Least-Squares Finite Element Methods for Nonlinear Hyperbolic PDEs

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Outline

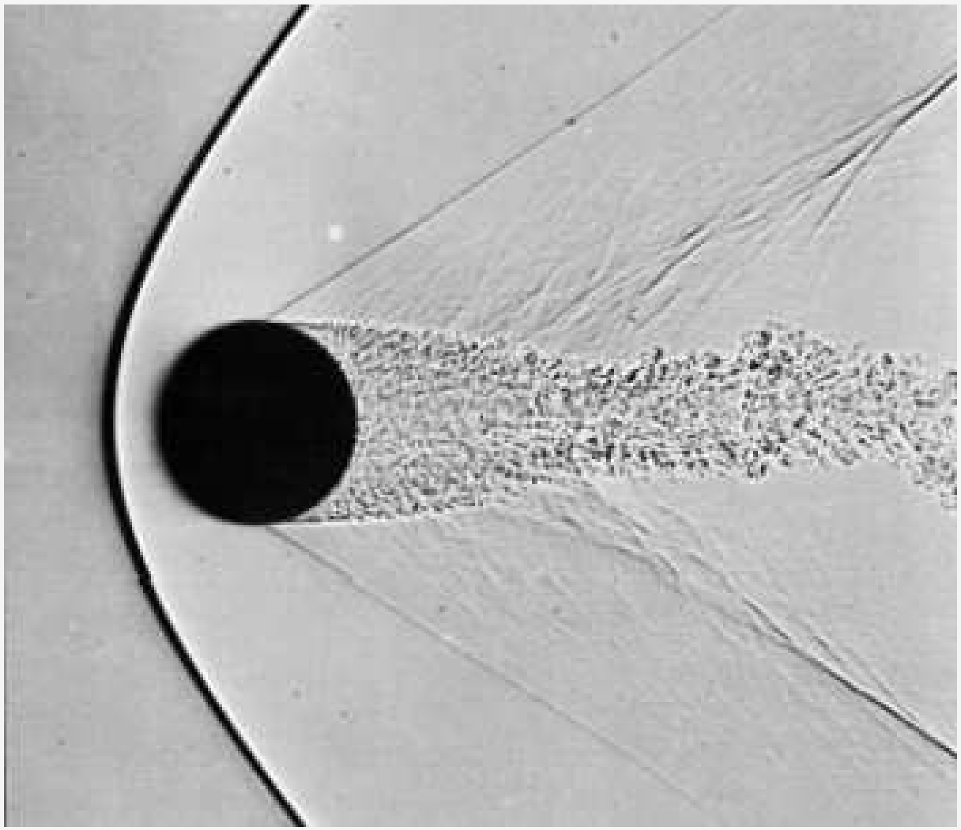
(1) Hyperbolic Conservation Laws: Introduction

(2) Least-Squares Finite Element Methods

(3) Fluid Dynamics Applications

(1) Numerical Simulation of Nonlinear Hyperbolic PDE Systems

Example application: gas dynamics



- supersonic flow of air over sphere (M=1.53)
- bow shock
- (An album of fluid motion, Van Dyke)

Nonlinear Hyperbolic Conservation Laws

- Euler equations of gas dynamics

$$\boxed{\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho\,\vec{v} \\ \rho\,e\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\,\vec{v} \\ \rho\,\vec{v}\,\vec{v} + p\,\vec{I} \\ (\,\rho\,e + p(\rho,e)\,)\;\vec{v}\end{bmatrix} = 0}$$

- nonlinear hyperbolic PDE system

$$\boxed{\frac{\partial U}{\partial t} + \nabla\cdot\vec{F}(U) = 0}$$

- conservation law

$$\frac{\partial}{\partial t}\left(\int_{\Omega} U\;dV\right) + \oint_{\partial\Omega} \vec{n}\cdot\vec{F}(U)\;dA = 0$$

Model Problem: Scalar Inviscid Burgers Equation

- scalar conservation law in 1D

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

- model problem: inviscid Burgers equation

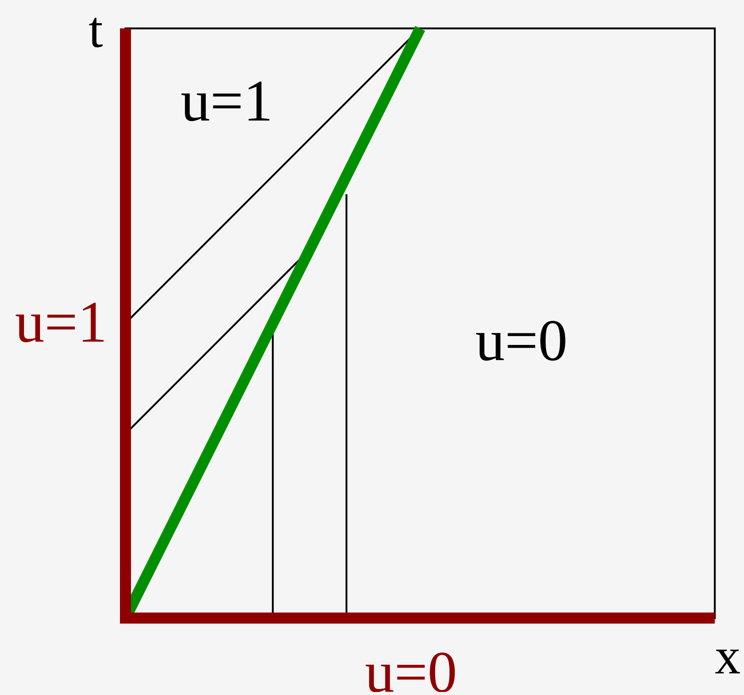
$$\frac{\partial u}{\partial t} + \frac{\partial u^2/2}{\partial x} = 0$$

Burgers Equation: Model Flow

$$\frac{\partial u}{\partial t} + \frac{\partial u^2/2}{\partial x} = 0$$

- hyperbolic PDE: information propagates along characteristic curves
- u is constant on characteristic
- u is slope of characteristic
- where characteristics cross:

shock formation (weak solution)



Space-Time Formulation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

- define $\nabla_{x,t} = (\partial_x, \partial_t)$
- define $\vec{f}_{x,t}(u) = (f(u), u)$

$$\begin{aligned} \nabla_{x,t} \cdot \vec{f}_{x,t}(u) = 0 \quad & \Omega \subset \mathbb{R}^2 \\ u = g \quad & \Gamma_I \end{aligned}$$

- conservation in space-time

$$\oint_\Gamma \vec{n}_{x,t} \cdot \vec{f}_{x,t}(u)\, dl = 0$$

Some Notation

- L_2 scalar product

$$\langle f, g \rangle_{0,\Omega} = \int_\Omega f\, g\, dxdt$$

- L_2 norm

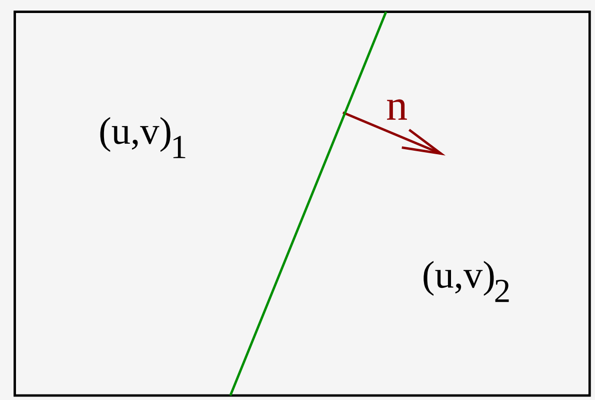
$$\| f \|_{0,\Omega} = \sqrt{\int_\Omega f^2\, dxdt}$$

- space $H(div, \Omega)$

$$\{\ (u, v) \in L_2 \times L_2 \quad | \quad \|\nabla \cdot (u, v)\|^2_{0,\Omega} < \infty\ \}$$

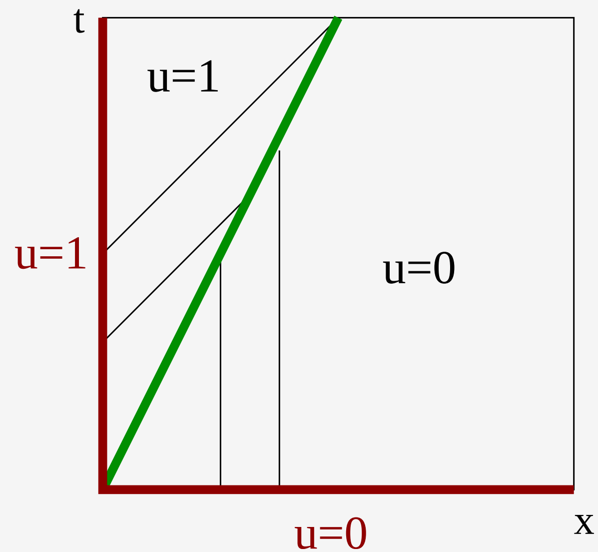
remark: (u, v) can be discontinuous, with normal component continuous:

$$\vec{n} \cdot ((u, v)_2 - (u, v)_1) = 0$$



Weak Solutions: Discontinuities

$$\nabla_{x,t} \cdot \vec{f}_{x,t}(u) = 0 \quad \Omega$$
$$u = g \quad \Gamma_I$$



(1) Rankine-Hugoniot relations: $\vec{n}_{x,t} \cdot (\vec{f}_{x,t}(u_2) - \vec{f}_{x,t}(u_1)) = 0$

(2) equivalent: $\vec{f}_{x,t}(u) \in H(div, \Omega)$ (solution regularity)

Burgers model flow: $\vec{f}_{x,t}(u) \in H(div, \Omega) \iff$ shock speed $s = \frac{1}{2}$

Numerical Approximation: Finite Differences

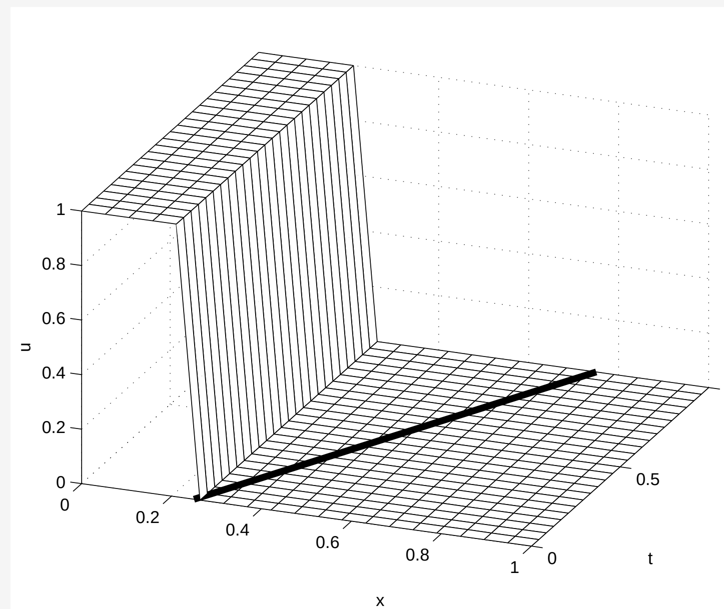
- derivatives $\Rightarrow$ use truncated Taylor series expansion

$$\Rightarrow \left.\frac{\partial u}{\partial x}\right|_i = \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x)$$

- Burgers: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad \frac{u^h_{i,n+1} - u^h_{i,n}}{\Delta t} + u^h_{i,n}\frac{u^h_{i,n} - u^h_{i-1,n}}{\Delta x} = 0$

$\Rightarrow$ convergence to wrong solution!

- reason: Taylor expansion not valid at shock!

