Least-Squares Finite Element Methods for Nonlinear Hyperbolic Conservation Laws

3 April 2003

Hans De Sterck

Luke Olson, Tom Manteuffel, Steve McCormick

Department of Applied Mathematics

University of Colorado at Boulder

Nonlinear Hyperbolic Conservation Laws

$$\nabla_{t,x} \cdot (u, f(u)) = 0 \quad in \quad \Omega$$
$$u = g \quad on \quad \Gamma_{in}$$

$$\nabla = (\partial_t, \partial_x) \qquad \nabla^{\perp} = (-\partial_x, \partial_t)$$

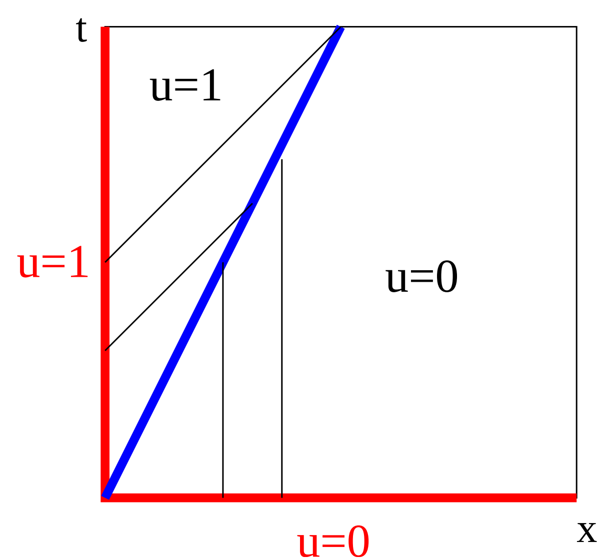
inviscid Burgers equation:

$$f(u) = u^2/2$$

Rankine-Hugoniot relation:

$$[\ (u, f(u))\ ] \cdot \vec{n} = 0$$

$$\Rightarrow (u, f(u)) \in H(div)$$



$$H(div) = \{(u,v) \in (L_2)^2 \mid \|\nabla \cdot (u,v)\|_0^2 < \infty\}$$

Outline

- Least-squares finite element method (LSFEM)
- (1) LSFEM for the Burgers equation
- (2) $H(div)$-conforming LSFEM
- (3) Dual $H(div)$-conforming LSFEM
- FEM convergence theory
- Numerical conservation
- Adaptivity
- Conclusions

Least-Squares Finite Element Method

- $Lu = f$
- define the functional
  $$\mathcal{F}(u; f) = \|Lu - f\|_0^2 = \langle Lu - f, Lu - f\rangle$$

$\Rightarrow$ minimization:
$$u_*^h = \underset{u^h \in \mathcal{U}^h}{arg\ min} \|Lu^h - f\|_0^2 = arg\ min\ \mathcal{F}(u^h; f)$$

- condition for stationary point:
  $$\frac{\partial \mathcal{F}(u_*^h + \alpha v^h; f)}{\partial \alpha}\Big|_{\alpha=0} = 0 \quad \forall\ v^h \in \mathcal{U}^h$$

  if L is linear: $\mathcal{F}(u_*^h + \alpha v^h; f) =$
  $$\left\langle Lu_*^h - f, Lu_*^h - f\right\rangle + 2\ \alpha\ \left\langle Lu_*^h - f, Lv^h\right\rangle + \alpha^2\ \left\langle v^h, v^h\right\rangle$$

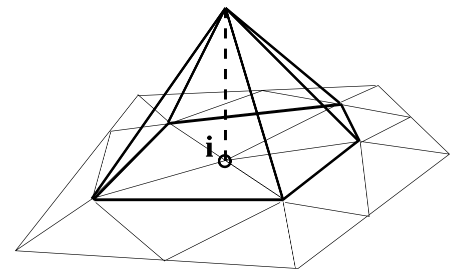
$\Rightarrow$ weak form:

find $u_*^h \in \mathcal{U}^h$, s.t. $\quad \left\langle Lu_*^h, Lv^h\right\rangle = \left\langle f, Lv^h\right\rangle \quad \forall v^h \in \mathcal{U}^h$

Finite Element Discretization

- approximate $u \in \mathcal{U}$ by $u^h \in \mathcal{U}^h$

$$u^h(t,x) = \sum_{i=1}^{n} u_i \, \phi_i(t,x)$$



- algebraic system from weak form:

$$\left\langle Lu^h, L\phi_j \right\rangle = \langle f, L\phi_j \rangle \qquad \forall \phi_j$$

equation j: $\quad \sum_{i=1}^{n} u_i \, \langle L\phi_i, L\phi_j \rangle = \langle f, L\phi_j \rangle$

(n equations in n unknowns)

(1) LSFEM for the Burgers Equation

- Gauss-Newton: first linearize the equation, then put the linearized equation into the LS functional, then minimize the LS functional
- Linearization:

$$F(u) := \nabla \cdot (u, u^2/2) = 0$$

Newton: $F(u) = 0 \;\Rightarrow\; \quad F(u_0) + dF|_{u_0}(u - u_0) = 0$

The Fréchet derivative $dF|_{u_0}(v)$ at u_0 in a direction v is

$$dF|_{u_0}(v) = \lim_{\varepsilon \to 0} \frac{F(u_0 + \varepsilon v) - F(u_0)}{\varepsilon}$$

