFEM for the Burgers equation

$$\begin{aligned} \nabla \cdot \vec{f}(u) &= 0 \quad \Omega \\ u &= g \quad \Gamma_I \end{aligned}$$

- LS functional

$$\mathcal{H}(u; g) := \|\nabla \cdot \vec{f}(u)\|^2_{0,\Omega} + \|u - g\|^2_{0,\Gamma_I}$$

- LSFEM

$$u^h_* = \underset{u^h \in \mathcal{U}^h}{\arg\min}\, \mathcal{H}(u^h; g)$$

$\mathcal{U}^h$: continuous bilinear finite elements on quadrilaterals

- Gauss-Newton minimization of LS functional

LSFEM for the Burgers equation

$$H(u) := \nabla \cdot \vec{f}(u) = 0 \quad \Omega$$
$$u = g \quad \Gamma_I$$

- Gauss-Newton minimization of LS functional:
  - first: Newton linearization of $H(u) = 0$

    $$H(u_i) + H'|_{u_i}(u_{i+1} - u_i) = 0$$

    with Fréchet derivative

    $$H'|_{u_i}(v) = \nabla \cdot (\vec{f}'|_{u_i}\, v)$$

  - then: LS minimization of linearized $H(u)$

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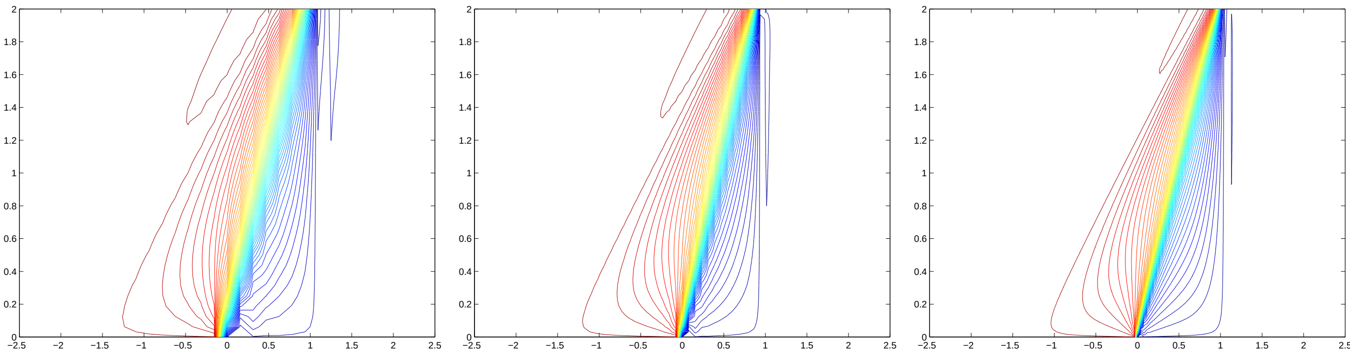
Numerical Results

shock flow: $u_{left} = 1$, $u_{right} = 0$, shock speed $s = 1/2$



32^2 64^2 128^2

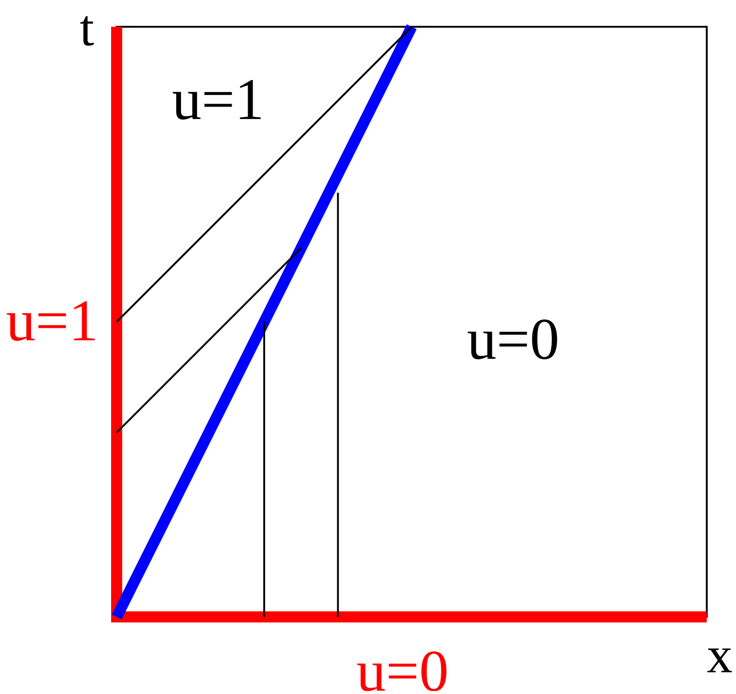
- correct shock speed, no oscillations
- on each grid, Newton process converges
- BUT: for $h \to 0$, nonlinear functional does not go to zero
- this means: for $h \to 0$, convergence to an incorrect solution!!! ($L^* L$ has a spurious stationary point)
- why does LSFEM produce wrong solution??

