Multi-Dimensional Upwind Constrained Transport (MUCT) of Divergence-Free Fields on Unstructured Grids

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Conservative form ideal MHD equations

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\vec{v}\\ \rho e\\ \vec{B}\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\vec{v}\\ \rho\vec{v}\vec{v}+\left(p+\frac{B^2}{2}\right)\vec{I}-\vec{B}\vec{B}\\ \left(\rho e+p+\frac{B^2}{2}\right)\vec{v}-(\vec{v}\cdot\vec{B})\vec{B}\\ \vec{v}\vec{B}-\vec{B}\vec{v}\end{bmatrix} = 0$$

- nonlinear system of hyperbolic conservation laws describing magnetized fluid

  - 8 waves (2 $\times$ fast, 2 $\times$ Alfvén, 2 $\times$ slow, entropy, spurious/divergence)

  - 3 types of shocks (fast, intermediate, slow)

  - non-classical, overcompressive, non-evolutionary shocks have conditionally stable viscous profiles, exist (Myong's talk)

  - constraint: $\nabla\cdot\vec{B}=0$ (my talk, Kroener's talk, Toth's talk)

Topic of my talk:

- **numerical schemes for the advection of divergence-free fields on unstructured grids**

$\Rightarrow$ divergence-free: $\nabla \cdot \vec{B} = 0$

- $\vec{B}$ magnetic field (plasma ...)
- no magnetic monopoles
- also numerically, avoid magnetic monopoles at the discrete level:

Constrained Transport (CT) approach

(known on structured grids, Evans & Hawley 1988, earlier for EM)

($\sim$ 'mimetic' schemes, Hyman & Shashkov 1997)

$\Rightarrow$ advection = hyperbolic (example: Magnetohydrodynamics (MHD))

$\Rightarrow$ unstructured grids: Multi-Dimensional Upwind (MU) schemes

Overview

(1) Constrained Transport on unstructured grids

(2) Multi-Dimensional Upwind Schemes

(3) Multi-Dimensional Upwind Constrained Transport (MUCT) Schemes for Faraday's equation

(4) MUCT Schemes for the 'Shallow Water' Magnetohydrodynamics Equations

(1) Constrained Transport on unstructured grids

Faraday's induction equation

(with constant $\vec{v}$):

- Faraday: $\dfrac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0$

  (ideal) MHD approximation: $\vec{E} = -\vec{v} \times \vec{B}$

$$\Rightarrow \qquad \frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})$$

- $\nabla \cdot \vec{B} = 0$ or $\oint \vec{B} \cdot \vec{n} dS = 0$

$$\Rightarrow \frac{\partial \nabla \cdot \vec{B}}{\partial t} = 0$$

(1) Constrained Transport on unstructured grids

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})$$

(1) $$\frac{\partial \vec{B}}{\partial t} = -\nabla \cdot (\vec{v}\vec{B} - \vec{B}\vec{v})$$

= conservation law form, use schemes for hyperbolic systems

$\Rightarrow$ problems with numerical stability, . . . , can be cured

- using *source term*, 8-wave formulation (Powell 1995)

but wrong jumps at shocks may persist (Toth 2000)

- using *projection* (Poisson solve)

- using *elliptic-hyperbolic-parabolic divergence cleaning* (Dedner, Muenz, Kroener et. al. 2001)

- using *divergence dissipation* (Linde & Malagoli)

(1) Constrained Transport on unstructured grids

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})$$

(2) $$\frac{\partial \int \vec{B} \cdot \vec{n} dS}{\partial t} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$\int \vec{B} \cdot \vec{n} dS = \bar{B}_n \Delta S \quad \Rightarrow \quad \frac{\partial \bar{B}_n}{\partial t} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l} \,/\, \Delta S$$

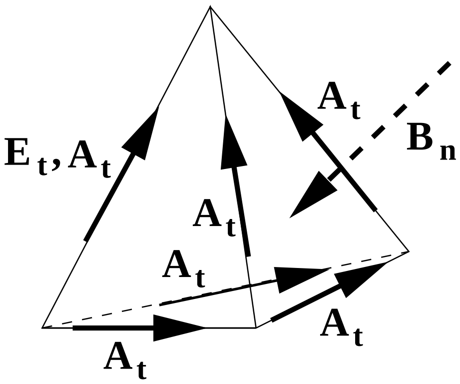
= time evolution of flux through surface