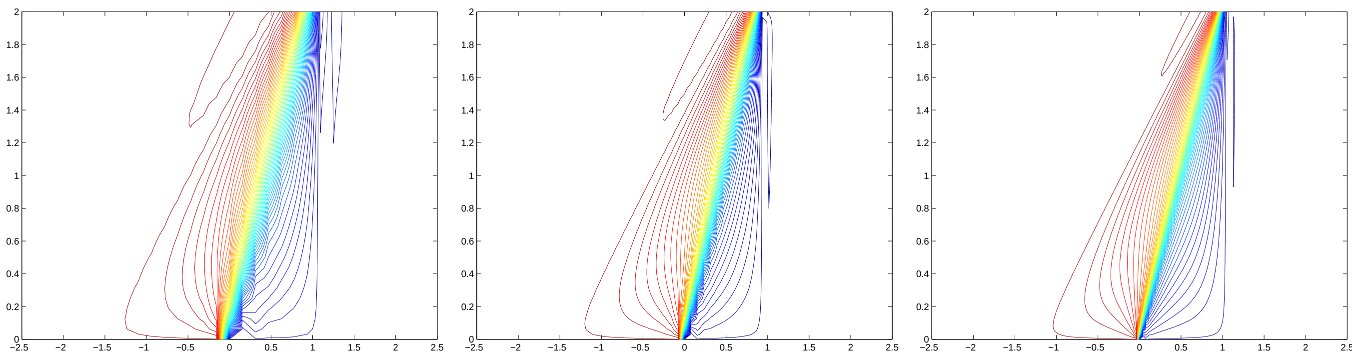
$$dF|_{u_0}(v) = \nabla \cdot ((1, u_0)\; v)$$

$$\Rightarrow\; \nabla \cdot (u_0, u_0^2/2) + \nabla \cdot ((1, u_0)\; (u - u_0)) = 0$$

Numerical Results

$$u_*^h = \arg\min_{u^h \in \mathcal{U}^h} \|\nabla \cdot (u, u^2/2)\|^2$$

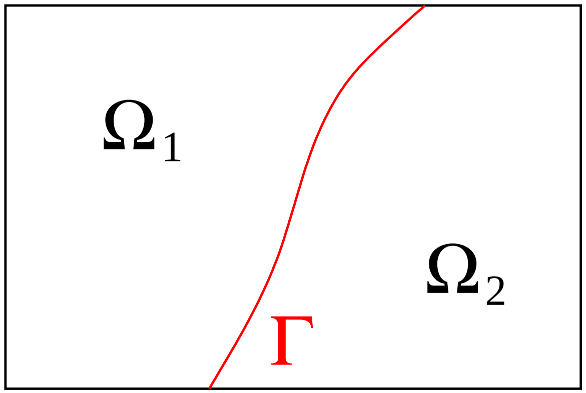
$u^h \in \mathcal{U}^h \subset H^1$, bilinear elements on quadrilaterals

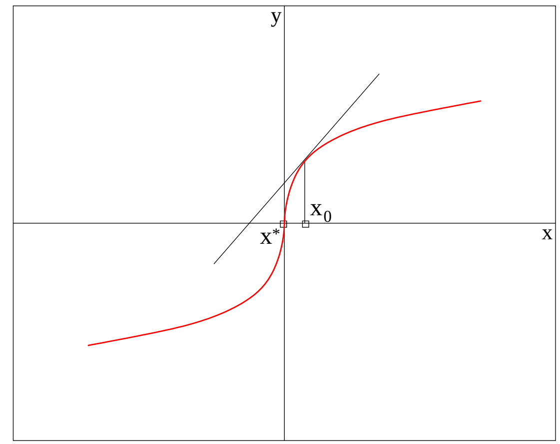


32^2 64^2 128^2

- right shock speed, no oscillations
- on each grid, newton process converges
- BUT: for decreasing h, functional does not go to zero
- this means: for decreasing h, convergence to an incorrect solution!!! (L^*L has a spurious stationary point)
- WHY NO CONVERGENCE??

Unbounded Fréchet Derivative





$$\nabla \cdot \vec{w} = \nabla \cdot \vec{w}_{1,2}^{(p)} + [\ \vec{w}\ ]_1^2 \cdot \vec{n}\ \delta_\Gamma$$

$$dF|_{u_0}(v) = \nabla \cdot ((1, u_0)\ v) = \nabla \cdot ((1, u_0)\ v)_{1,2}^{(p)} + [\ (1, u_0)\ v\ ]_1^2 \cdot \vec{n}\ \delta_\Gamma$$

- when u_0 is discontinuous (e.g. u_0 is the exact solution), for almost all functions v the Fréchet derivative is unbounded!! (recall: $[\ (u_0, u_0^2/2)\ ]_1^2 \cdot \vec{n} = 0$)

- Newton with $f'(x_*) = \infty$ may have empty basin of attraction (*e.g.* $f(x) = |x|^{1/3} \ \Rightarrow x_1 = -2x_0$)

$\Rightarrow$ this may explain why LSFEM fails to converge (convergence to a spurious stationary point)