Divergence of Newton's method

- LS functional

$$\mathcal{H}(u;g) := \|\nabla \cdot \vec{f}(u)\|_{0,\Omega}^2 + \|u - g\|_{0,\Gamma_I}^2$$

- reason: functional is not continuous for $u \in H^{1/2-\epsilon}(\Omega)$



Divergence of Newton's method

- consequence: Fréchet derivative operator is unbounded

Burgers: $H'|_{u_0}(v) = \nabla \cdot ((u_0, 1)\ v)$

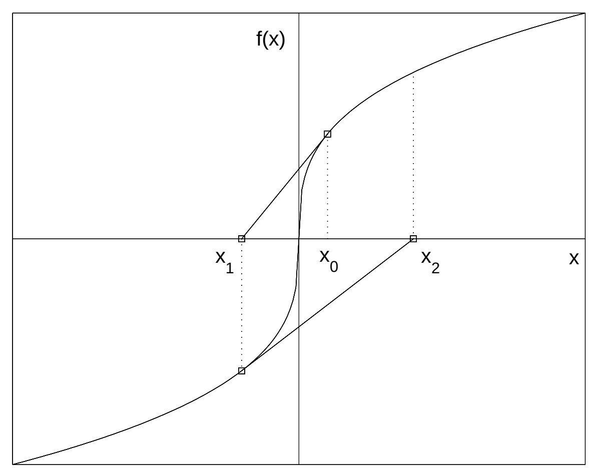
operator $H'|_{u_0} : v \in H^{1/2-\epsilon}(\Omega) \rightarrow L^2(\Omega)$

$$\Rightarrow \|\ H'|_{u_0}\ \|_{0,\Omega} = \infty$$

because $\forall\ u_0 \in H^{1/2-\epsilon}(\Omega), \exists\ v \in H^{1/2-\epsilon}(\Omega)$:

$$((u_0, 1)\ v) \notin H(div, \Omega)$$

example: $h(x) = \mp|x|^{1/3}$
$\Rightarrow x_1 = -2x_0$
Newton with $h'(x_*) = \infty$ may have empty basin of attraction



(3) $H(div)$-conforming LSFEM

- reformulate conservation law in terms of flux vector $\vec{w}$:

$$\begin{aligned} \nabla \cdot \vec{f}(u) &= 0 \quad \Omega \\ u &= g \quad \Gamma_I \end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned} \nabla \cdot \vec{w} &= 0 && \Omega \\ \vec{w} &= \vec{f}(u) && \Omega \\ \vec{n} \cdot \vec{w} &= \vec{n} \cdot \vec{f}(g) && \Gamma_I \\ u &= g && \Gamma_I \end{aligned}}$$

- Gauss-Newton applied to

$$\begin{aligned} \mathcal{F}(\vec{w}^h, u^h; g) =& \|\nabla \cdot \vec{w}^h\|_{0,\Omega}^2 + \|\vec{w}^h - \vec{f}(u^h)\|_{0,\Omega}^2 \\ &+ \|\vec{n} \cdot (\vec{w}^h - \vec{f}(g))\|_{0,\Gamma_I}^2 + \|u^h - g\|_{0,\Gamma_I}^2 \end{aligned}$$

- $\vec{w}^h \in RT_0 \subset H(div, \Omega)$, and u^h continuous bilinear

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(4) Potential $H(div)$-conforming LSFEM

- $\nabla \cdot \vec{f}(u) = 0$ implies $\vec{f}(u) = \nabla^{\perp}\psi$ for some $\psi \in H^1(\Omega)$