= time evolution of average normal component $\bar{B}_n$ of $\vec{B}$

$\Rightarrow \oint \vec{B} \cdot \vec{n} dS = 0$ on discrete level!!

because boundary of boundary vanishes (or contributions cancel)

= CT (Evans & Hawley 1988)

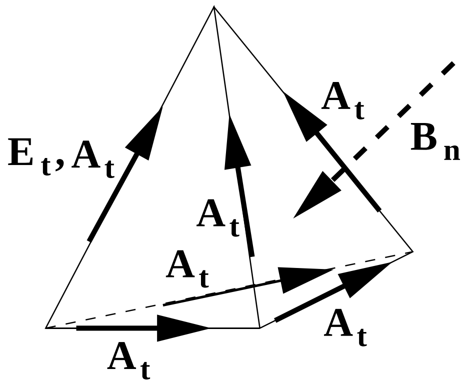


(1) Constrained Transport on unstructured grids

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})$$

(3) $$\frac{\partial \vec{A}}{\partial t} = \vec{v} \times (\nabla \times \vec{A})$$

with $\vec{B} = \nabla \times \vec{A}$ and $\vec{A}$ = vector potential



= system Hamilton-Jacobi equation for vector potential (Londrillo & Del Zanna 2000)

(remark: only $\vec{v}$ determines upwind direction)

$\Rightarrow \bar{B}_n$ from $\int \vec{B} \cdot \vec{n} dS = \bar{B}_n \Delta S = \oint \vec{A} \cdot d\vec{l}$

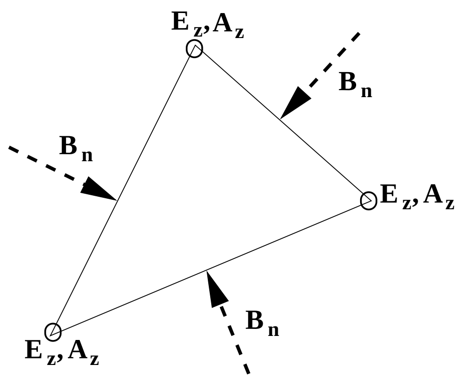
$\Rightarrow \oint \vec{B} \cdot \vec{n} dS = 0$ on discrete level!!

because boundary of boundary vanishes (or contributions cancel)

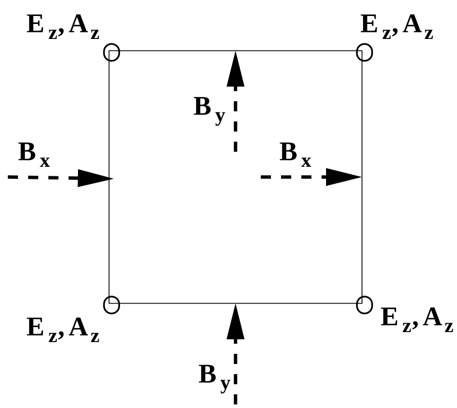
= CT (Evans & Hawley 1988)

in 2D:

$$\frac{\partial \int_1^2 \vec{B} \cdot \vec{n} dl}{\partial t} = \frac{\partial \bar{B}_n}{\partial t} \Delta l = (\vec{v} \times \vec{B})_2 - (\vec{v} \times \vec{B})_1 \quad \text{or} \quad \frac{\partial A_z}{\partial t} = -\vec{v} \cdot \nabla A_z$$



triangle = difficult: how to get $\vec{B}$ in nodes from $\bar{B}_n$?