(2) $H(div)$-conforming Reformulation

$$\nabla \cdot (u, u^2/2) = 0 \quad \Omega$$
$$u = g \quad \Gamma_{in}$$

$\Rightarrow$

$$\nabla \cdot \vec{w} = 0 \quad \Omega$$
$$\vec{w} - (u, u^2/2) = 0$$
$$\vec{n} \cdot (\vec{w} - (g, g^2/2)) = 0 \quad \Gamma_{in}$$

$$\vec{F}(\vec{w}, u) = 0 \quad \Rightarrow \quad \vec{F}(\vec{w}_0, u_0) + \vec{F}'|_{\vec{w}_0, u_0}(\vec{w} - \vec{w}_0, u - u_0) = 0$$

$$\vec{F}'|_{\vec{w}_0, u_0}(\vec{w} - \vec{w}_0, u - u_0) = \begin{bmatrix} \nabla \cdot & 0 \\ I & \begin{matrix} -1 \\ -u_0 \end{matrix} \end{bmatrix} \cdot \begin{bmatrix} \vec{w} - \vec{w}_0 \\ u - u_0 \end{bmatrix}$$

$\Rightarrow$ Fréchet derivative is bounded!
(we choose $\vec{w} \in H(div)$)

(3) Dual $H(div)$-conforming Reformulation

$$\begin{aligned} \nabla \cdot (u, u^2/2) &= 0 \quad \Omega \\ u &= g \quad \Gamma_{in} \end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned} \nabla^\perp p - (u, u^2/2) &= 0 \quad \Omega \\ \vec{n} \cdot (\nabla^\perp p - (g, g^2/2)) &= 0 \quad \Gamma_{in} \end{aligned}}$$

$$\vec{F}(p, u) = 0 \quad \Rightarrow \quad \vec{F}(p_0, u_0) + \vec{F}'|_{p_0, u_0}(p - p_0, u - u_0) = 0$$

$$\vec{F}'|_{p_0, u_0}(p - p_0, u - u_0) = \begin{bmatrix} \nabla^\perp & -1 \\ & -u_0 \end{bmatrix} \cdot \begin{bmatrix} p - p_0 \\ u - u_0 \end{bmatrix}$$

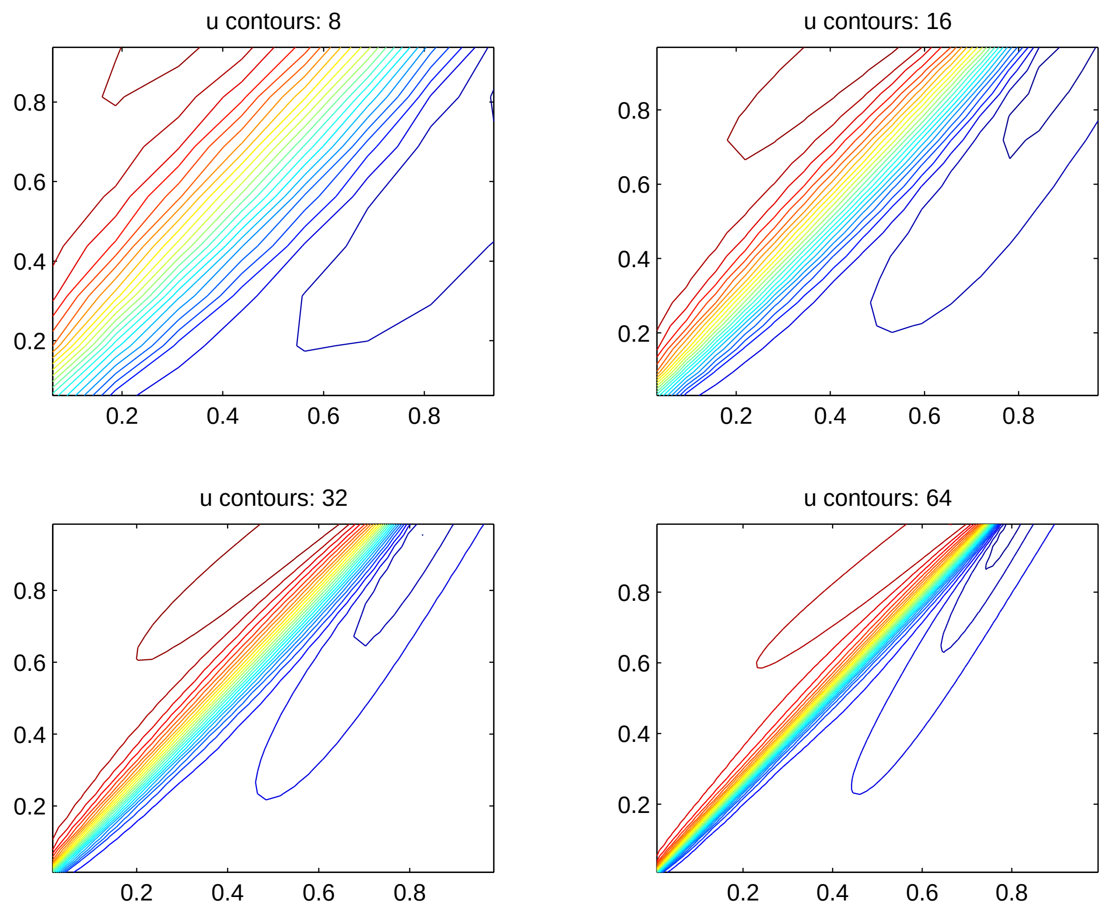
$\Rightarrow$ Fréchet derivative is bounded! (we choose $p \in H(curl)$)

- p is De Rham-dual of $\vec{w}$
  - move up one space to the left in De Rham-diagram of differential forms
  - similar to potential formulations