$\Rightarrow$ reformulate conservation law in terms of flux potential ψ:

$$\begin{aligned} \nabla \cdot \vec{f}(u) &= 0 \quad \Omega \\ u &= g \quad \Gamma_I \end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned} \nabla^{\perp}\psi - \vec{f}(u) &= 0 && \Omega \\ \vec{n} \cdot \nabla^{\perp}\psi &= \vec{n} \cdot \vec{f}(g) && \Gamma_I \\ u &= g && \Gamma_I \end{aligned}}$$

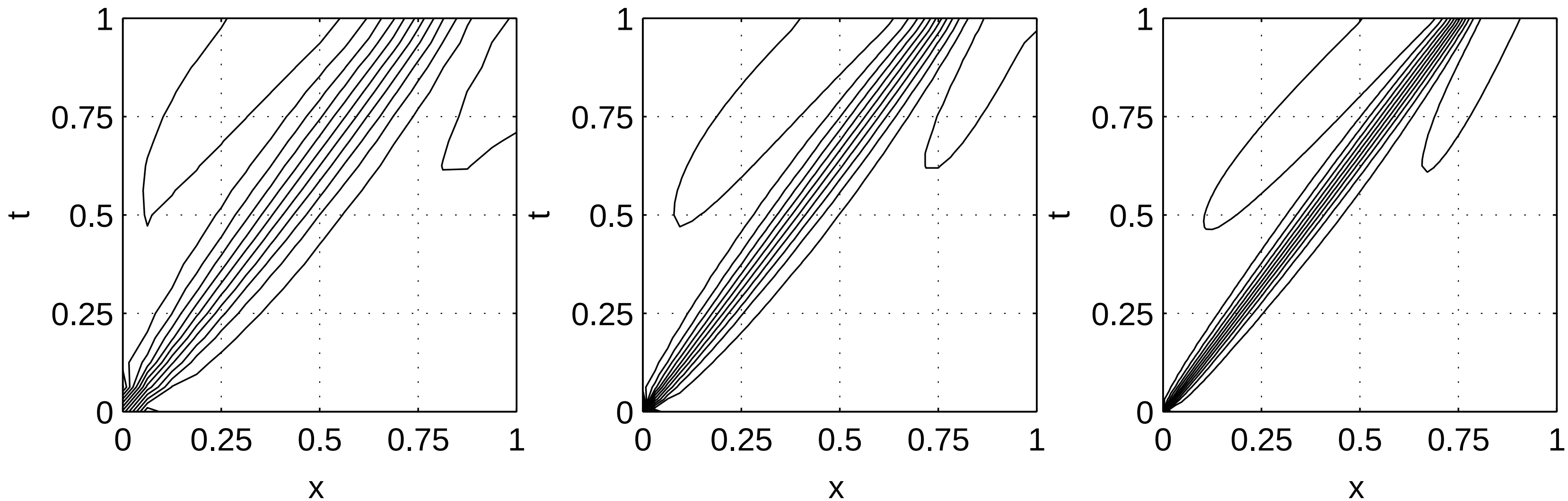
- Gauss-Newton applied to

$$\mathcal{G}(\psi^h, u^h; g) := \|\nabla^{\perp}\psi^h - \vec{f}(u^h)\|^2_{0,\Omega} + \|\vec{n} \cdot (\nabla^{\perp}\psi^h - \vec{f}(g))\|^2_{0,\Gamma_I} + \|u^h - g\|^2_{0,\Gamma_I}$$