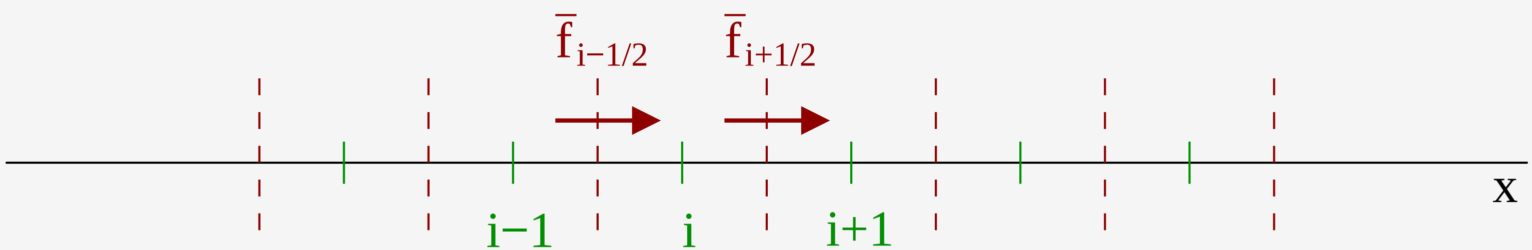


Conservative Finite Difference Schemes

THEOREM. Lax-Wendroff (1960).

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \quad \rightarrow \quad \frac{u^h_{i,n+1} - u^h_{i,n}}{\Delta t} + \frac{\bar{f}_{i+1/2,n}(u^h) - \bar{f}_{i-1/2,n}(u^h)}{\Delta x} = 0$$

theorem: conservative finite difference scheme guarantees convergence to a correct weak solution (assuming convergence of u^h to some $\hat{u}$)



$\Rightarrow$ 'conservative' form is a *sufficient* condition for convergence to a weak solution (but it may not be *necessary*! . . .)

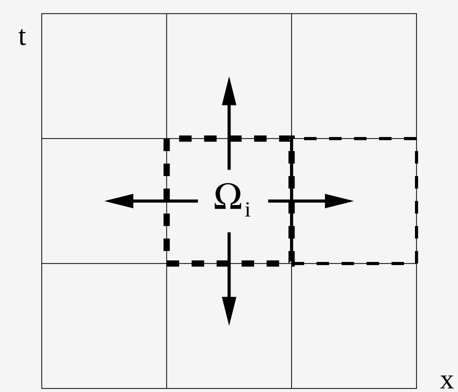
Why the Name 'Conservative Scheme'?

$$\frac{u^h_{i,n+1} - u^h_{i,n}}{\Delta t} + \frac{\bar{f}_{i+1/2,n}(u^h) - \bar{f}_{i-1/2,n}(u^h)}{\Delta x} = 0$$

$$\oint_{\partial\Omega_i} \vec{n}_{x,t} \cdot (\bar{f}(u^h), u^h)\ dl = 0 \quad \forall\ \Omega_i$$

- recall conservation in space-time $\displaystyle\oint_{\partial\Omega} \vec{n}_{x,t} \cdot \vec{f}_{x,t}(u)\ dl = 0$

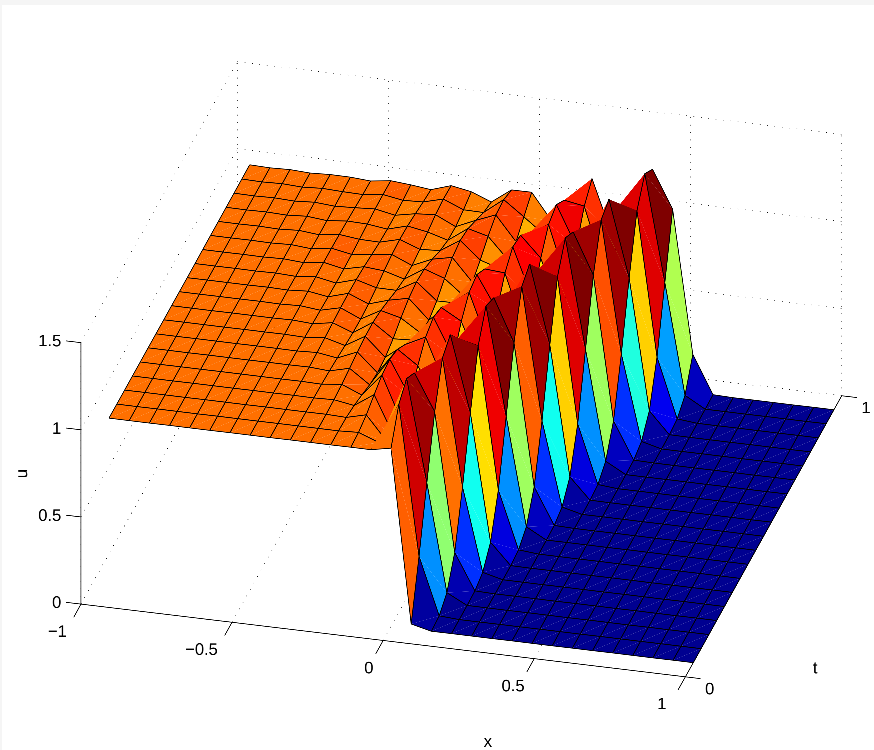
$\Rightarrow$ exact discrete conservation in every discrete cell Ω_i



- exact discrete conservation constrains the solution, s.t. convergence to a solution with wrong shock speed cannot happen

Lax-Wendroff Scheme

$$\bar{f}_{i+1/2} = \frac{1}{2}\left(\left(\frac{u_{i+1}}{2}\right)^2 + \left(\frac{u_i}{2}\right)^2 - \frac{\Delta t}{\Delta x}\left(\frac{u_i + u_{i+1}}{2}\right)^2 (u_{i+1} - u_i)\right)$$



- conservative
- $O(\Delta x^2)$ (Taylor)
- correct shock speed
- ...oscillations!

Possible Remedy: Numerical Diffusion

- add numerical diffusion

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \eta_{num} \frac{\partial^2 u}{\partial x^2}$$

  - $\eta_{num} = O(\Delta x^2)$, e.g.
  - problem: need nonlinear limiters
  - problem: higher-order difficult

- this 'stabilization by numerical diffusion' approach is employed in

  - upwind schemes
  - finite volume schemes
  - most existing finite element schemes

Alternative: Solution Control through Functional Minimization

- minimize the error in a continuous norm

$$u_*^h = \underset{u^h \in \mathcal{U}^h}{arg\ min}\ \|\nabla_{x,t} \cdot \vec{f}_{x,t}(u^h)\|_{0,\Omega}^2$$

- goal:

  - control oscillations
  - control convergence to weak solution
  - control numerical stability (no need for time step limitation)
  - higher-order finite elements

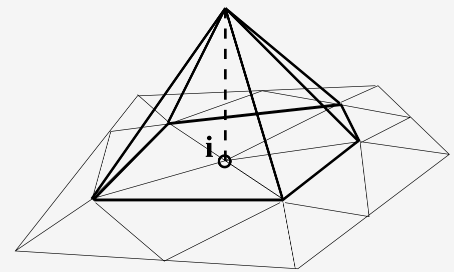
  $\Rightarrow$ achieve through norm minimization

(remark: $h = \Delta x$)

(2) Least-Squares Finite Element (LSFEM) Discretizations

with Luke Olson, Tom Manteuffel, Steve McCormick, Applied Math CU Boulder

- finite element method: approximate $u \in \mathcal{U}$ by $u^h \in \mathcal{U}^h$



$$u^h(x,t) = \sum_{i=1}^{n} u_i \, \phi_i(x,t)$$

- abstract example: solve $Lu = 0$ (assume L linear PDE operator)

- define the functional $\mathcal{F}(u) = \|Lu\|_{0,\Omega}^2$

Least-Squares Finite Element (LSFEM) Discretizations

$\Rightarrow$ minimization:

$$u_*^h \; = \; \underset{u^h \in \, \mathcal{U}^h}{arg\; min} \; \|Lu^h\|_{0,\Omega}^2 \; = \; arg\; min\; \mathcal{F}(u^h)$$

- condition for u^h stationary point:

$$\frac{\partial \mathcal{F}(u^h + \alpha v^h)}{\partial \alpha} \,\Big|_{\alpha=0} = 0 \quad \forall\, v^h \in \mathcal{U}^h$$

Least-Squares Finite Element Discretizations

- algebraic system of linear equations:

$$\sum_{i=1}^{n} u_i \langle L\phi_i, L\phi_j \rangle_{0,\Omega} = 0$$

(n equations in n unknowns, $\mathsf{A}\,\mathsf{u} = 0$)

(actually, with boundary conditions, $\mathsf{A}\,\mathsf{u} = \mathsf{f}$)

- Symmetric Positive Definite (SPD) matrices A

$H(div)$-Conforming LSFEM for Hyperbolic Conservation Laws

- reformulate conservation law in terms of flux vector $\vec{w}$:

$$\begin{aligned} \nabla_{x,t} \cdot \vec{f}_{x,t}(u) &= 0 \quad \Omega \\ u &= g \quad \Gamma_I \end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned} \nabla_{x,t} \cdot \vec{w} &= 0 && \Omega \\ \vec{w} &= \vec{f}_{x,t}(u) && \Omega \\ \vec{n}_{x,t} \cdot \vec{w} &= \vec{n}_{x,t} \cdot \vec{f}_{x,t}(g) && \Gamma_I \\ u &= g && \Gamma_I \end{aligned}}$$

- functional

$$\mathcal{F}(\vec{w}^h, u^h; g) = \|\nabla_{x,t} \cdot \vec{w}^h\|^2_{0,\Omega} + \|\vec{w}^h - \vec{f}(u^h)\|^2_{0,\Omega} + \|\vec{n}_{x,t} \cdot (\vec{w}^h - \vec{f}(g))\|^2_{0,\Gamma_I} + \|u^h - g\|^2_{0,\Gamma_I}$$

- Newton linearization: minimize functional with linearized equation

Finite Element Spaces

- weak solution: $\vec{f}_{x,t} \in H(div, \Omega)$

  $\Rightarrow$ choose $\vec{w}^h \in H(div, \Omega)$

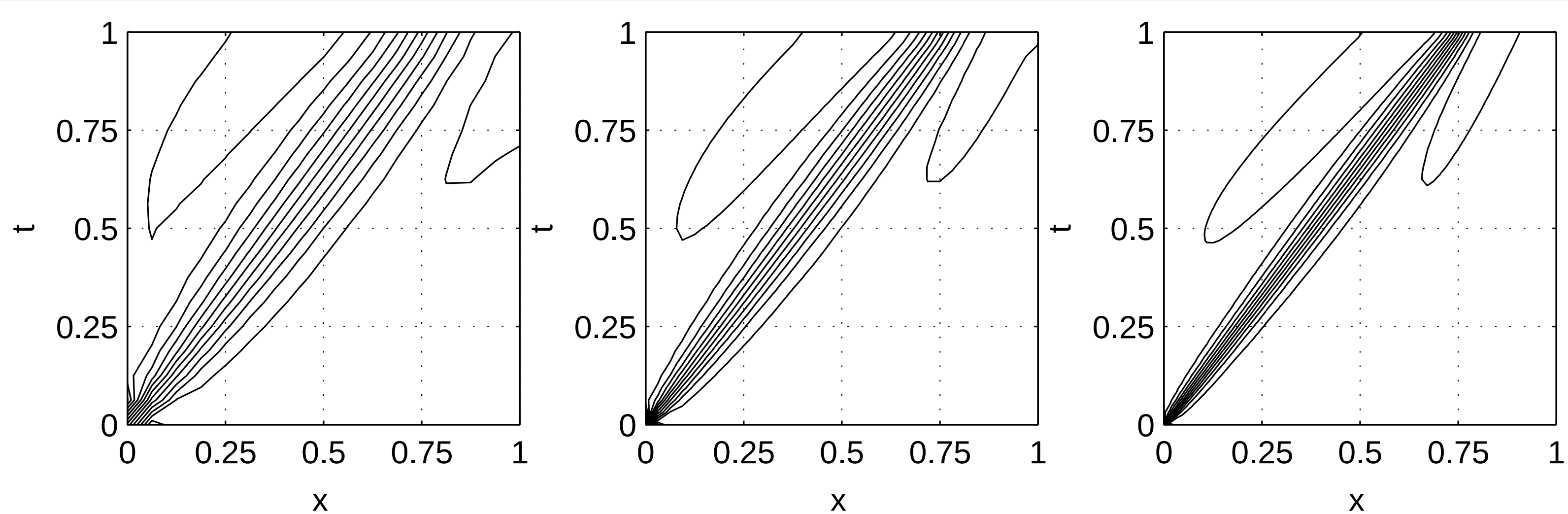
- Raviart-Thomas elements: the normal components of $\vec{w}^h$ are continuous