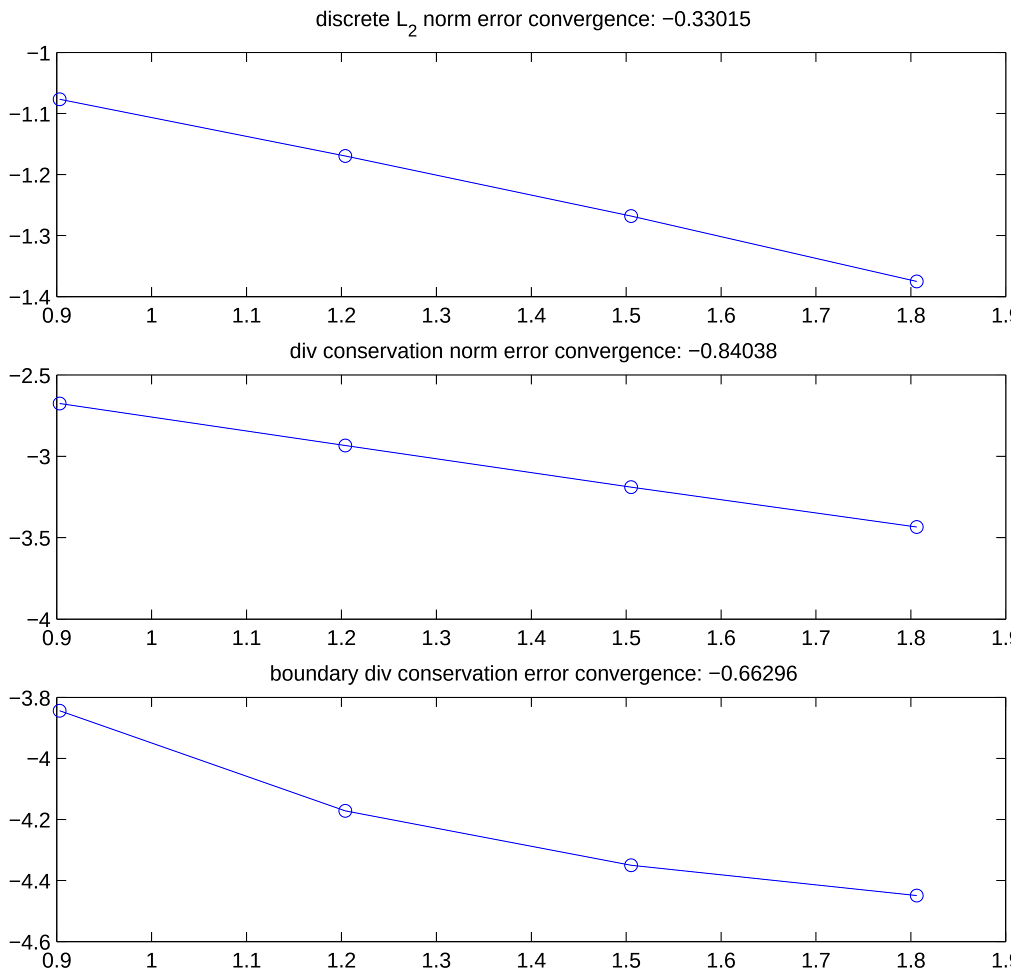
Results for $H(div)$-conforming LSFEM

$$(\vec{w}_*^h, u_*^h) = \underset{\vec{w}^h \in \mathcal{W}_g^h, u^h \in \mathcal{U}^h}{\arg\min} \|\nabla \cdot \vec{w}\|^2 + \|\vec{w} - (u, u^2/2)\|^2$$

$\vec{w}^h$ Raviart-Thomas elements $\subset H(div)$, u^h bilinear elements on quadrilaterals, strong boundary conditions