- ψ^h and u^h continuous bilinear

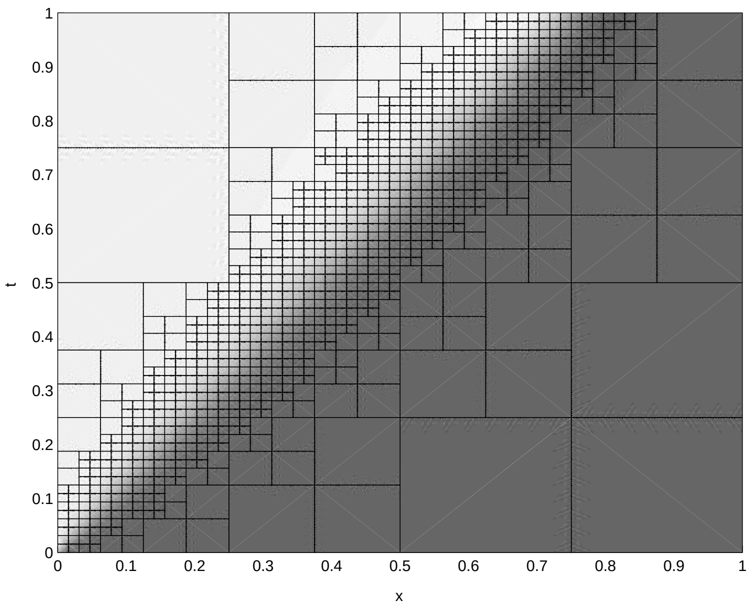
Numerical results

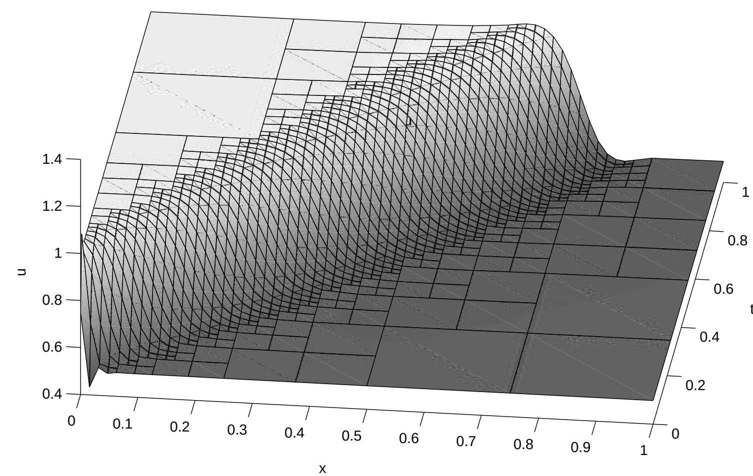
- shock flow: $u_{left} = 1.0$, $u_{right} = 0.5$, shock speed $s = 0.75$
- $H(div)$-conforming LSFEM:



Numerical results

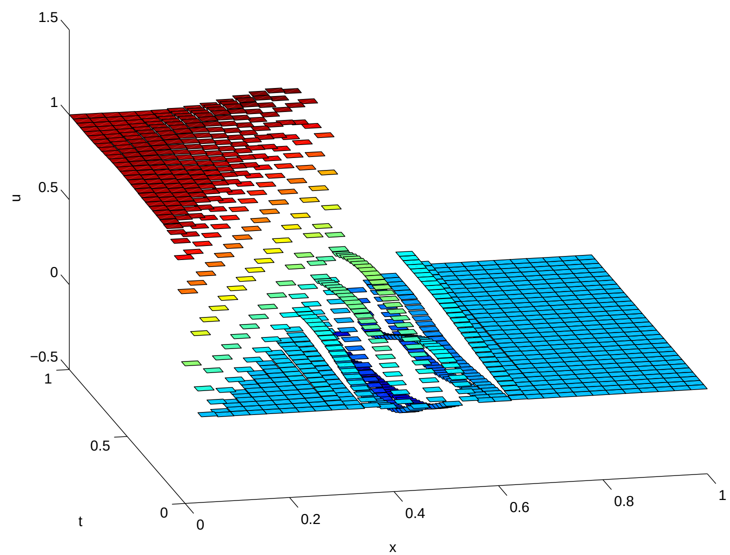
- potential $H(div)$-conforming LSFEM:





Numerical results – choice of spaces

- for u^h piecewise constant (discontinuous): oscillations!



- reason: the functionals are not uniformly coercive
- for right choices of FE spaces (*e.g.*, u^h continuous bilinear), numerical evidence suggests FE convergence
- we have some heuristic understanding of this, but rigorous proofs not yet obtained
- potential formulation is equivalent to H^{-1} minimization