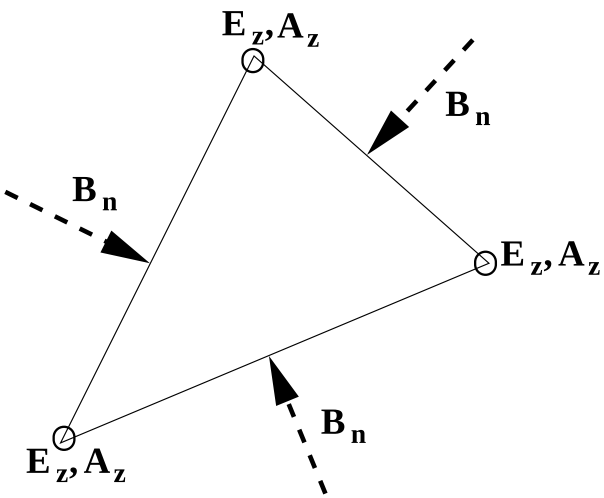


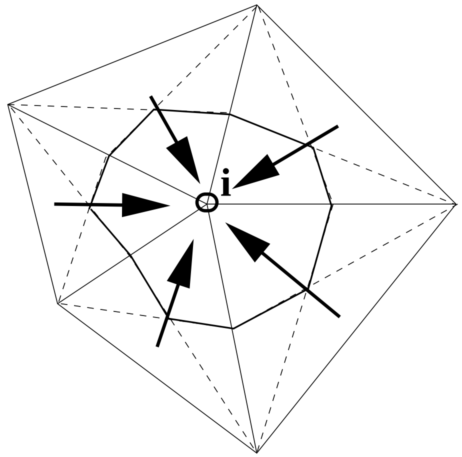
cartesian quadrilateral = easy:

B_x and B_y reconstruct $\vec{B}$ in nodes = CT (Evans & Hawley 1988)

Need vector basis functions: $\vec{P}_j$

($\sim$ face elements, from EM, e.g. Jin 93; Robinson & Bochev 2001 for MHD)





(1) reconstruct $\vec{B}$ in cell from $\bar{B}_n$ as

$$\vec{B}_{cell} = \sum_{j=1}^{3} \vec{P}_j \, B_{n,j}$$

(2) average $\vec{B}_{cell}$ to nodal $\vec{B}_i$ in upwind way

e.g. $\vec{P}_1$: normal component $\vec{P}_{1,n}$ constant on edge 1, vanishing on other edges

Vector basis functions $\vec{P}_i$

$$\vec{P}_1 = l_1 \,(L_3\, \nabla \times L_2 - L_2\, \nabla \times L_3) = \frac{l_1}{2\Delta}(L_3\, l_2\, \vec{t}_2 - L_2\, l_3\, \vec{t}_3)$$

$$\vec{P}_2 = l_2 \,(L_1\, \nabla \times L_3 - L_3\, \nabla \times L_1) = \frac{l_2}{2\Delta}(L_1\, l_3\, \vec{t}_3 - L_3\, l_1\, \vec{t}_1)$$

$$\vec{P}_3 = l_3 \,(L_2\, \nabla \times L_1 - L_1\, \nabla \times L_2) = \frac{l_3}{2\Delta}(L_2\, l_1\, \vec{t}_1 - L_1\, l_2\, \vec{t}_2)$$

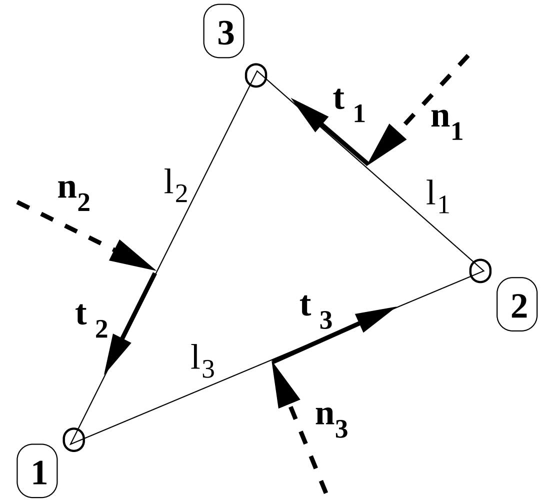
($\sim$ face elements; Raviart-Thomas or Nedelec)

e.g. $\vec{P}_1$: normal component $\vec{P}_{1,n}$ constant on edge 1, vanishing on other edges

