Least-Squares Finite Element Methods for Nonlinear Hyperbolic Conservation Laws

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Hans De Sterck

Luke Olson, Tom Manteuffel, Steve McCormick

Department of Applied Mathematics
University of Colorado at Boulder
Nonlinear hyperbolic conservation law

\[ \nabla \cdot \vec{f}(u) = 0 \quad \Omega \]
\[ u = g \quad \Gamma_I \]

- \( \Omega \subset \mathbb{R}^2 \) \( \Gamma_I \) inflow boundary
- space-time domains: \( \nabla = (\partial_x, \partial_t) \)
- \( \vec{f}(u) \) Lipschitz continuous:
  \[ \exists K \text{ s.t. } |f_i(u_1) - f_i(u_2)| \leq K |u_1 - u_2| \]
  \[ \forall u_1, u_2, \quad i = 1, 2 \]
- inviscid Burgers equation: \( \vec{f}(u) = (u^2/2, u) \)
Nonlinear hyperbolic conservation law

\[ \nabla \cdot \vec{f}(u) = 0 \quad \Omega \]
\[ u = g \quad \Gamma_I \]

- weak solutions:
  \[ -\langle \vec{f}(u), \nabla \phi \rangle_{0,\Omega} + \langle \vec{n} \cdot \vec{f}(g), \phi \rangle_{0,\Gamma_I} = 0 \quad \forall \phi \in C^1_{\Gamma_o}(\Omega) \]

- restrict to piecewise \( C^1 \) functions with jump discontinuities
  \[ \Rightarrow \quad u \in H^{1/2-\epsilon}(\Omega) \quad \forall \epsilon > 0 \]
  \[ \Rightarrow \quad \text{THEOREM:} \quad \vec{f}(u) \in H(div, \Omega) \]
Outline

• (1) Standard LSFEM for the Burgers equation

• (2) $H(\text{div})$-conforming LSFEM

• (3) Potential $H(\text{div})$-conforming LSFEM

• Numerical results – convergence study

• Numerical conservation – Weak conservation proofs

• Conclusions
(1) LSFEM for the Burgers equation

\[ \nabla \cdot \vec{f}(u) = 0 \quad \Omega \]

\[ u = g \quad \Gamma_I \]

- LS functional
  \[ \mathcal{H}(u; g) := \| \nabla \cdot \vec{f}(u) \|_{0,\Omega}^2 + \| u - g \|_{0,\Gamma_I}^2 \]

- LSFEM
  \[ u^*_h = \arg \min_{u^h \in \mathcal{U}^h} \mathcal{H}(u^h; g) \]

\[ \mathcal{U}^h: \text{continuous bilinear finite elements on quadrilaterals} \]

- Gauss-Newton minimization of LS functional
LSFEM for the Burgers equation

\[ H(u) := \nabla \cdot \vec{f}(u) = 0 \quad \Omega \]
\[ u = g \quad \Gamma_I \]

- Gauss-Newton minimization of LS functional:
  - first: Newton linearization of \( H(u) = 0 \)
    \[ H(u_i) + H'_i(u_i)(u_{i+1} - u_i) = 0 \]
    with Fréchet derivative
    \[ H'_i(u_i)(v) = \nabla \cdot (\vec{f}'_{u_i} v) \]
  - then: LS minimization of linearized \( H(u) \)
Numerical Results

shock flow: \( u_{left} = 1 \), \( u_{right} = 0 \), shock speed \( s = 1/2 \)

- correct shock speed, no oscillations
- on each grid, Newton process converges
- BUT: for \( h \to 0 \), nonlinear functional does not go to zero
- this means: for \( h \to 0 \), convergence to an incorrect solution!!!(\(L^*L\) has a spurious stationary point)
- why does LSFEM produce wrong solution??
Divergence of Newton’s method

- reason: Fréchet derivative operator is unbounded

Burgers: \[ H'_{|u_0}(v) = \nabla \cdot ((u_0, 1) \cdot v) \]

operator \( H'_{|u_0} : v \in H^{1/2-\epsilon}(\Omega) \rightarrow L^2(\Omega) \)

\[ \Rightarrow \| H'_{|u_0} \|_{0, \Omega} = \infty \]

because \( \forall u_0 \in H^{1/2-\epsilon}(\Omega), \exists v \in H^{1/2-\epsilon}(\Omega) : ((u_0, 1) \cdot v) \notin H(div, \Omega) \)

example: \( h(x) = \mp |x|^{1/3} \)

\[ \Rightarrow x_1 = -2x_0 \]