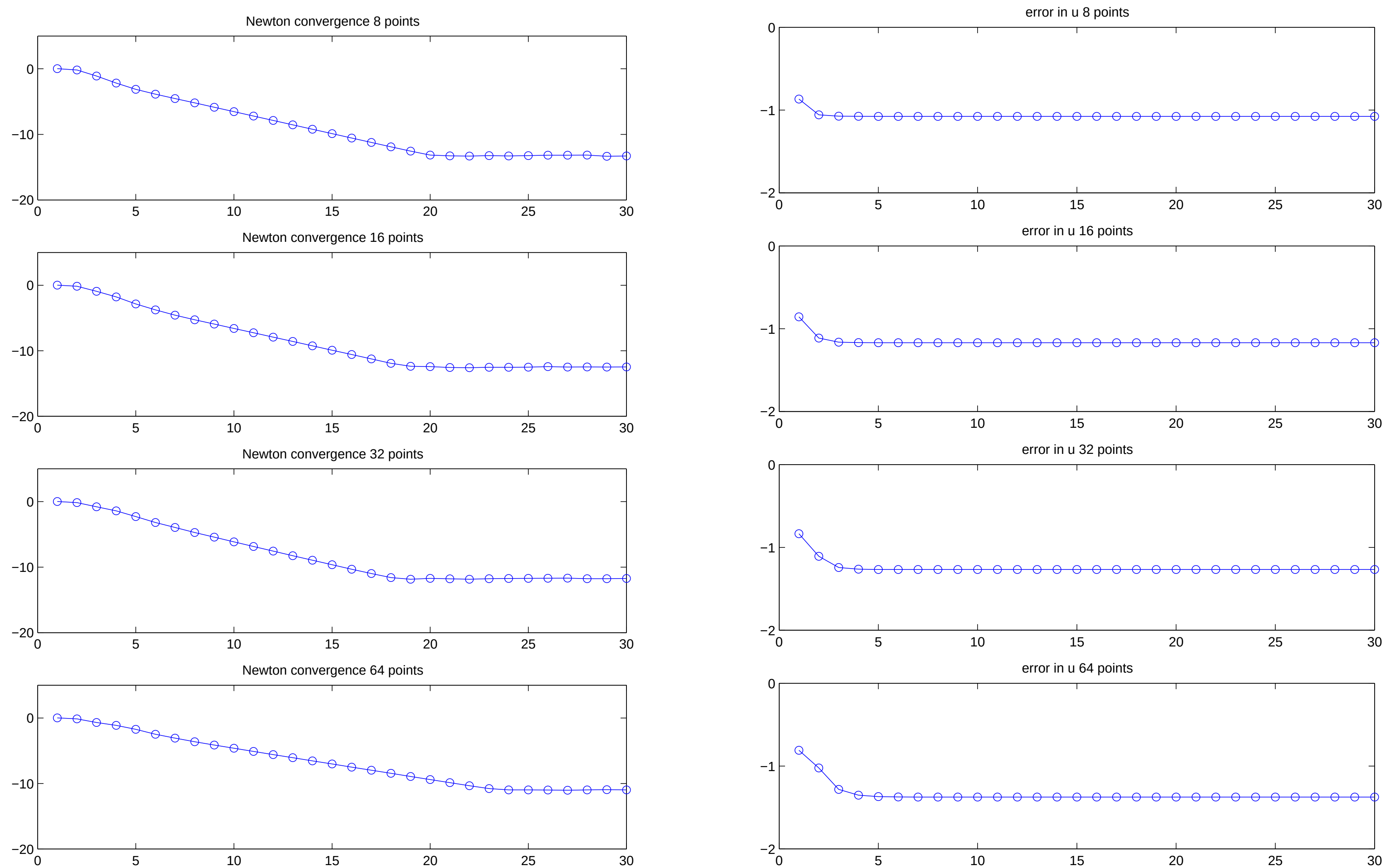
L2 convergence rate to exact solution



$$\|u^h - u_{exact}\|_0^2 = O(h^{0.66}) \quad (optimal : O(h^1))$$

$$\|\nabla \cdot \vec{w}\|_0^2 = O(h^{1.68})$$

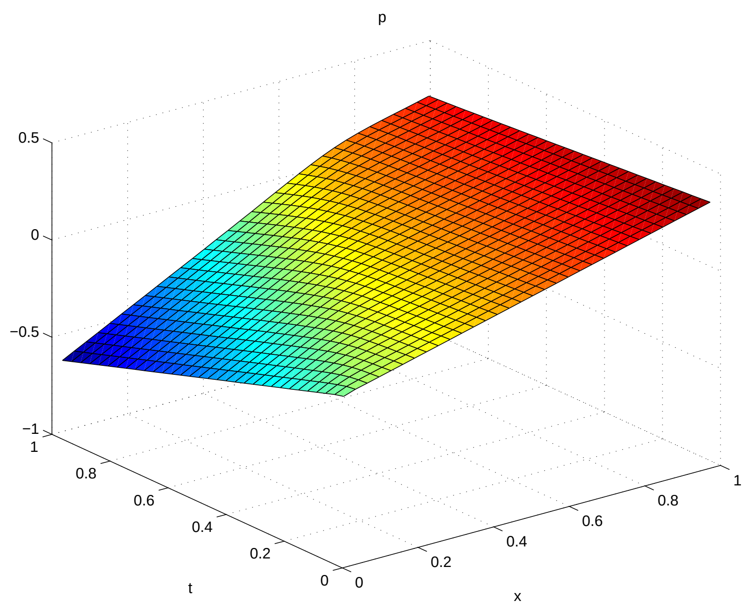
Gauss-Newton Convergence



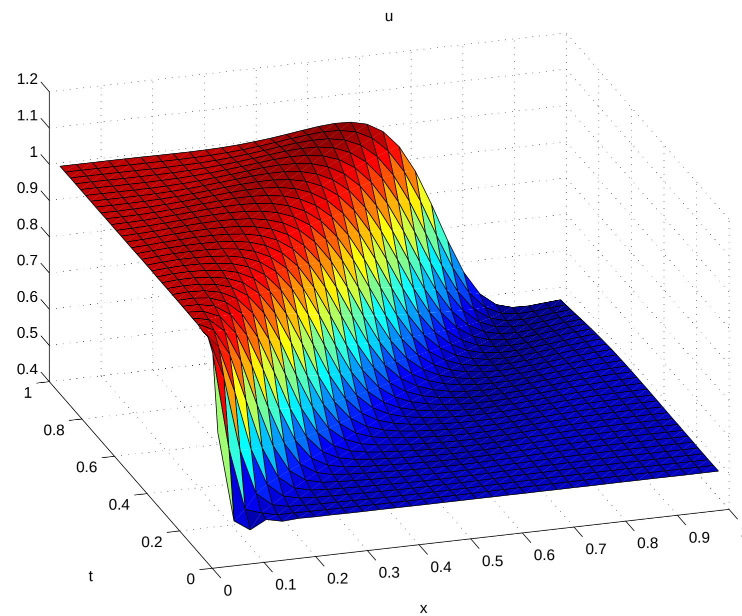
Results for dual $H(div)$-conforming LSFEM