$\nabla \cdot \vec{P}_j =$ constant

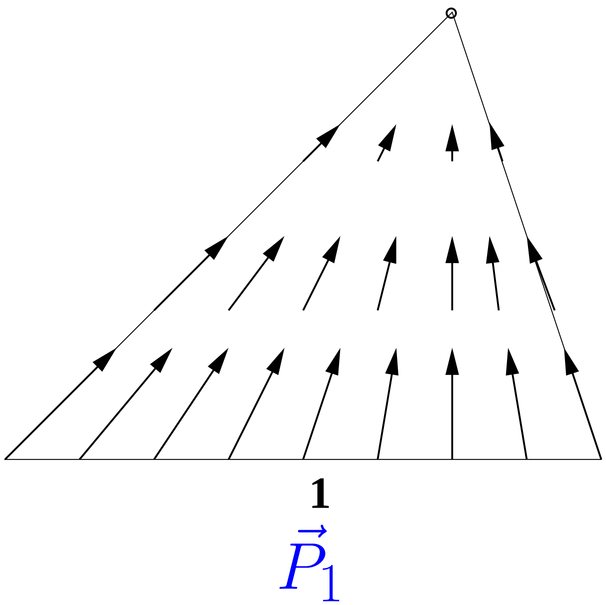
$\nabla \times \vec{P}_j = 0$ (on triangle)

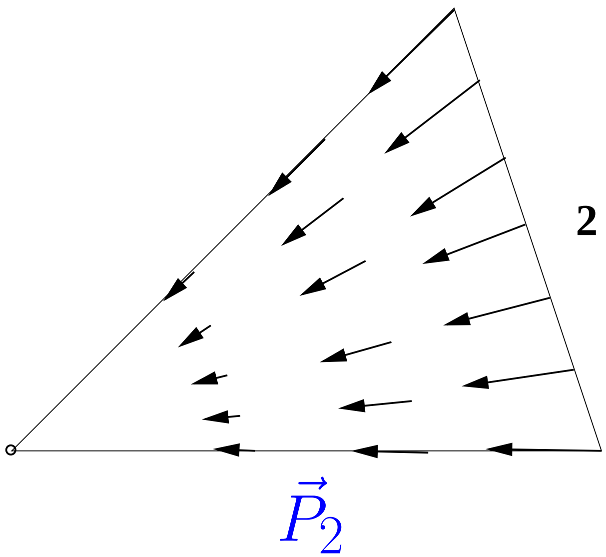
$\Rightarrow$ diverging, non-sheared vector basis functions



(1) Constrained Transport on unstructured grids

$$\vec{B}_{cell} = \sum_{j=1}^{3} \vec{P}_j \, B_{n,j}$$





e.g. $\vec{P}_1$: normal component $\vec{P}_{1,n}$ constant on edge 1, vanishing on other edges

(also higher order, quads, . . .: general concept)

(1) Constrained Transport on unstructured grids

$$\vec{B}_{cell} = \sum_{j=1}^{3} \vec{P}_j \, B_{n,j}$$

- $B_{n,j}$ such that $\nabla \cdot \vec{B} =$ constant $\equiv 0$ everywhere inside element
- B_n is continuous at element interfaces, so there also $\nabla \cdot \vec{B} = 0$

$\Rightarrow$ finite-element reconstructed solution satisfies $\nabla \cdot \vec{B} = 0$ everywhere!