Newton with \( h'(x_\ast) = \infty \)

may have empty basin of attraction
(2) \( H(div) \)-conforming LSFEM

- reformulate conservation law in terms of flux vector \( \vec{w} \):

\[
\nabla \cdot \vec{f}(u) = 0 \quad \Omega \\
u = g \quad \Gamma_I
\]

\[
\Rightarrow
\begin{aligned}
\nabla \cdot \vec{w} &= 0 \quad \Omega \\
\vec{w} &= \vec{f}(u) \quad \Omega \\
\vec{n} \cdot \vec{w} &= \vec{n} \cdot \vec{f}(g) \quad \Gamma_I \\
u &= g \quad \Gamma_I
\end{aligned}
\]

- Gauss-Newton applied to

\[
\mathcal{F}(\vec{w}^h, u^h; g) = \| \nabla \cdot \vec{w}^h \|_{0,\Omega}^2 + \| \vec{w}^h - \vec{f}(u^h) \|_{0,\Omega}^2 + \| \vec{n} \cdot (\vec{w}^h - \vec{f}(g)) \|_{0,\Gamma_I}^2 + \| u^h - g \|_{0,\Gamma_I}^2
\]

- \( \vec{w}^h \in RT_0 \subset H(div, \Omega) \), and \( u^h \) continuous bilinear
\( H(\text{div}) \)-conforming LSFEM

- nonlinear operator

\[
F(\vec{w}, u) := \begin{bmatrix}
\nabla \cdot \vec{w} \\
\vec{w} - \vec{f}(u)
\end{bmatrix} = 0
\]

- Fréchet derivative:

\[
F'|_{(\vec{w}_0, u_0)}(\vec{w}_1 - \vec{w}_0, u_1 - u_0) = \begin{bmatrix}
\nabla \cdot \\
I
\end{bmatrix}
= \begin{bmatrix}
0 \\
-f'|_{u_0}
\end{bmatrix}
\cdot \begin{bmatrix}
\vec{w}_1 - \vec{w}_0 \\
u_1 - u_0
\end{bmatrix}
\]

**LEMMA.** Fréchet derivative operator

\[
F'|_{(\vec{w}_0, u_0)} : H(\text{div}, \Omega) \times L^2(\Omega) \rightarrow L^2(\Omega)
\]

is bounded:

\[
\| F'|_{(\vec{w}_0, u_0)} \|_{0, \Omega} \leq \sqrt{1 + K^2}
\]
(3) Potential $H(\text{div})$-conforming LSFEM

- $\nabla \cdot \vec{f}(u) = 0$ implies $\vec{f}(u) = \nabla^\perp \psi$ for some $\psi \in H^1(\Omega)$

$\Rightarrow$ reformulate conservation law in terms of flux potential $\psi$:

$$
\begin{align*}
\nabla \cdot \vec{f}(u) &= 0 \quad \Omega \\
u &= g \quad \Gamma_I
\end{align*}
$$

$\Rightarrow$

$$
\begin{align*}
\nabla^\perp \psi - \vec{f}(u) &= 0 \quad \Omega \\
\vec{n} \cdot \nabla^\perp \psi &= \vec{n} \cdot \vec{f}(g) \quad \Gamma_I \\
u &= g \quad \Gamma_I
\end{align*}
$$

- Gauss-Newton applied to

$$
G(\psi^h, u^h; g) := \sum \left[ \left\| \nabla^\perp \psi^h - \vec{f}(u^h) \right\|_{0, \Omega}^2 + \left\| \vec{n} \cdot (\nabla^\perp \psi^h - \vec{f}(g)) \right\|_{0, \Gamma_I}^2 + \left\| u^h - g \right\|_{0, \Gamma_I}^2 \right]
$$

- $\psi^h$ and $u^h$ continuous bilinear
Potential $H(\text{div})$-conforming LSFEM

- nonlinear operator
  \[ G(\psi, u) := \nabla^\perp \psi - \vec{f}(u) = 0 \]

- Fréchet derivative:
  \[ G''|_{(\psi_0,u_0)}(\psi_1 - \psi_0, u_1 - u_0) = \begin{bmatrix} \nabla^\perp & -\vec{f}'|_{u_0} \end{bmatrix} \cdot \begin{bmatrix} \psi_1 - \psi_0 \\ u_1 - u_0 \end{bmatrix} \]

**LEMMA.** Fréchet derivative operator
\[ G''|_{(\psi_0,u_0)} : H^1(\Omega) \times L^2(\Omega) \to L^2(\Omega) \] is bounded:
\[ \| G''|_{(\psi_0,u_0)} \|_{0,\Omega} \leq \sqrt{1 + K^2} \]
Numerical results