$$(p_*^h, u_*^h) = \operatorname*{arg\,min}_{p^h \in \mathcal{P}^h, u^h \in \mathcal{U}^h} \|\nabla^\perp p - (u, u^2/2)\|^2 + \|\vec{n} \cdot (\nabla^\perp p - (g, g^2/2))\|_B^2$$

p^h, u^h bilinear elements onquadrilaterals, weak boundary conditions

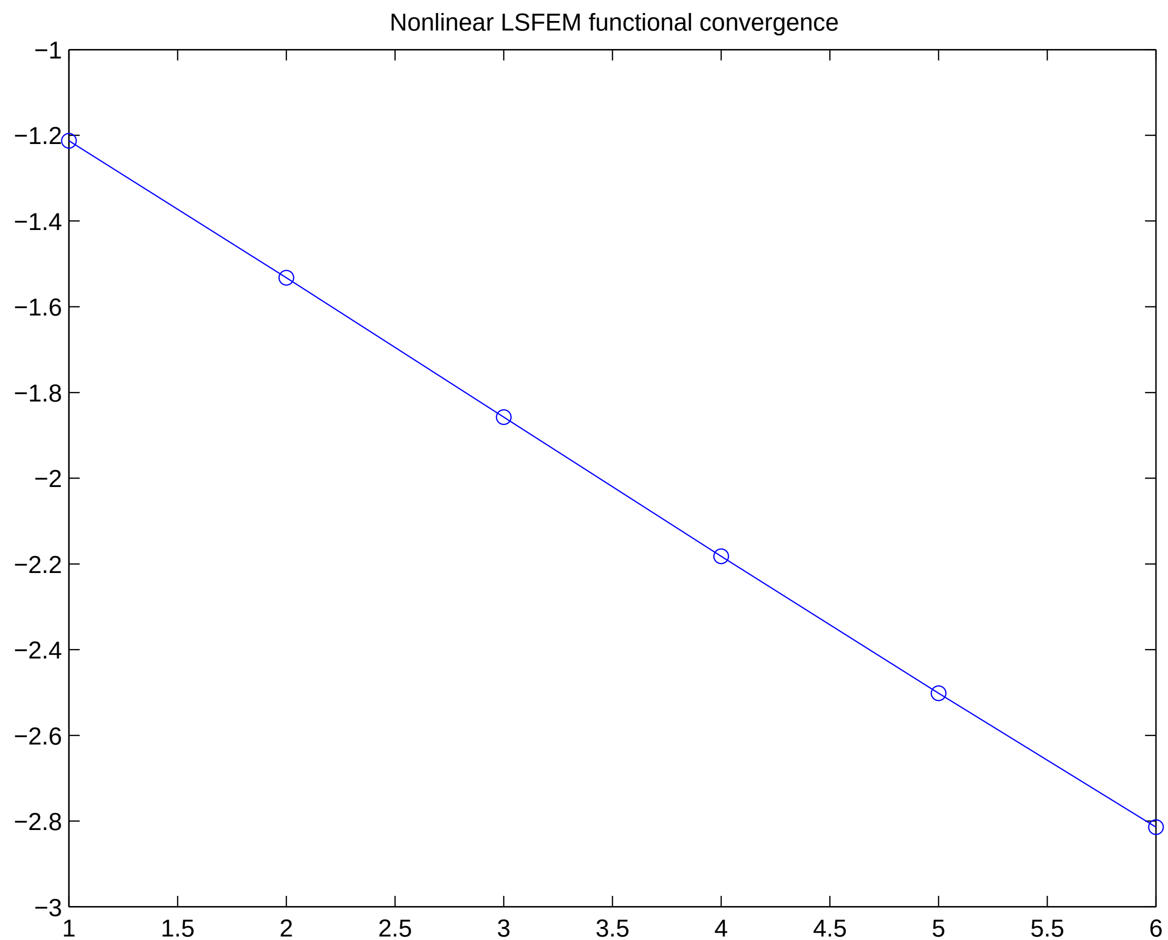


p



u

Nonlinear functional convergence



$$\|\nabla^{\perp} p - (u, f(u))\|^2 + \|\vec{n} \cdot (\nabla^{\perp} p - (g, f(g)))\|_B^2 = O(h^{1.1})$$

Convergence of $H(div)$-conforming LSFEM

- is the LS functional uniformly coercive?

  $$\exists\, c \text{ s.t. } \|\nabla^{\perp} p - (u, f(u))\|^2 \geq c\,(\|\nabla^{\perp} p\|^2 + \|u\|^2)$$

  NO: there are high-frequency error modes for which
  $c = O((1/n)^2)$

- compatible spaces: if we choose both p^h and u^h bilinear, we find (1D heuristics and numerical evidence) that $c = O(h)$ (only for the low-frequency modes, high-frequency error is filtered out)

  remark: if we choose u^h piecewise constant, high-frequency error contaminates solution

Convergence of $H(div)$-conforming LSFEM

- we observe that $\|\nabla^{\perp} p^h - (u^h, f(u^h))\|^2$ converges faster than $\|\nabla^{\perp} p^h\|^2 + \|u^h\|^2$

$\Rightarrow$ conjecture:

$$\|\nabla^{\perp} p^h - (u^h, f(u^h))\|^2 \geq c\, h\, (\|\nabla^{\perp} p^h\|^2 + \|u^h\|^2)$$

with

$$\|\nabla^{\perp} p^h - (u^h, f(u^h))\|^2 = O(h^{1+\beta})$$

$$\|\nabla^{\perp} p^h\|^2 + \|u^h\|^2 = O(h^{\alpha})$$

- to be investigated further

Numerical Conservation

THEOREM. Lax-Wendroff (1960).