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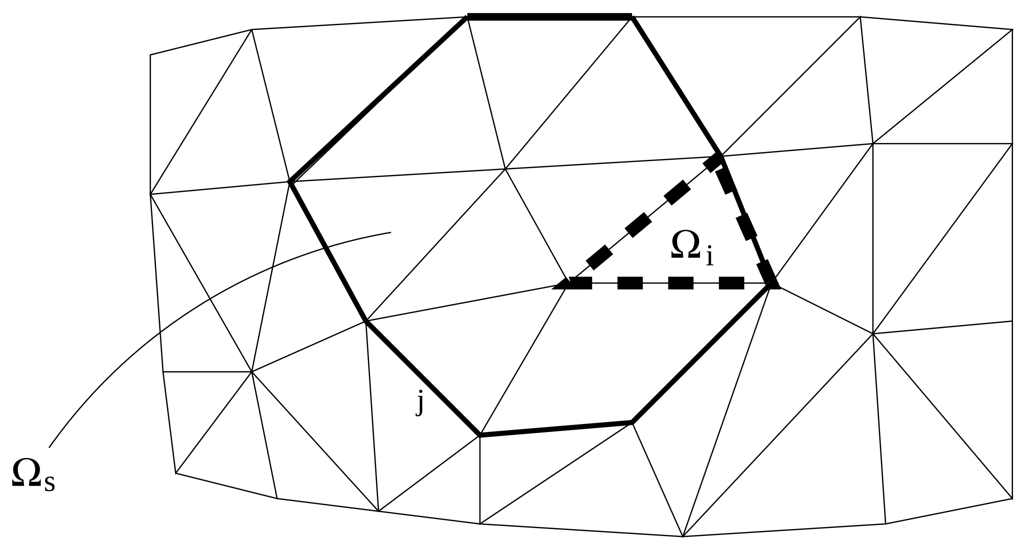
Numerical conservation

- Lax-Wendroff theorem: exact discrete conservation

$$\nabla_{discrete} \cdot \vec{f}(u^h) := \oint_{\partial\Omega_i} \vec{n} \cdot \vec{\bar{f}}(u^h)\, dl = 0 \quad \forall\, \Omega_i$$

guarantees convergence to a weak solution

(assuming convergence of u^h to $\hat{u}$ boundedly a.e.)



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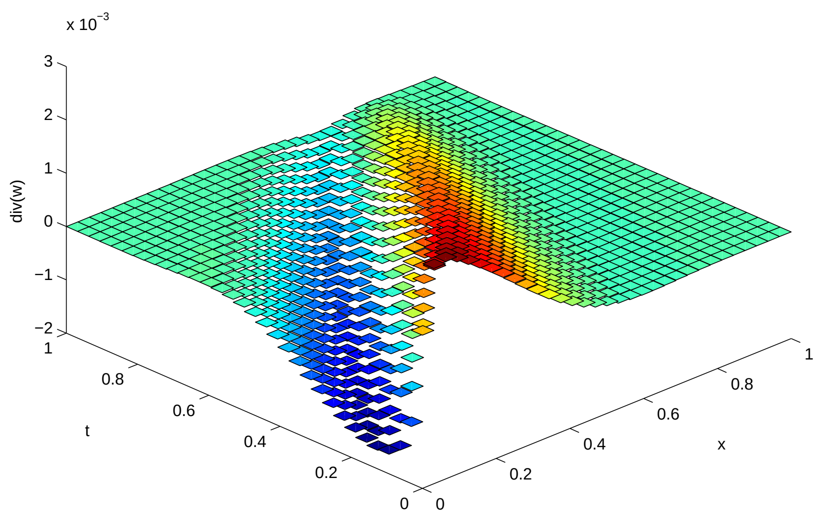
Numerical conservation

- our $H(div)$-conforming LSFEM do not satisfy the exact discrete conservation property of Lax and Wendroff
- $H(div)$-conforming LSFEM:

  $\nabla \cdot \vec{f}(u^h) \neq 0$, and also $\nabla \cdot \vec{w}^h \neq 0$



$\nabla \cdot \vec{w}^h$

- potential $H(div)$-conforming LSFEM:

  $\nabla \cdot \vec{f}(u^h) \neq 0$, but $\nabla \cdot \nabla^{\perp} \psi^h \equiv 0$

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Numerical conservation

- however, we can prove:

THEOREM. [Conservation for $H(div)$-conforming LSFEM]

If finite element approximation u^h converges in the L^2 sense to $\hat{u}$ as $h \longrightarrow 0$, then $\hat{u}$ is a weak solution of the conservation law.