- shock flow: $u_{left} = 1.0$, $u_{right} = 0.5$, shock speed $s = 0.75$
- $H(div)$-conforming LSFEM:
Numerical results

- potential $H(\text{div})$-conforming LSFEM:
Numerical results – convergence study

- estimate $\alpha$ in $\|u^h - u\|_{0,\Omega}^2 \approx \mathcal{O}(h^\alpha)$
  
  $u \in H^{1/2-\varepsilon}(\Omega)$ discontinuous $\Rightarrow$ optimal $\alpha = 1.0$
  
  i.e., $\|u^h - u\|_{0,\Omega}^2 \approx \mathcal{O}(h)$, or $\|u^h - u\|_{0,\Omega} \approx \mathcal{O}(h^{1/2})$

- estimate $\alpha$ in $\mathcal{F}(\tilde{w}^h, u^h; g) \approx \mathcal{O}(h^\alpha)$

- estimate $\alpha$ in $\mathcal{G}(\psi^h, u^h; g) \approx \mathcal{O}(h^\alpha)$
Numerical results – convergence study

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|u^h - u|_{0,\Omega}^2$</th>
<th>$\alpha$</th>
<th>$\mathcal{F}(\bar{w}^h, u^h)$</th>
<th>$\alpha$</th>
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</table>
FMG Newton $\|u^h - u\|_{0,\Omega}$ convergence
Numerical results – choice of spaces

- for $u^h$ piecewise constant (discontinuous): oscillations!

- reason: the functionals are not uniformly coercive

- for right choices of FE spaces (e.g., $u^h$ continuous bilinear), numerical evidence suggests FE convergence

- we have some heuristic understanding of this, but rigorous proofs not yet obtained

- potential formulation is equivalent to $H^{-1}$ minimization
Numerical conservation

- Lax-Wendroff theorem: exact discrete conservation

\[ \nabla_{\text{discrete}} \cdot \vec{f}(u^h) := \oint_{\partial \Omega_i} \vec{n} \cdot \vec{f}(u^h) \, dl = 0 \quad \forall \Omega_i \]

guarantees convergence to a weak solution

(assuming convergence of \( u^h \) to \( \hat{u} \) boundedly a.e.)
Numerical conservation

- our $H(div)$-conforming LSFEM do not satisfy the exact discrete conservation property of Lax and Wendroff

- $H(div)$-conforming LSFEM:
  \[ \nabla \cdot \vec{f}(u^h) \neq 0, \text{ and also } \nabla \cdot \vec{w}^h \neq 0 \]

- potential $H(div)$-conforming LSFEM:
  \[ \nabla \cdot \vec{f}(u^h) \neq 0, \text{ but } \nabla \cdot \nabla^\perp \psi^h \equiv 0 \]
Numerical conservation

- however, we can prove:

**THEOREM.** [Conservation for $H(\text{div})$-conforming LSFEM]

If finite element approximation $u_h$ converges in the $L^2$ sense to $\hat{u}$ as $h \to 0$, then $\hat{u}$ is a weak solution of the conservation law.

**THEOREM.** [Conservation for potential $H(\text{div})$-conforming LSFEM]

If finite element approximation $u_h$ converges in the $L^2$ sense to $\hat{u}$ as $h \to 0$, then $\hat{u}$ is a weak solution of the conservation law.

$\Rightarrow$ exact discrete conservation is not a necessary condition for numerical conservation!

(can be replaced by minimization in a suitable continuous norm)
Numerical conservation
Conclusions

we have developed two classes of $H(div)$-conforming LSFEM for hyperbolic conservation laws

- disadvantages
  - extra variables are introduced ($\bar{w}$ or $\psi$)
  - smearing of LSFEM at shocks

- advantages of LSFEM
  - optimal solution within finite element space
  - SPD linear systems (iterative methods, AMG)
  - error estimator (efficient adaptive refinement)
  - convergence to weak solution
  - no spurious oscillations at discontinuities (without need to add numerical diffusion)
  - extension to linear higher order schemes
Conclusions

- advantages of flux vector/flux potential reformulations
  - bounded Fréchet derivative $\Rightarrow$ Newton converges
  - smoothness of the solution ($\vec{f}(u) \in H(div)$) is made explicit, also at the discrete level using Raviart-Thomas elements ($\Rightarrow H(div)$-conforming LSFEM)
  - differential part of operator is linear
  - optimal multigrid exists for $H(div)$

- FE convergence theory needs to be worked out further
- promising initial AMG results, to be developed further
- methods can be extended to multiple spatial dimensions (using de Rham diagram), and to systems of equations