'conservative' finite difference formula:

$$\frac{u_i^{h,n+1} - u_i^{h,n}}{\Delta t} + \frac{\bar{f}_{i+1/2}^{h,n} - \bar{f}_{i-1/2}^{h,n}}{\Delta x} = 0,$$

scheme converges $\Longrightarrow$ convergence to weak solution

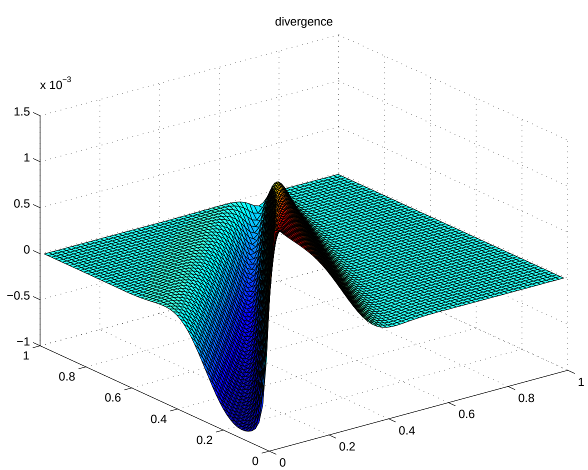
$\Rightarrow$ exact discrete conservation ($\nabla \cdot (u, f(u))_d \equiv 0$) is a *sufficient* condition for convergence to a weak solution, but is often erroneously considered as *necessary*

- popular FEM for hyperbolic conservation laws (e.g. Discontinuous Galerkin) are discretely conservative in the Lax-Wendroff sense

Numerical Conservation

THEOREM. Conservation theorem for $H(div)$-conforming LSFEM.
$H(div)$-conforming LSFEM converges $\Longrightarrow$ convergence to weak solution

THEOREM. Conservation theorem for dual $H(div)$-conforming LSFEM.
dual $H(div)$-conforming LSFEM converges $\Longrightarrow$ convergence to weak solution



- $H(div)$-conforming LSFEM does not impose strict discrete numerical conservation, but converges to weak solution! $\Rightarrow$ discrete numerical conservation is not necessary

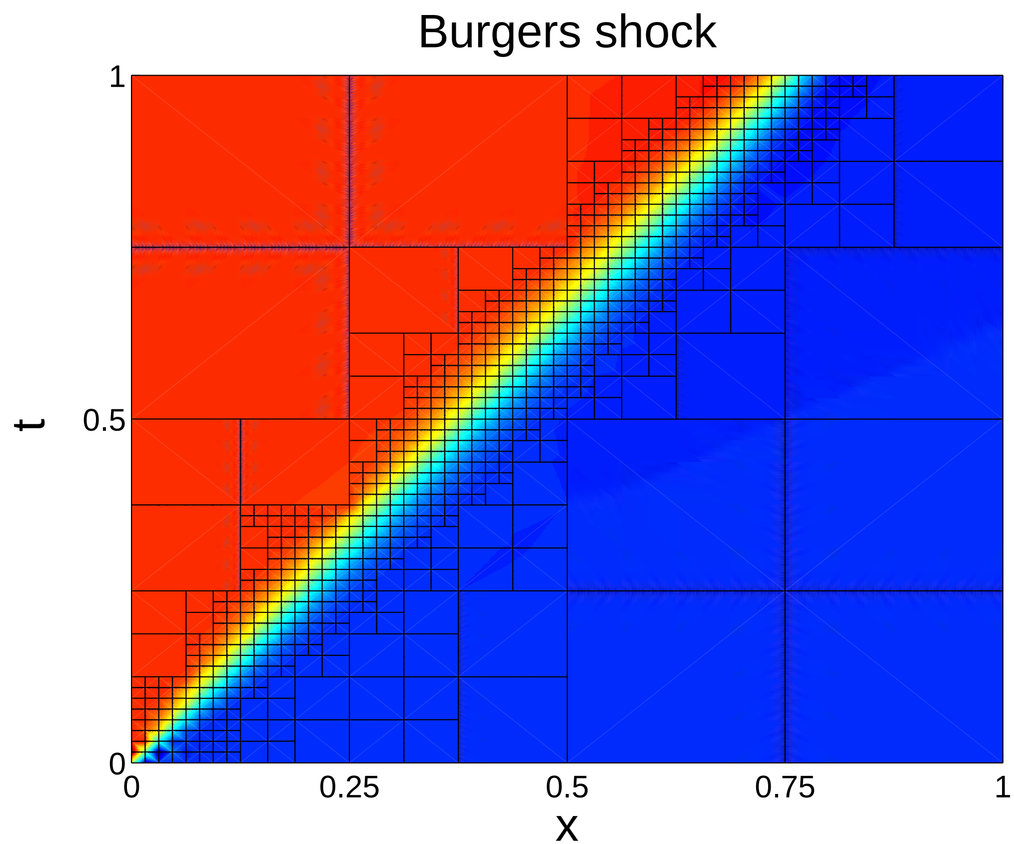
- dual $H(div)$-conforming LSFEM has stronger, *pointwise* discrete conservation property: $\nabla \cdot (\nabla^{\perp} p^h) \equiv 0$