  $\Rightarrow \vec{w}^h \in H(div, \Omega)$

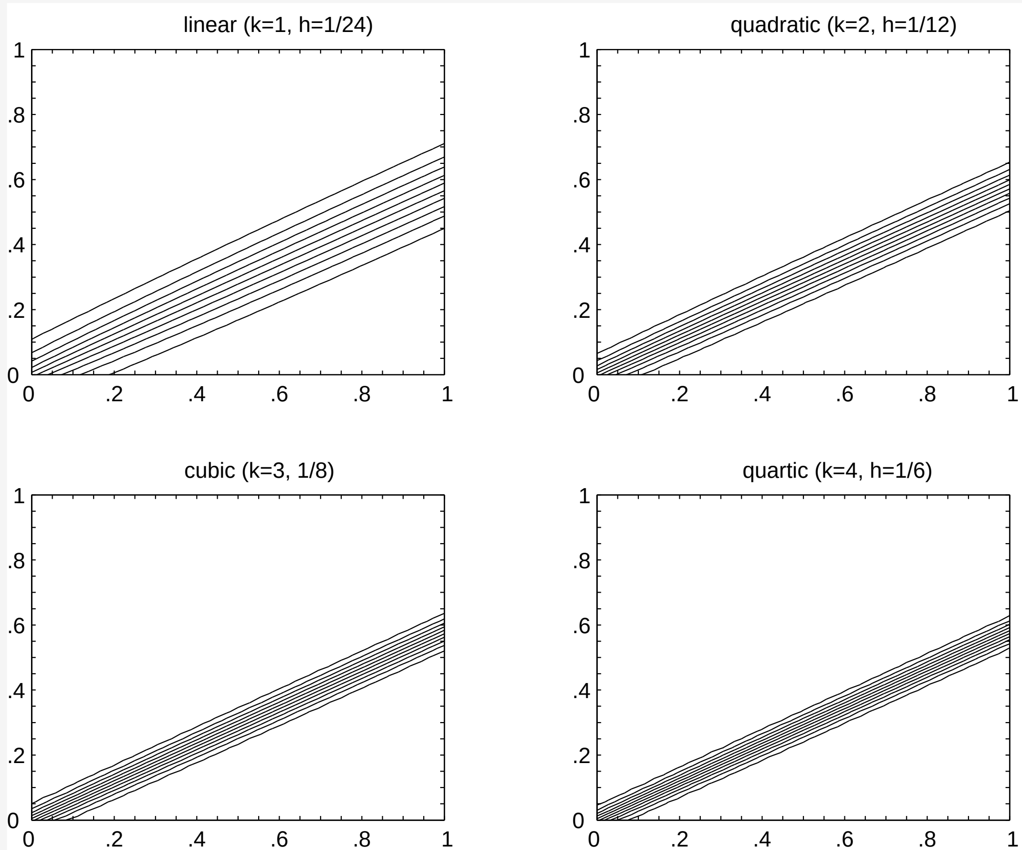
$\Rightarrow$ $H(div)$-conforming LSFEM

Numerical Results

- shock flow: $u_{left} = 1.0$, $u_{right} = 0.5$, shock speed $s = 0.75$
- convergence to correct weak solution with optimal order
- no oscillations, correct shock speed, no CFL limit



Linear Advection – Higher-Order Elements



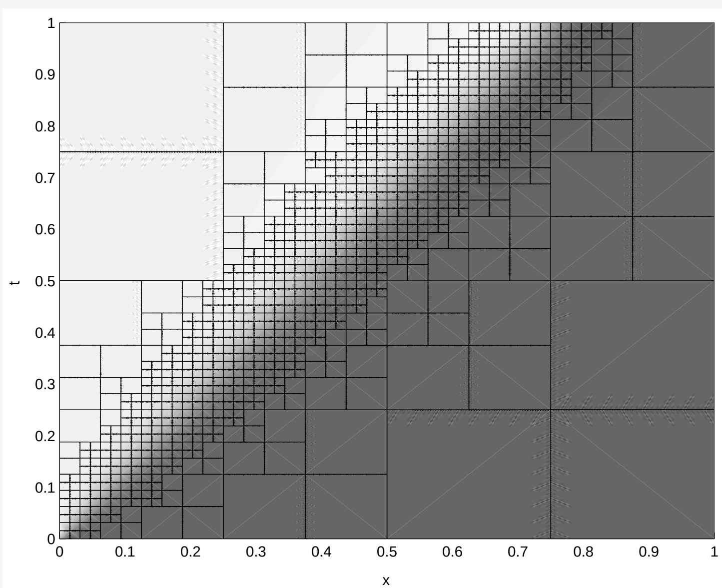
- order $k = 1,\ 2,\ 3,\ 4$: sharper shock for same dof
- remark: also discontinuous finite elements for u^h

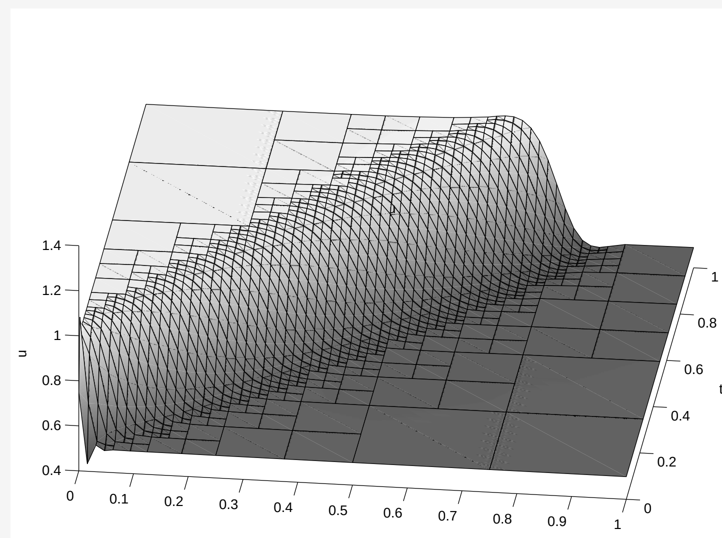
  (SIAM J. Sci. Comput., accepted)

Solution-Adaptive Refinement

- LS functional is sharp *a posteriori* error estimator:

$$
\begin{aligned}
\mathcal{F}(u^h) &= \|Lu^h\|^2_{0,\Omega} \\
&= \|Lu^h - Lu_{exact}\|^2_{0,\Omega} \\
&= \|L(u^h - u_{exact})\|^2_{0,\Omega} \\
&= \|Le^h\|^2_{0,\Omega}
\end{aligned}
$$



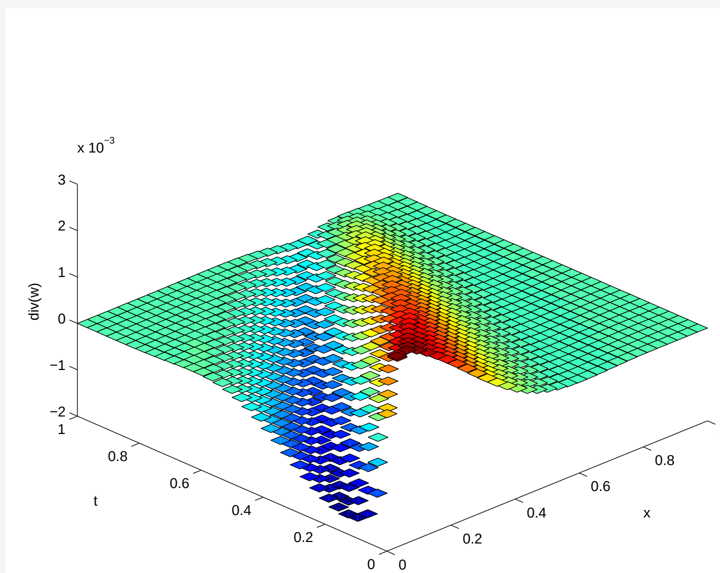


Numerical Conservation

- we minimize

$$\mathcal{F}(\vec{w}^h, u^h; g) = \|\nabla_{x,t} \cdot \vec{w}^h\|^2_{0,\Omega} + \|\vec{w}^h - \vec{f}(u^h)\|^2_{0,\Omega}$$
$$+ \|\vec{n}_{x,t} \cdot (\vec{w}^h - \vec{f}(g))\|^2_{0,\Gamma_I} + \|u^h - g\|^2_{0,\Gamma_I}$$

- our $H(div)$-conforming LSFEM does not satisfy the exact discrete conservation property of Lax and Wendroff



$\nabla \cdot \vec{w}^h$

Numerical Conservation

$$\mathcal{F}(\vec{w}^h, u^h; g) = \|\nabla_{x,t} \cdot \vec{w}^h\|^2_{0,\Omega} + \|\vec{w}^h - \vec{f}(u^h)\|^2_{0,\Omega}$$
$$+ \|\vec{n}_{x,t} \cdot (\vec{w}^h - \vec{f}(g))\|^2_{0,\Gamma_I} + \|u^h - g\|^2_{0,\Gamma_I}$$

- however, we can prove: (submitted to SIAM J. Sci. Comput.)

THEOREM. [Conservation for $H(div)$-conforming LSFEM]
If fi nite element approximation u^h converges in the L_2 sense to $\hat{u}$ as $h \rightarrow 0$, then $\hat{u}$ is a weak solution of the conservation law.

$\Rightarrow$ exact discrete conservation is not a necessary condition for numerical conservation!

(can be replaced by minimization in a suitable continuous norm)

LSFEM for Nonlinear Hyperbolic PDEs: Status

- Burgers equation:

  - nonlinear

  - scalar

  - 2D domains

- extensions, in progress:

  - systems of equations

  - higher-dimensional domains

- need efficient solvers for $A u = f$

Scalable Linear Solvers

with Ulrike Yang, Center for Applied Scientific Computing, Lawrence Livermore National Laboratory

$$\mathsf{A\,u} = \mathsf{f} \qquad (n \text{ dof})$$

- scalable, or $O(n)$, solver:

  $\Rightarrow$ for a twice larger problem, you only need twice the work

  $\Rightarrow$ 'optimal' solvers for sparse matrix problems

  (not easy: Gaussian elimination $O(n^3)$ . . .)