THEOREM. [Conservation for potential $H(div)$-conforming LSFEM]

If finite element approximation u^h converges in the L^2 sense to $\hat{u}$ as $h \longrightarrow 0$, then $\hat{u}$ is a weak solution of the conservation law.

$\Rightarrow$ exact discrete conservation is not a necessary condition for numerical conservation!

(can be re placed by minimization in a suitable continuous norm)

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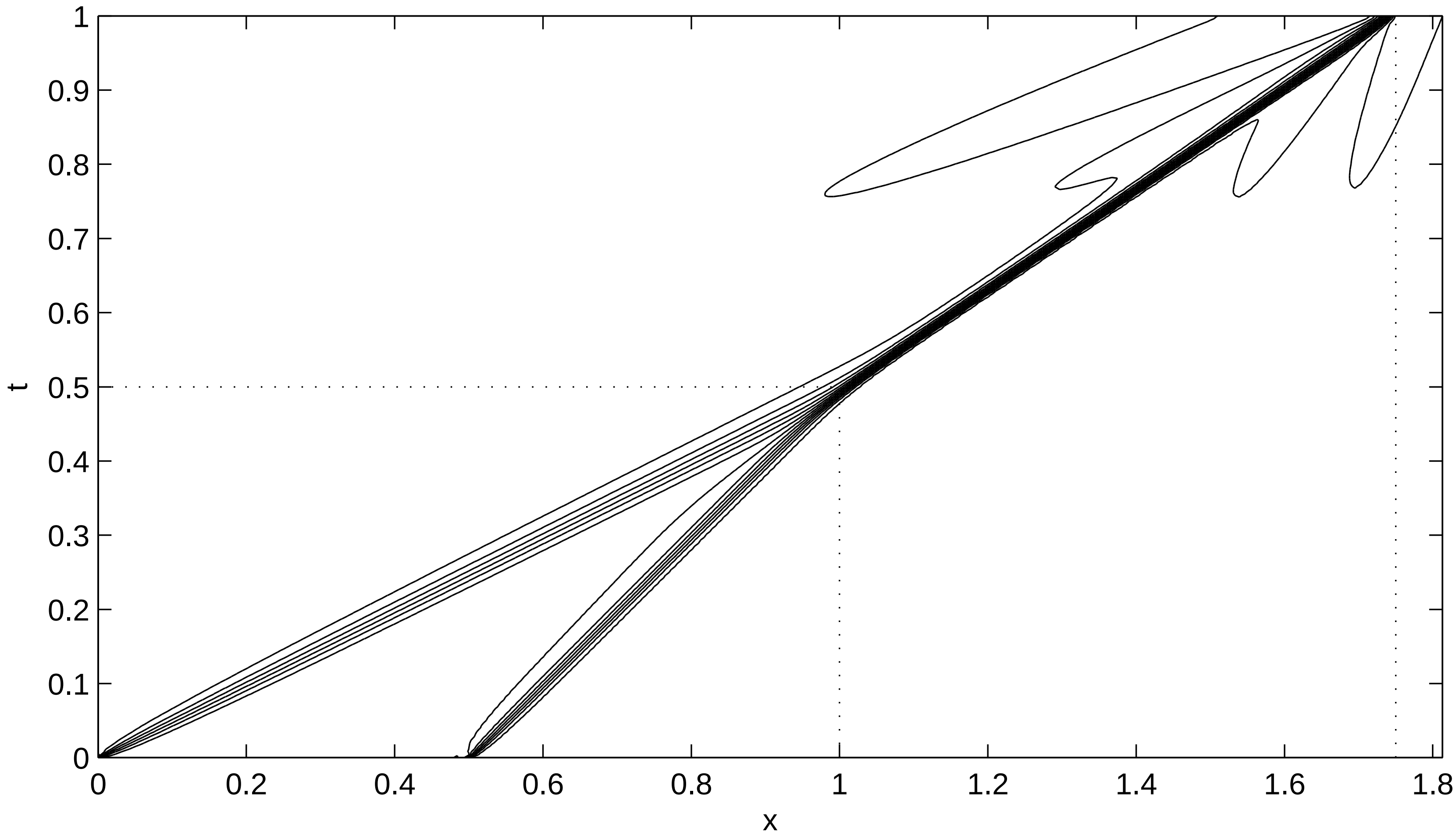
Numerical conservation