Error Estimator and Adaptive Refinement

$$
\begin{aligned}
\mathcal{F}(u^h; f) &= \|Lu^h - f\|_0^2 \\
&= \|Lu^h - Lu_{exact}\|_0^2 \\
&= \|L(u^h - u_{exact})\|_0^2 \\
&= \|Le^h\|_0^2
\end{aligned}
$$

- functional value gives sharp local a posteriori error estimator
- use error estimator for adaptive refinement in space–time

Error Estimator and Adaptive Refinement



- sharp fronts at shocks
- with optimal $O(n)$ solver, work per grid point is bounded
- promises to be competitive with other methods (also explicit timemarching)

Conclusions

we have developed two classes of $H(div)$-conforming LSFEM for hyperbolic conservation laws

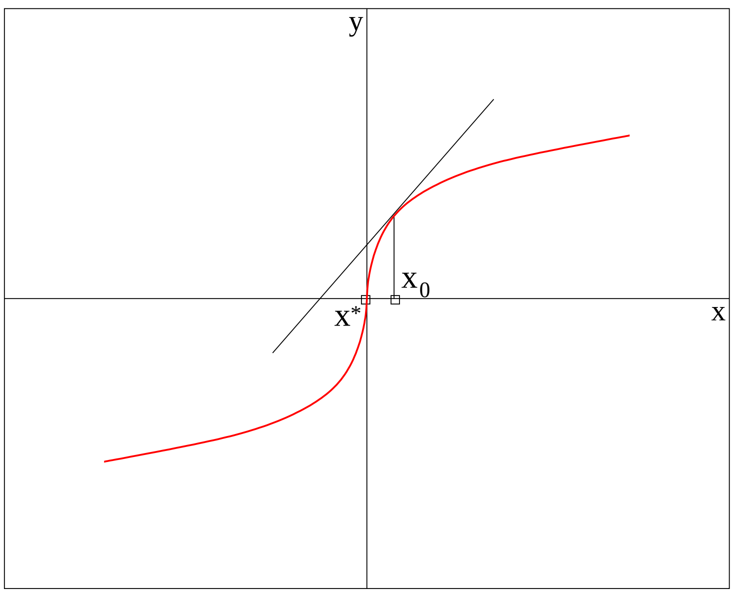
- disadvantages
  - extra variables are introduced ($\vec{w}$ or p)
  - high diffusion of LSFEM at shocks
- advantages of LSFEM
  - optimal solution within finite element space
  - SPD linear systems (iterative methods, AMG)
  - error estimator (efficient adaptive refinement)
  - conservation (either weaker or stronger than Lax-Wendroff discrete conservation)
  - no spurious oscillations at discontinuities (without need to add numerical diffusion)
  - *linear* higher order schemes

Conclusions

- advantages of $H(div)$-conforming reformulation
  - linear differential operator
  - bounded Fréchet derivative $\Rightarrow$ Newton converges
  - nullspace of operator can be represented exactly
  - differential boundary conditions help AMG (dual)
  - regularity of the solution ($\vec{w} \in H(div)$) is made explicit, also at the discrete level using Raviart-Thomas elements
- FE convergence theory remains to be worked out further
- promising initial AMG results, to be developed further
- methods can be extended to multiple spatial dimensions, and to systems of equations

Newton with Unbounded Derivative

$$f(x) = 0 \;\Rightarrow f(x_0) + f'(x_0)(x - x_0) = 0 \;\Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$



$$f(x) = x^\alpha \qquad x > 0$$
$$= -|x|^\alpha \qquad x \le 0$$
$$\alpha \in (0,1)$$
$$f'(x) = \alpha\, x^{\alpha-1} \qquad x > 0$$
$$f'(0) = \infty$$
$$x = x_0 - \frac{x_0^\alpha}{\alpha\, x_0^{\alpha-1}}$$
$$x = (1 - 1/\alpha)\, x_0$$

divergence $\forall\, \alpha < 1/2$, *e.g.* for $\alpha = 1/3$, $x = -2\, x_0$

$\Rightarrow$ if $f'(x^*) = \infty$, basin of attraction can be empty!

Algebraic Multigrid

- use approach investigated for linear problems for $H(div)$-conforming LSFEM
- e.g., dual LSFEM: $L = \nabla^{\perp} p - \vec{b} u$

$$< \nabla^{\perp} p^h, \nabla^{\perp} q^h > + < -\vec{b} u^h, \nabla^{\perp} q^h > = 0 \qquad \forall \quad q^h \in \mathcal{P}^h$$

$$< \nabla^{\perp} p^h, -\vec{b} s^h > + < -\vec{b} u^h, -\vec{b} s^h > = 0 \qquad \forall \quad s^h \in \mathcal{U}^h$$

$$A = \begin{bmatrix} A_{pp} & A_{pu} \\ A_{up} & A_{uu} \end{bmatrix}$$

- A is symmetric positive definite
- standard AMG efficient for A_{pp}
- A_{uu} mass matrix, but A_{up} strong off-diagonal coupling
- some promising initial results, but work in progress

Numerical Conservation

$$\partial_t(u_i^h\ S_i) + \sum_{j \in \partial\Omega_i} \bar{f}_j^h \cdot \vec{n}_j\ l_j = 0.$$

$$\partial_t(\sum_{i \in \Omega_s} u_i^h\ S_i) + \sum_{j \in \partial\Omega_s} \bar{f}_j^h \cdot \vec{n}_j\ \ l_j = 0.$$