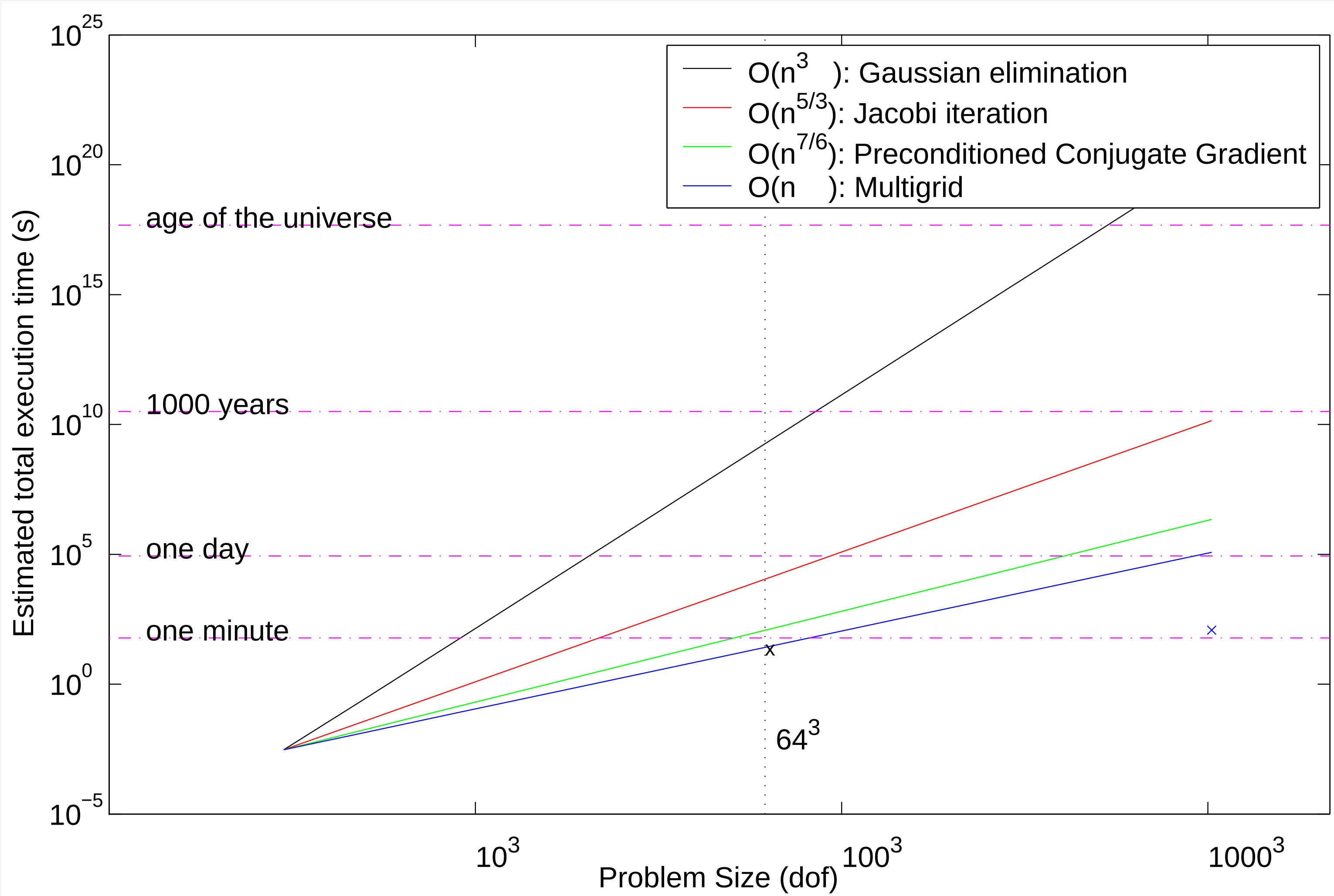
- parallel algebraic multigrid solvers

  - scalability for very large problems ($\sim$ 1000s of processors)

  - scalability for hyperbolic PDEs

Scalable Linear Solvers

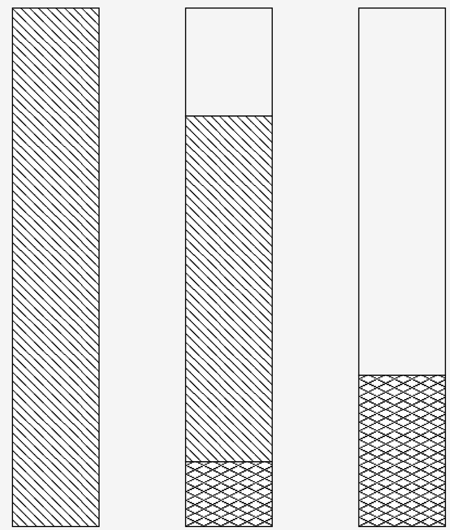
example: elliptic PDE problem in 3D



(3) Fluid Dynamics Applications

(A) Soil Sedimentation (Civil Engineering)

with Gert Bartholomeeusen, Mechanical Engineering, University of Oxford



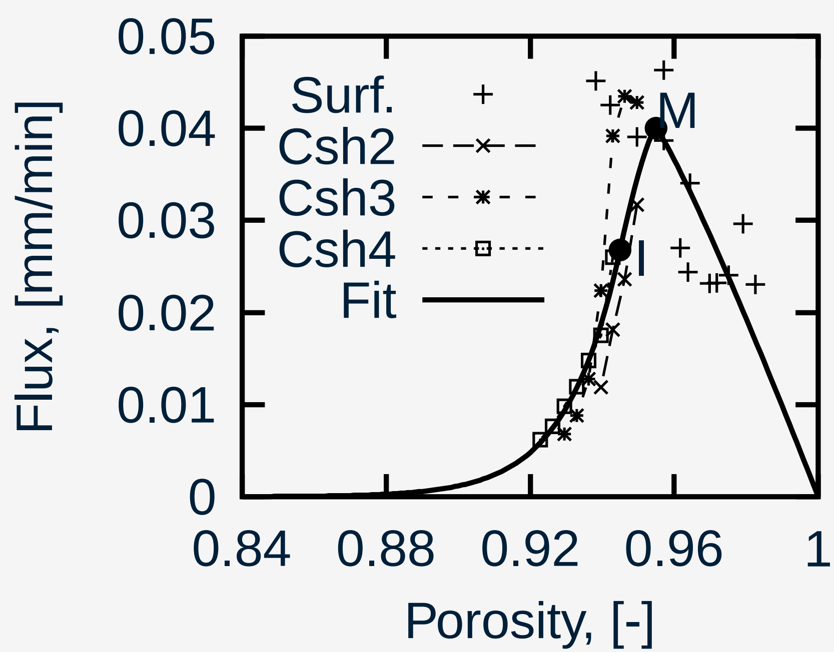
- settling column experiments: soil particles settle
- nonlinear waves, modeled by

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

Soil Sedimentation

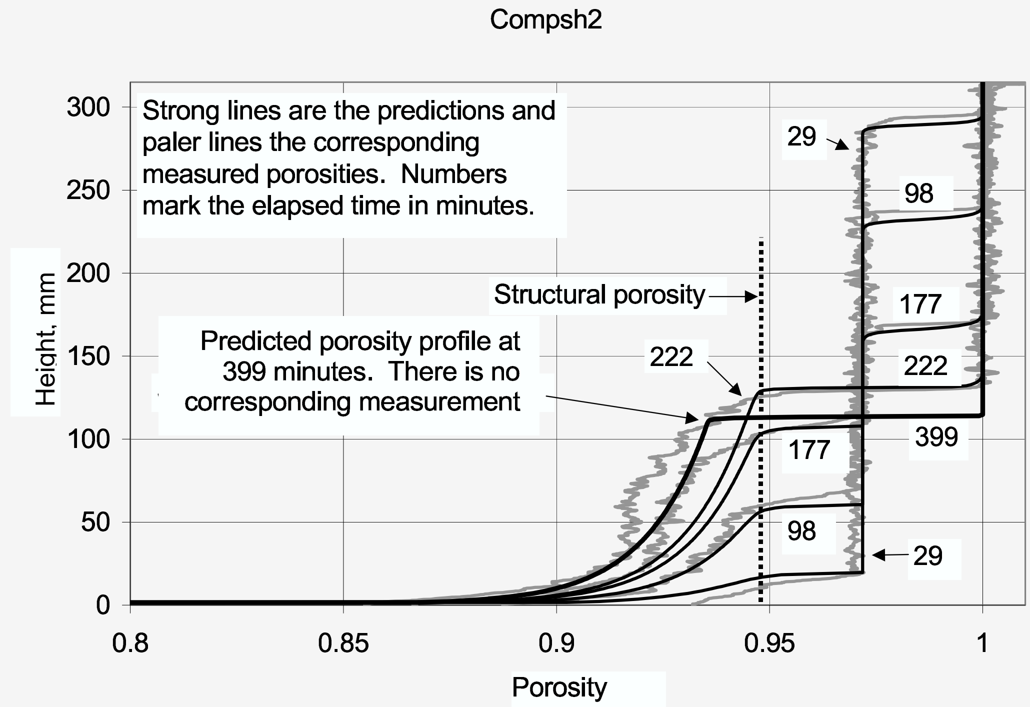
- experimental determination of flux function $f(u)$, nonconvex

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$



(kaolinite soil suspension)

Soil Sedimentation



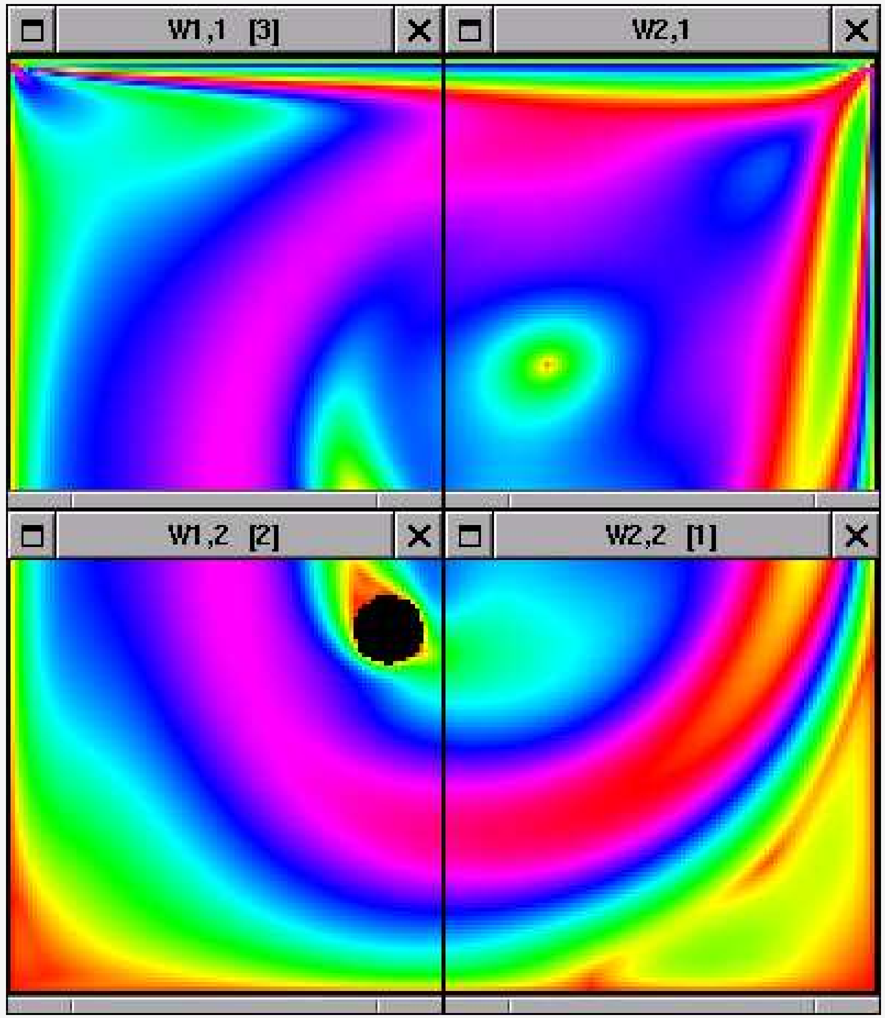
- simulation using flux function
- observation of compound shock waves = shock + sonic rarefaction
- new theory for transition between sedimentation and consolidation

(Proceedings of the 2002 Conference on Hyperbolic Systems)