Conclusions

we have developed two classes of $H(div)$-conforming LSFEM for hyperbolic conservation laws

- disadvantages
  - extra variables are introduced ($\vec{w}$ or ψ)
  - smearing of LSFEM at shocks
- advantages of LSFEM
  - optimal solution within finite element space
  - SPD linear systems (iterative methods, AMG)
  - error estimator (efficient adaptive refinement)
  - convergence to weak solution
  - no spurious oscillations at discontinuities (without need to add numerical diffusion)
  - extension to *linear* higher order schemes

Conclusions

- advantages of flux vector/flux potential reformulations

  - bounded Fréchet derivative $\Rightarrow$ Newton converges

  - smoothness of the solution ($\vec{f}(u) \in H(div)$) is made explicit, also at the discrete level using Raviart-Thomas elements ($\Rightarrow H(div)$-conforming LSFEM)

  - differential part of operator is linear

- L^2 convergence theory?? coercivity? suitable FE spaces?

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Error Estimator and Adaptive Refinement

$$
\begin{aligned}
\mathcal{F}(u^h) &= \|Lu^h\|^2_{0,\Omega} \\
&= \|Lu^h - Lu_{exact}\|^2_{0,\Omega} \\
&= \|L(u^h - u_{exact})\|^2_{0,\Omega} \\
&= \|Le^h\|^2_{0,\Omega}
\end{aligned}
$$

- functional value gives sharp local a posteriori error estimator
- use error estimator for adaptive refinement in space–time
- error estimator is significant advantage of LSFEM

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Numerical results – convergence study

- estimate α in $\|u^h - u\|_{0,\Omega}^2 \approx \mathcal{O}(h^\alpha)$

  $u \in H^{1/2-\epsilon}(\Omega)$ discontinuous $\Rightarrow$ optimal $\alpha = 1.0$

  i.e., $\|u^h - u\|_{0,\Omega}^2 \approx \mathcal{O}(h)$, or $\|u^h - u\|_{0,\Omega} \approx \mathcal{O}(h^{1/2})$

- estimate α in $\mathcal{F}(\vec{w}^h, u^h; g) \approx \mathcal{O}(h^\alpha)$

- estimate α in $\mathcal{G}(\psi^h, u^h; g) \approx \mathcal{O}(h^\alpha)$

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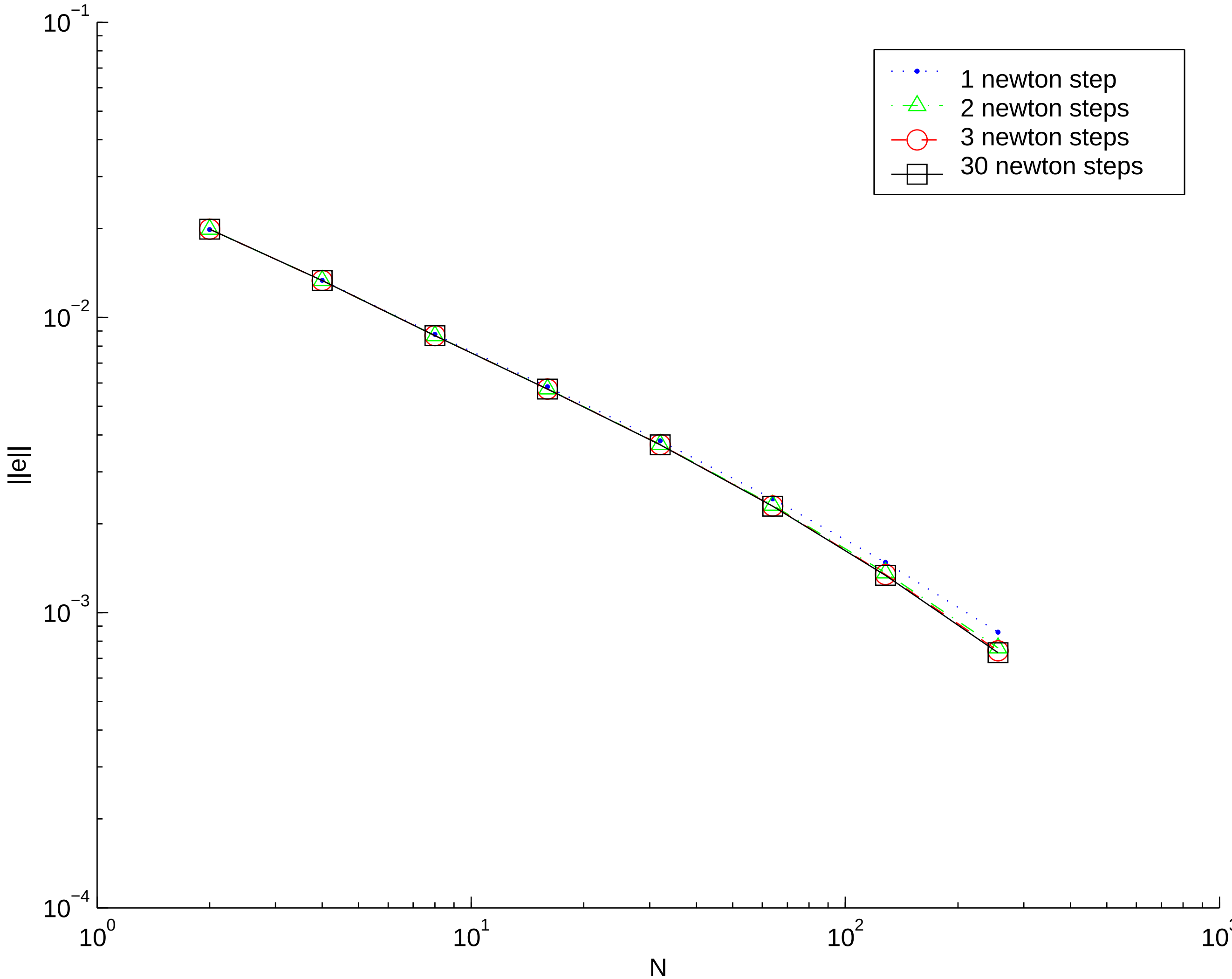
Numerical results – convergence study