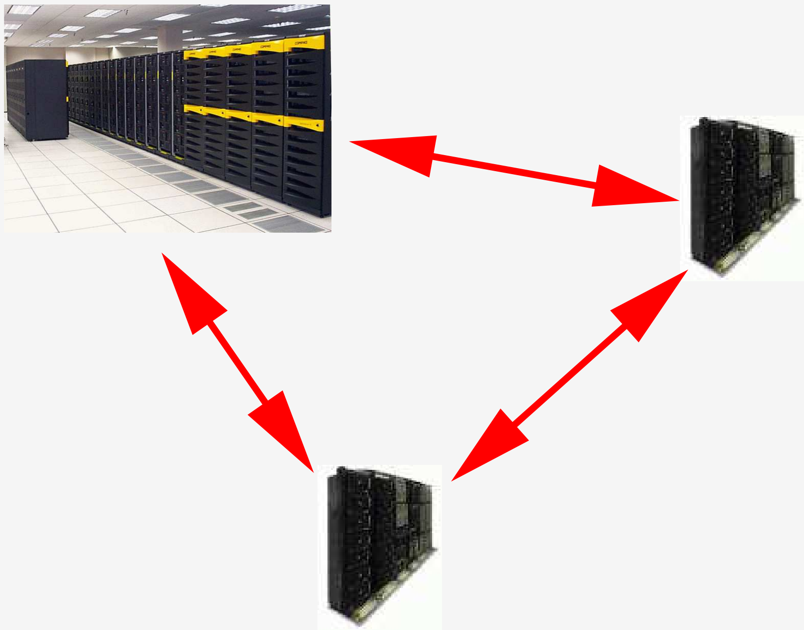
(B) Driven Cavity Navier-Stokes Flow on Computational Grids

with Thomas Pohl, Computer Science, University of Erlangen

with Rob Markel, Scientifi c Computing Division, NCAR



Computational Grids for Scientific Computing

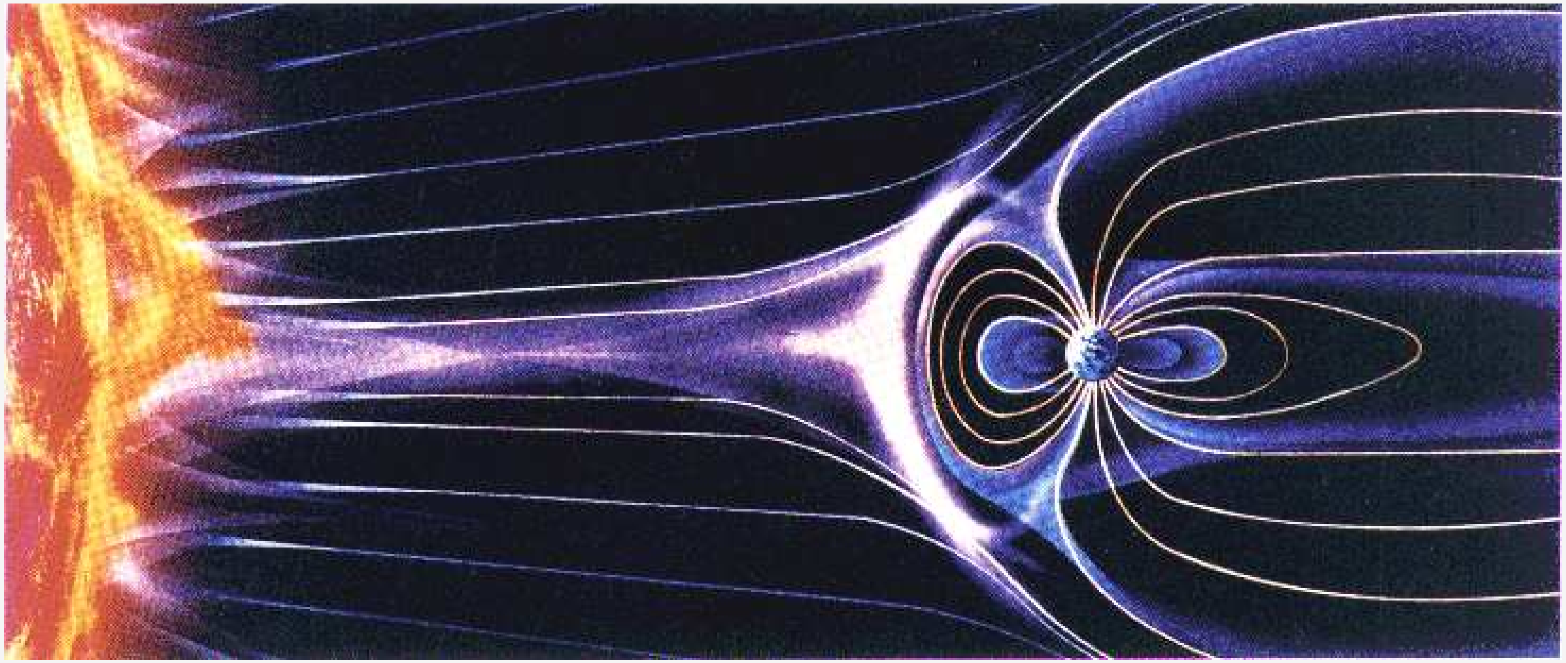


- use several parallel computers at the same time ($\sim$ power grid)

- developed Java-based grid computing framework

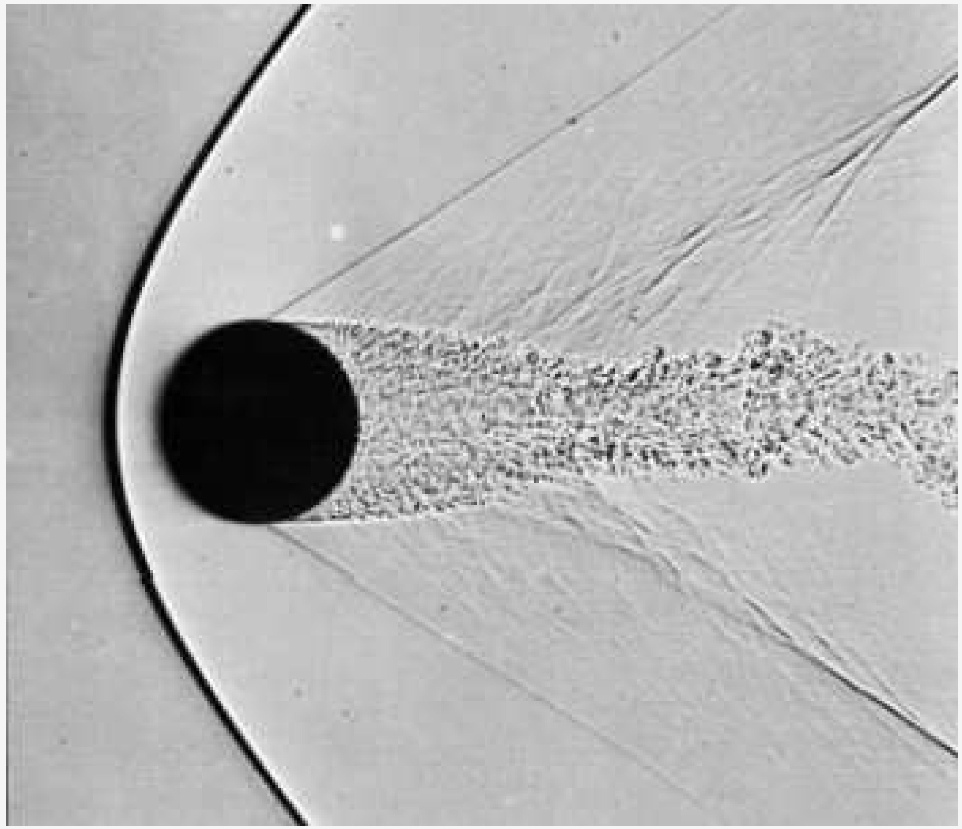
- applications:
  - fluid dynamics: driven cavity problem
  - iterative solver: scalable on grid
  - parallel bioinformatics problem (RNA folding)

(C) Bow Shock Flows in Solar-Terrestrial Plasmas



- supersonic solar wind plasma induces quasi-steady bow shock in front of earth's magnetosphere
- plasma = gas + magnetic field B
- described by Magnetohydrodynamics (MHD), hyperbolic system

Recall: Gas Dynamics Bow Shock

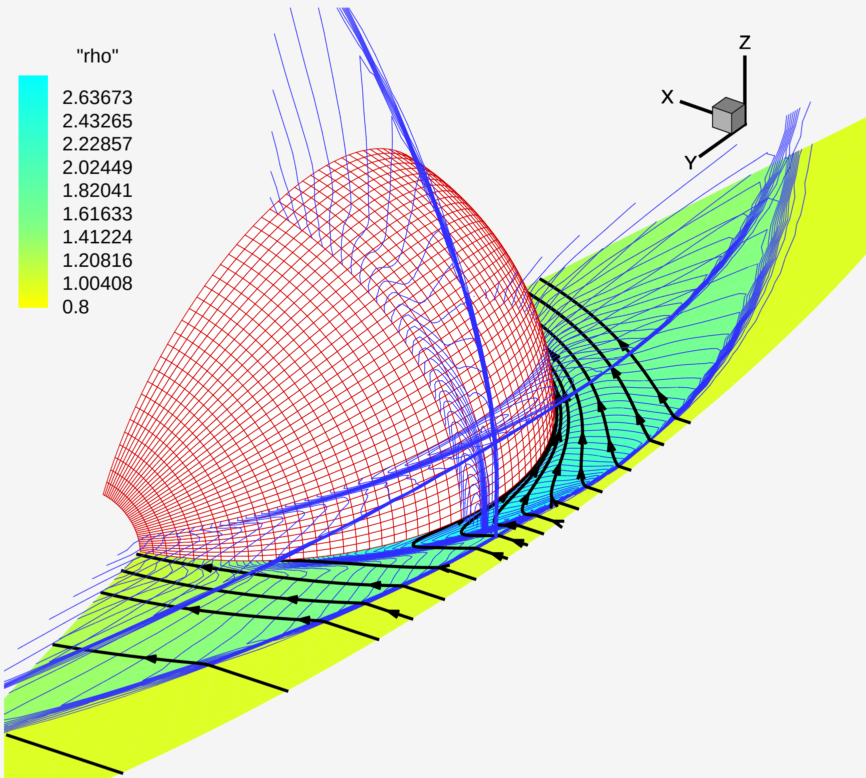


Bow Shock Flows in Solar-Terrestrial Plasmas

- simulation:

  for large upstream B:

  multiple shock fronts!



- reason: MHD has multiple waves
- also: compound shocks (like in sedimentation application)

  (Phys. Rev. Lett. 2000)

- predictive result:
  - not observed yet
  - confirmed in several other MHD codes
  - new spacecraft may allow observation

Collaborators

- LSFEM for Hyperbolic PDEs

  Luke Olson

  Tom Manteuffel

  Steve McCormick

  Applied Math, CU Boulder

- Scalable Solvers

  Ulrike Yang

  CASC, LLNL

  John Ruge

  Applied Math, CU Boulder

- Fluid Dynamics Applications

  Gert Bartholomeeusen, Thomas Pohl, Rob Markel

  Oxford, Erlangen, NCAR

Hyperbolic PDE Systems

$$\boxed{\frac{\partial U}{\partial t} + \nabla \cdot \vec{F}(U) = 0}$$

- PDE of hyperbolic type: consider 1D

$$\frac{\partial U}{\partial t} + \frac{\partial F_x(U)}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial U}{\partial t} + \frac{\partial F_x(U)}{\partial U} \cdot \frac{\partial U}{\partial x} = 0$$

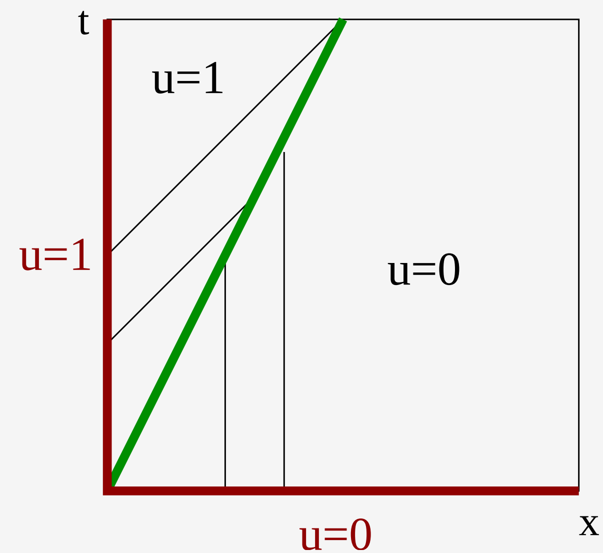
- define Flux Jacobian matrix $\mathsf{G}(U)$

$$\mathsf{G}(U) = \frac{\partial F_x(U)}{\partial U}$$

- PDE is hyperbolic $\iff$ $\mathsf{G}(U)$ has real eigenvalues and a complete set of eigenvectors
- the eigenvalues λ_i of $\mathsf{G}(U)$ are *wave speeds* of the system, and define *characteristic directions*
- nonlinear waves can steepen into discontinuities: *shock waves*

Burgers Equation: Characteristic Curves

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0$$



- define curve $x(t)$ in xt-plane
  with slope u:

$$x(t):\quad \frac{dx(t)}{dt} = u$$

$$\Rightarrow\quad \frac{\partial u(x(t),t)}{\partial t} + \frac{dx(t)}{dt}\,\frac{\partial u(x(t),t)}{\partial x} = 0 \quad \text{or} \quad \frac{du(x(t),t)}{dt} = 0$$

- characteristic curve $x(t)$

u is constant on $x(t)$ hyperbolic PDE reduces to ODE along $x(t)$

u is the slope of $x(t)$ u is also called the *wave speed*

- characteristics cross $\Rightarrow$ shock formation (weak solution)

Numerical Results – Convergence Study

- solution regularity: $u \in H^{1/2-\epsilon} \quad \forall\, \epsilon > 0$

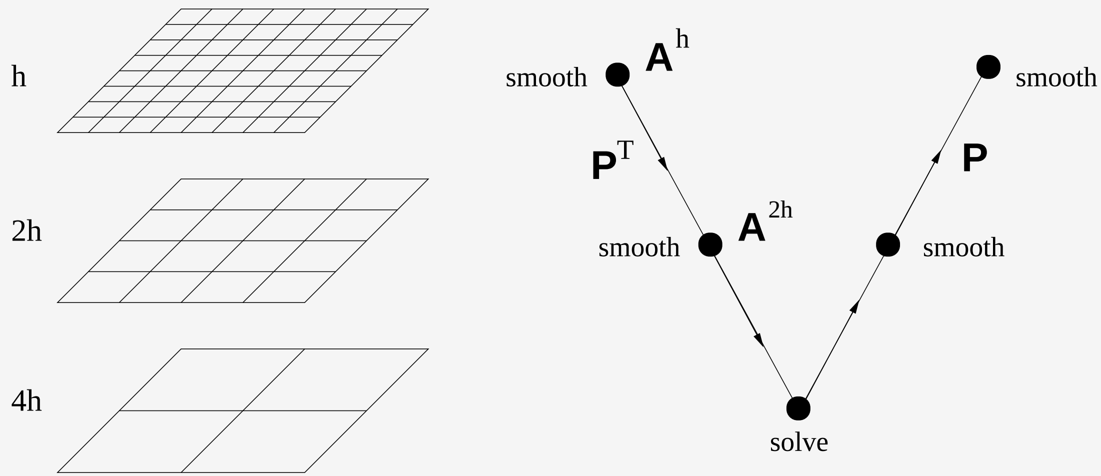
  $\Rightarrow \|u - u^h\|_{0,\Omega} \ \leq c\, h^{1/2-\epsilon}\, \|u\|_{1/2-\epsilon,\Omega} \quad \forall \epsilon,$ optimally

- $\|u^h - u\|^2_{0,\Omega} = O(h^\alpha)$ and $\mathcal{F}(\vec{w}^h, u^h) = O(h^\alpha)$, measure α

N	$\|u^h - u\|^2_{0,\Omega}$	α	$\mathcal{F}(\vec{w}^h, u^h)$	α
16	5.96e-3		1.89e-2	
		0.58		1.03
32	3.81e-3		9.25e-3	
		0.69		1.02
64	2.36e-3		4.56e-3	
		0.77		1.01
128	1.38e-3		2.26e-3	
		0.85		1.01
256	7.66e-4		1.12e-3	

Optimal $O(n)$ Solver: Multigrid Iterative Method

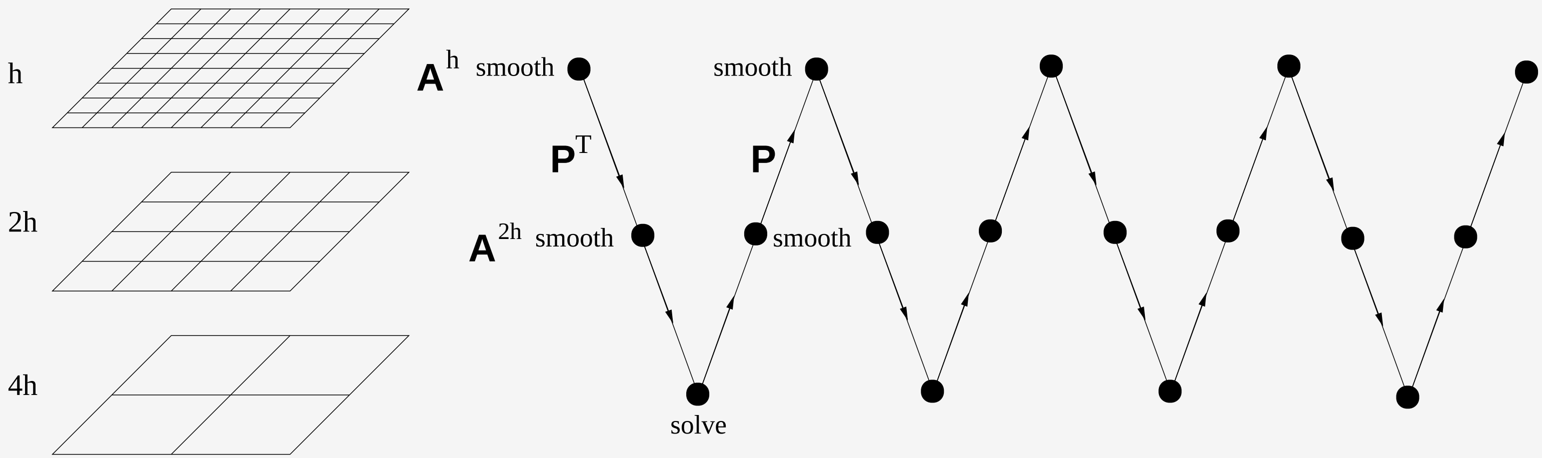
- multigrid V-cycle:



- residual reduction per cycle: convergence factor $\rho = \dfrac{\|A\,u_{i+1} - f\|}{\|A\,u_i - f\|}$
- work per cycle $W_{1\ cycle} = O(n)$

Optimal $O(n)$ Solver: Multigrid Method

- m multigrid V-cycles:



- residual reduction per cycle: convergence factor $\rho = \frac{\|A\,u_{i+1} - f\|}{\|A\,u_i - f\|}$
- work per cycle $W_{1\ cycle}$
- scalable method if $W_{1\ cycle} = O(n)$ and ρ is independent of n

Algebraic Multigrid Work in Progress

with Ulrike Yang, Center for Applied Scientifi c Computing, Lawrence Livermore National Laboratory

problem: hyperbolic PDEs: growth of convergence factor ρ as a function of n (not scalable)

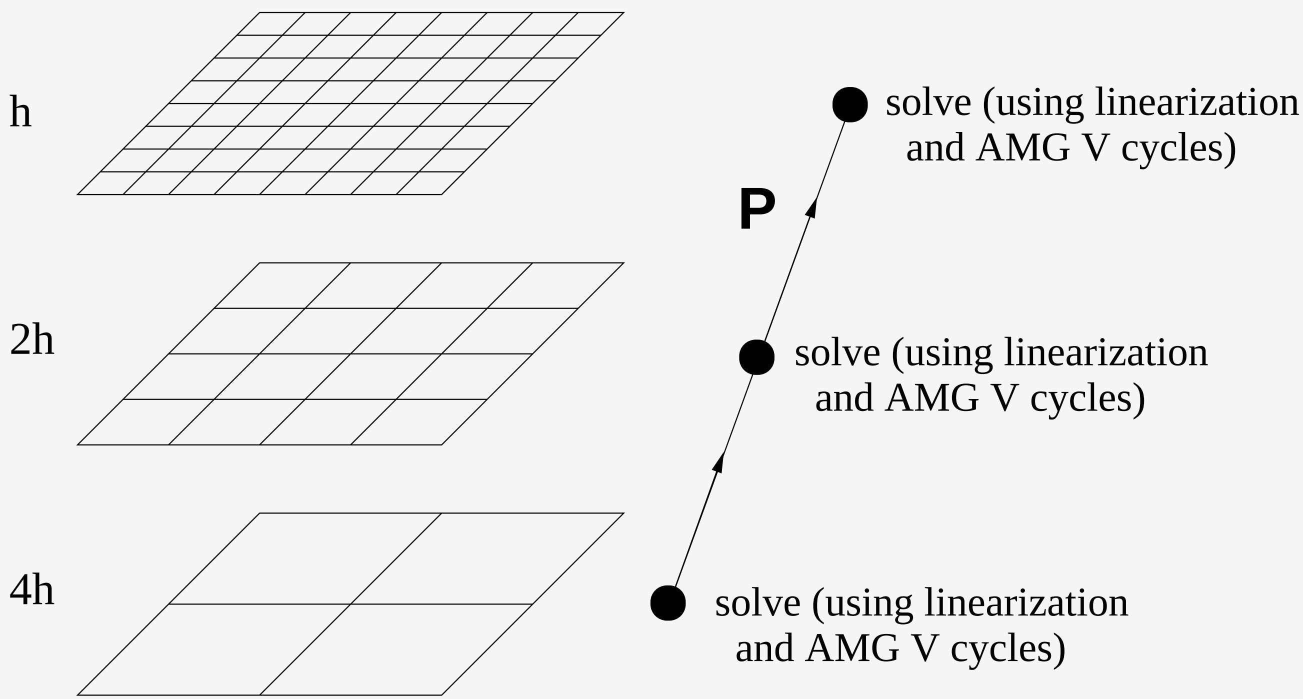
processors	n (dof)	ρ_{AMG}
1	131,072	0.83
4	524,288	0.87
16	2,097,152	0.88
64	8,388,608	0.92
256	33,554,432	0.96
1,024	134,217,728	0.98

(256^2 nodes per processor)

our approach: reformulate equations (SPD matrices), and more robust ways to choose coarse grids, interpolation matrix, relaxation

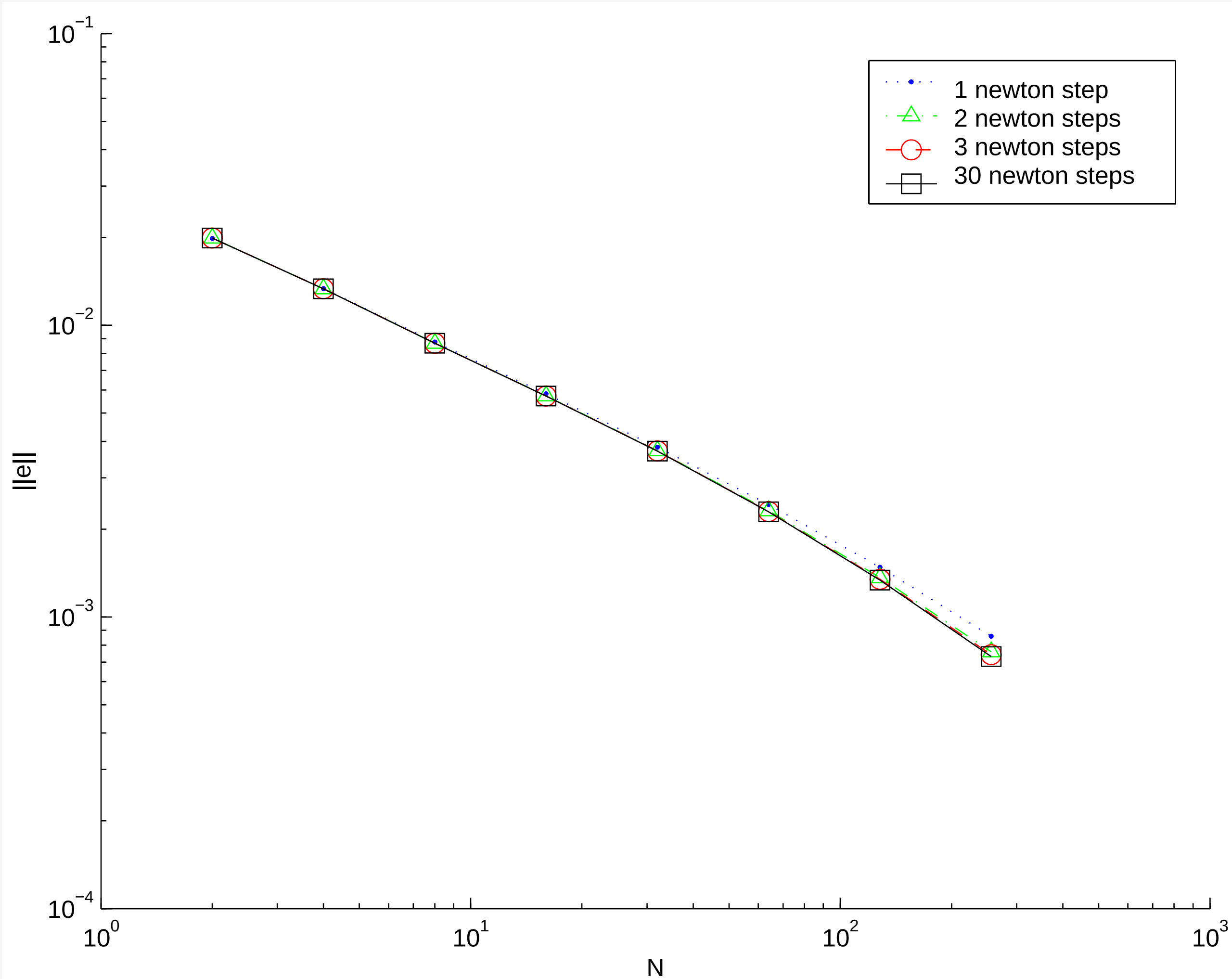
Scalable Nonlinear Solver – Newton Nested Iteration

- for many methods, number of Newton steps required grows with n
- use nested iteration:



Burgers: nested iteration with only one Newton step per level required!