N	$\|u^h - u\|^2_{0,\Omega}$	α	$\mathcal{F}(\vec{w}^h, u^h)$	α
16	5.96e-3		1.89e-2	
		0.58		1.03
32	3.81e-3		9.25e-3	
		0.69		1.02
64	2.36e-3		4.56e-3	
		0.77		1.01
128	1.38e-3		2.26e-3	
		0.85		1.01
256	7.66e-4		1.12e-3	

FMG Newton $\|u^h - u\|_{0,\Omega}$ convergence



$H(div)$-conforming LSFEM

- nonlinear operator

$$F(\vec{w}, u) := \begin{bmatrix} \nabla \cdot \vec{w} \\ \vec{w} - \vec{f}(u) \end{bmatrix} = 0$$

- Fréchet derivative:

$$F'|_{(\vec{w}_0,u_0)}(\vec{w}_1 - \vec{w}_0, u_1 - u_0) = \begin{bmatrix} \nabla\cdot & 0 \\ I & -\vec{f}'|_{u_0} \end{bmatrix} \cdot \begin{bmatrix} \vec{w}_1 - \vec{w}_0 \\ u_1 - u_0 \end{bmatrix}$$

LEMMA. Fréchet derivative operator
$F'|_{(\vec{w}_0,u_0)} : H(div, \Omega) \times L^2(\Omega) \longrightarrow L^2(\Omega)$ is bounded:

$$\| \, F'|_{(\vec{w}_0,u_0)} \, \|_{0,\Omega} \leq \sqrt{1 + K^2}$$

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Potential $H(div)$-conforming LSFEM

- nonlinear operator

$$G(\psi, u) := \nabla^{\perp}\psi - \vec{f}(u) = 0$$

- Fréchet derivative:

$$G'|_{(\psi_0,u_0)}(\psi_1 - \psi_0, u_1 - u_0) = \begin{bmatrix} \nabla^{\perp} & -\vec{f'}|_{u_0} \end{bmatrix} \cdot \begin{bmatrix} \psi_1 - \psi_0 \\ u_1 - u_0 \end{bmatrix}$$

LEMMA. Fréchet derivative operator $G'|_{(\psi_0,u_0)} : H^1(\Omega) \times L^2(\Omega) \to L^2(\Omega)$ is bounded:

$$\| \, G'|_{(\psi_0,u_0)} \, \|_{0,\Omega} \leq \sqrt{1 + K^2}$$

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