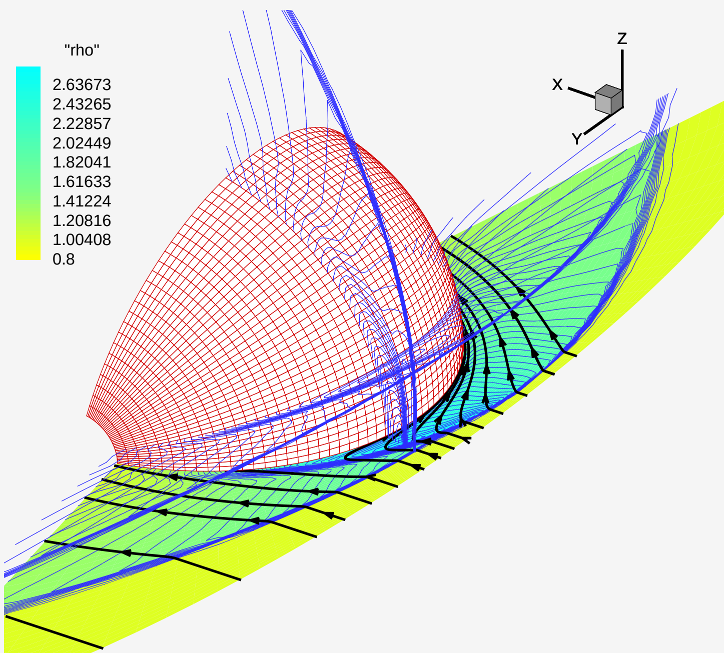
(6) Scalable nonlinear solver – Newton FMG



$\|u^h - u\|_{0,\Omega}$ convergence: grid continuation (FMG) with only one Newton step per level required!

3D MHD bow shock flows

- PhD thesis research (1999)
- 3D Finite Volume code
- MPI, F90 (64 procs)
- ‘shock-capturing’



- explicit time marching towards steady state
- problems:

(1) small timesteps, many iterations (many 100,000s): need implicit solvers

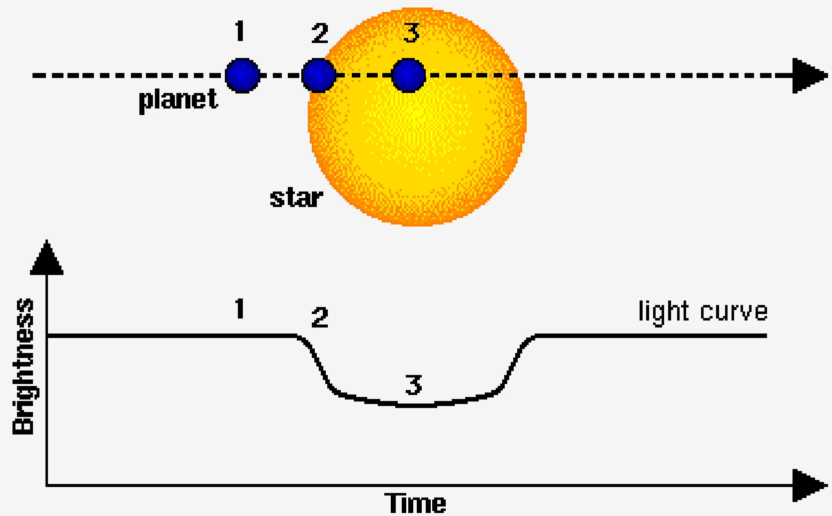
(2) algorithm not scalable

(3) low order of discretization accuracy (2nd order)

(4) robustness

(D) Supersonic Outflow from Exoplanet Atmospheres

with Feng Tian, PhD student, Astrophysics, CU Boulder



- extrasolar planets, as of 13 January 2004

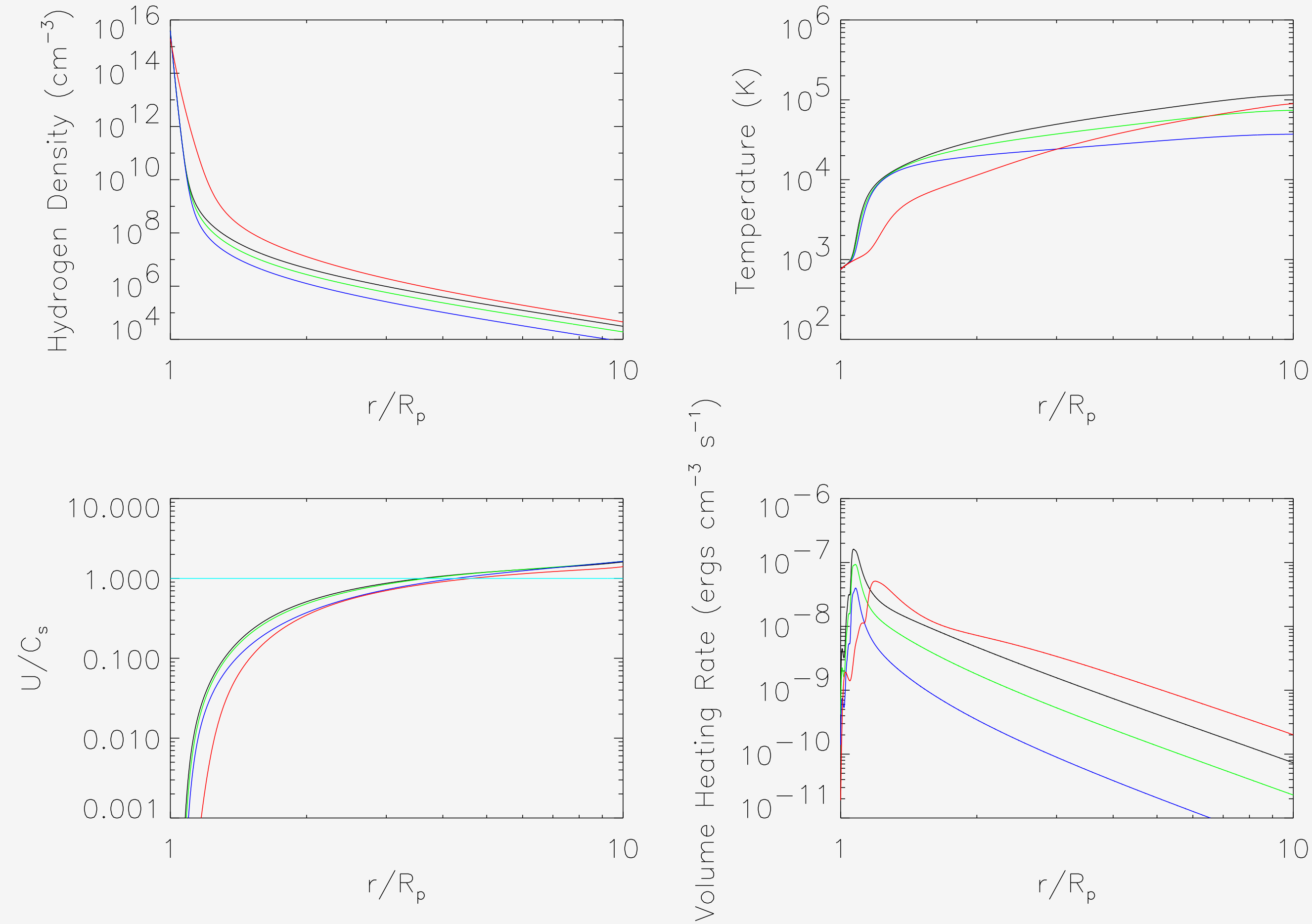
- 104 planetary systems
- 119 planets
- 13 multiple planet systems

- gas giants ('hot Jupiters')
- very close to star (~ 0.05 AU)

$\Rightarrow$ supersonic hydrogen escape (like the solar wind), Euler

Supersonic Outflow from Exoplanet Atmospheres

Supersonic Outflow from Exoplanet Atmospheres



Supersonic Outflow from Exoplanet Atmospheres

- planet around HD209458

  - 0.67 Jupiter masses, 0.05 AU

  - hydrogen atmosphere and escape observed
    (Vidal-Madjar, Nature March 2003)

- Feng's simulations show:

  - extent and temperature of Hydrogen atmosphere
    consistent with observations

  - atmosphere is stable (1% mass loss in 12 billion years)

- 'Mercury-type' planet with gas atmosphere would lose
  10% of mass in 8.5 million years

(2) LSFEM for the Burgers equation

$$\nabla \cdot \vec{f}(u) = 0 \quad \Omega$$

$$u = g \quad \Gamma_I$$

- LS functional

$$\mathcal{H}(u; g) := \|\nabla \cdot \vec{f}(u)\|_{0,\Omega}^2 + \|u - g\|_{0,\Gamma_I}^2$$

- LSFEM

$$u_*^h = \arg\min_{u^h \in \mathcal{U}^h} \mathcal{H}(u^h; g)$$

LSFEM for the Burgers equation

$$H(u) := \nabla \cdot \vec{f}(u) = 0 \quad \Omega$$
$$u = g \quad \Gamma_I$$

- Gauss-Newton minimization of LS functional:
  - first: Newton linearization of $H(u) = 0$

    $$H(u_i) + H'|_{u_i}(u_{i+1} - u_i) = 0$$

    with Fréchet derivative

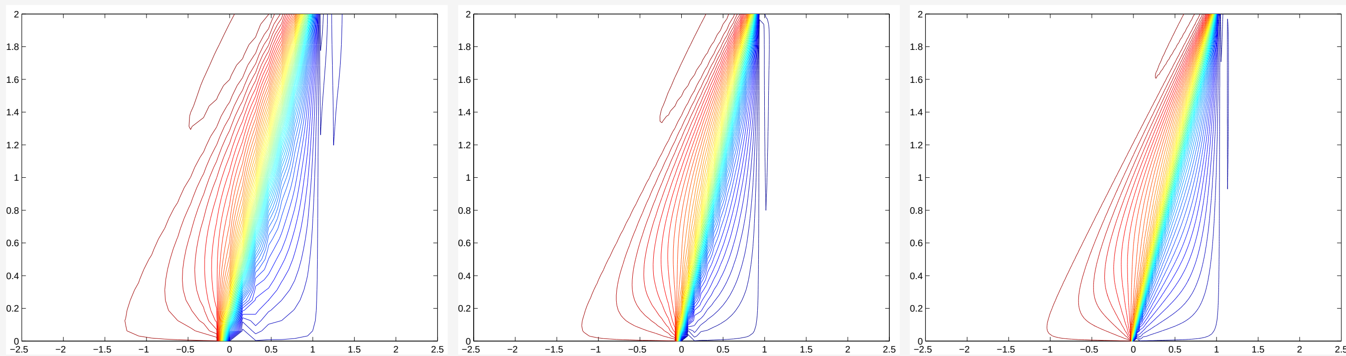
    $$H'|_{u_i}(v) = \lim_{\varepsilon \to 0} \frac{H(u_i + \varepsilon v) - H(u_i)}{\varepsilon}$$
    $$= \nabla \cdot (\vec{f}'|_{u_i}\, v)$$

  - then: LS minimization of linearized $H(u)$

    continuous bilinear finite elements on quads for u^h

Numerical Results

shock flow: $u_{left} = 1$, $u_{right} = 0$, shock speed $s = 1/2$



32^2 64^2 128^2

- correct shock speed, no oscillations
- on each grid, Newton process converges
- BUT: for $h \to 0$, nonlinear functional does not go to zero
- this means: for $h \to 0$, convergence to an incorrect solution!!! (L^*L has a spurious stationary point)
- why does LSFEM produce wrong solution??

Divergence of Newton's method

- reason: Fréchet derivative operator is unbounded

Burgers: $H'|_{u_0}(v) = \nabla \cdot ((u_0, 1)\ v)$

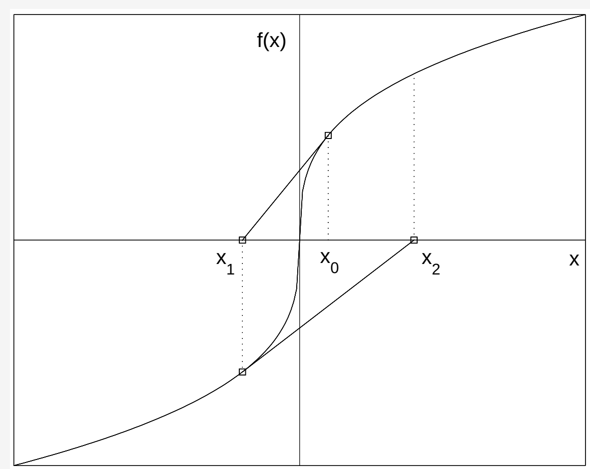
operator $H'|_{u_0}$:

$$\Rightarrow \|\ H'|_{u_0}\ \|_{0,\Omega} = \infty$$

because for most v

$$((u_0, 1)\ v) \notin H(div, \Omega)$$

example: $h(x) = \mp|x|^{1/3}$
$\Rightarrow x_1 = -2x_0$
Newton with $h'(x_*) = \infty$ may have empty basin of attraction



$H(div)$-conforming LSFEM

- nonlinear operator

$$F(\vec{w}, u) := \begin{bmatrix} \nabla \cdot \vec{w} \\ \vec{w} - \vec{f}(u) \end{bmatrix} = 0$$

- Fréchet derivative:

$$F'|_{(\vec{w}_0, u_0)}(\vec{w}_1 - \vec{w}_0, u_1 - u_0) = \begin{bmatrix} \nabla\cdot & 0 \\ I & -\vec{f}'|_{u_0} \end{bmatrix} \cdot \begin{bmatrix} \vec{w}_1 - \vec{w}_0 \\ u_1 - u_0 \end{bmatrix}$$

LEMMA. Fréchet derivative operator

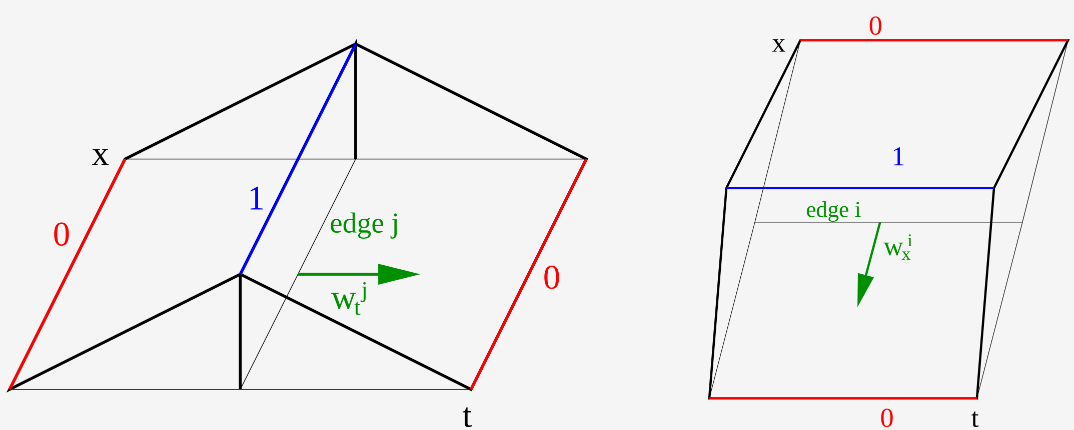
$F'|_{(\vec{w}_0, u_0)} : H(div, \Omega) \times L^2(\Omega) \to L^2(\Omega)$ is bounded:

$$\| \, F'|_{(\vec{w}_0, u_0)} \, \|_{0,\Omega} \leq \sqrt{1 + K^2}$$

Finite Element Discretization

- discretize $\vec{w}$ with face elements on quads (Raviart-Thomas in 2D):
  $\vec{w}^h = (w_t^h, w_x^h) \in (V_t^h, V_x^h)$

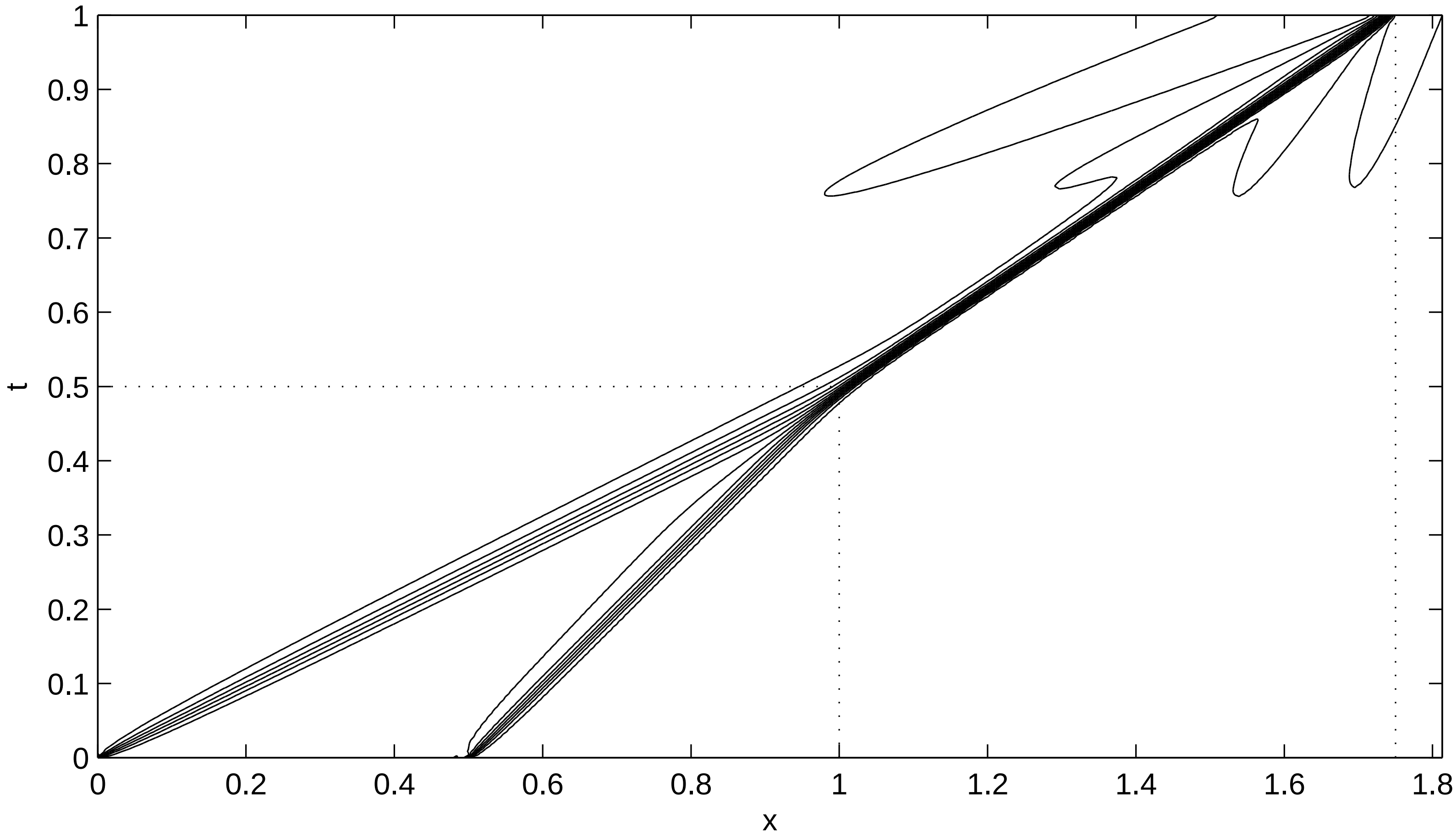
  face elements: normal vector components are degrees of freedom



  normal components of $\vec{w}^h$ are continuous $\Rightarrow \vec{w}^h \in RT_0 \subset H(div, \Omega)$

- continuous bilinear finite elements on quads for u^h

Numerical conservation



Hyperbolic PDEs – Conservation Laws

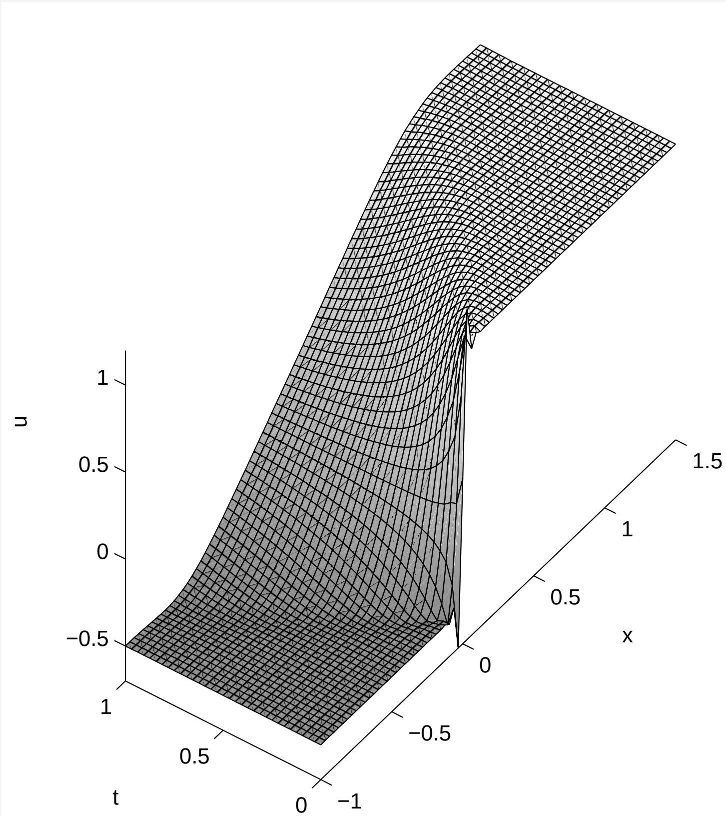
$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{F}(U) = 0$$

- e.g., compressible gases and plasmas
- example: ideal magnetohydrodynamics

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho\vec{v} \\ \rho e \\ \vec{B} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\vec{v} \\ \rho\vec{v}\vec{v} + \left(p + \frac{B^2}{2}\right)\vec{I} - \vec{B}\vec{B} \\ \left(\rho e + p + \frac{B^2}{2}\right)\vec{v} - (\vec{v}\cdot\vec{B})\vec{B} \\ \vec{v}\vec{B} - \vec{B}\vec{v} \end{bmatrix} = 0$$

(fusion plasmas, space plasmas, . . .)

Convergence to entropy solution



- transonic rarefaction
- many weak solutions
- one stable, entropy solution (rarefaction)
- LSFEM converges to entropy solution
- observed in numerical results, no theoretical proof yet