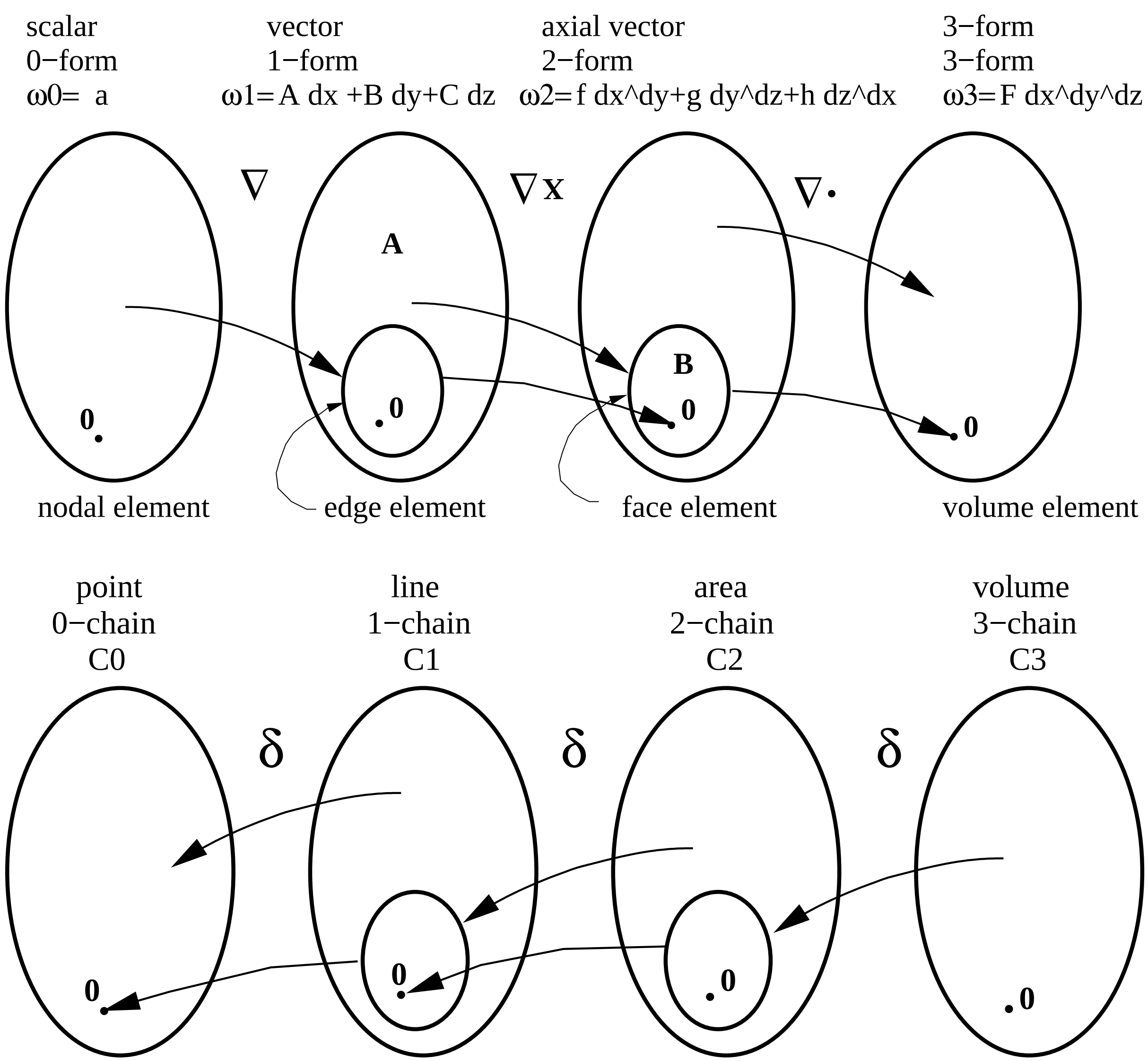
in triangle, for lowest order element:

$\vec{B}$ constant in space, B_n continuous

(on quad, or for higher order vector basis function:

$\vec{B}$ not constant in space, B_n continuous)

Interpretation: differential geometry



- physics = geometry
- numerics = geometry

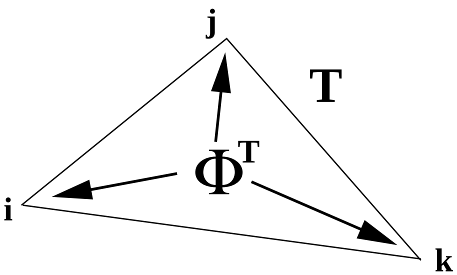
$\Rightarrow$ in a consistent way!

(2) Multi-Dimensional Upwind Schemes

(Roe, Deconinck, Barth, Abgrall, Sidilkover, . . . , 1990–2001)

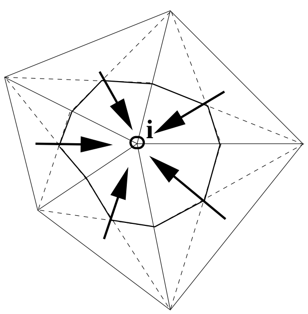
$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0$$

define residual $\Phi^T = \int_T \nabla \cdot f(u^h)\, d\Omega$



The cell residual Φ^T is distributed to the nodes u_i of the cell:

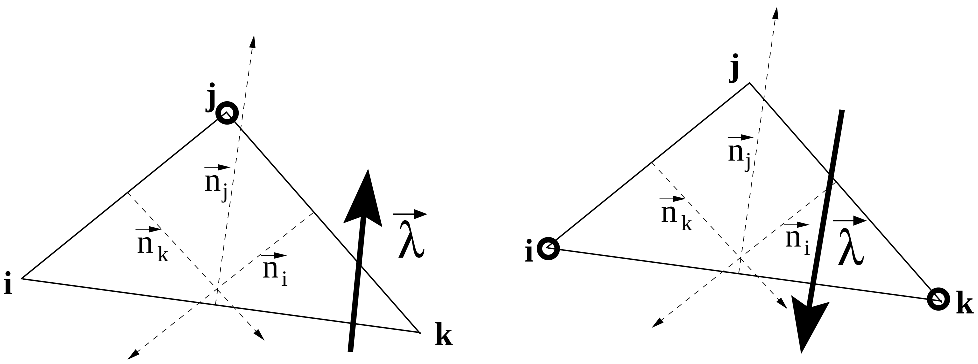
$$\Phi_i^T = \beta_i^T \Phi^T$$



The contributions to node i are assembled from all surrounding triangles: *compact* scheme

$$\frac{\partial u_i}{\partial t} = -\frac{1}{S_i} \sum_{T:i \in T} \beta_i^T \Phi^T,$$

(2) Multi-Dimensional Upwind Schemes



One-target (top) and two-target (bottom) situations for an upwind scheme. Define the upwind parameters k_i as $k_i = \vec{\lambda}^T \cdot \vec{n}_i/2$.

- Galerkin or central scheme: $\beta_i = 1/3$ (*unstable for advection*)
- N scheme: $\beta_i = k_i^+ \dfrac{\sum_j k_j^- (u_i - u_j)}{\left(\sum_j k_j^-\right) \Phi^T}$.

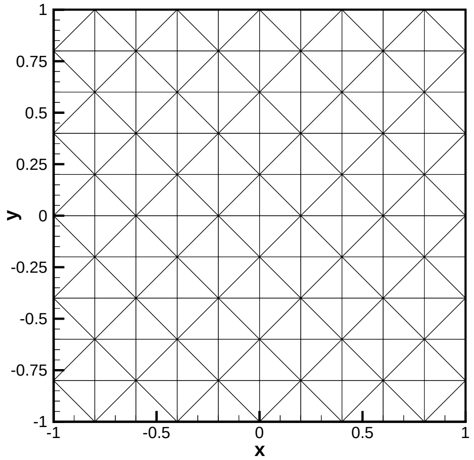
  (*only first order accurate, but positive, and upwind*)

  remark: need nodal values

- Lax-Wendroff scheme, LDA scheme: *second order for steady flow, not positive*
- Blended scheme (N+LDA): *(almost) second order for steady flow, positive*

(3) MUCT Schemes for Faraday's equation

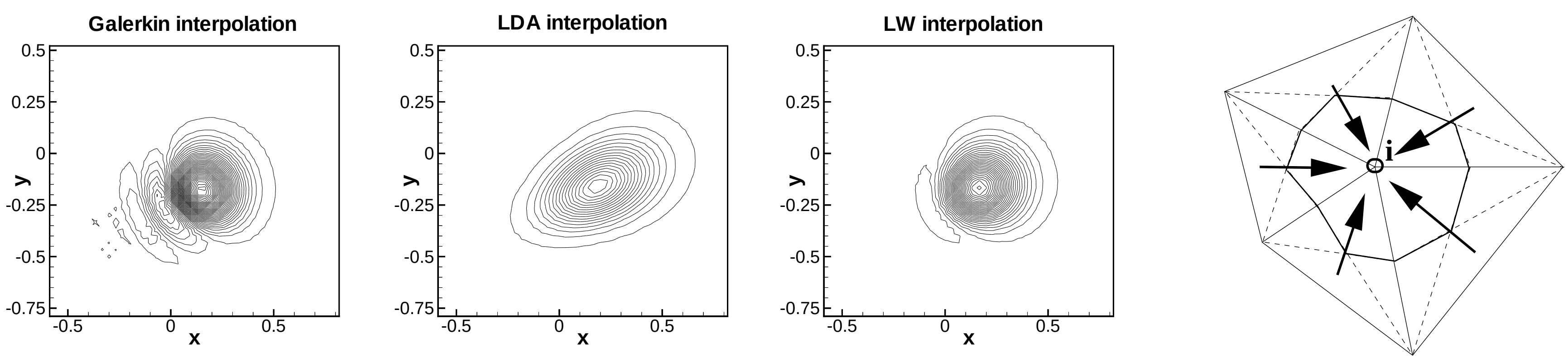
Interpolation MUCT schemes



$$\frac{\partial \int_1^2 \vec{B} \cdot \vec{n} dl}{\partial t} = \frac{\partial \bar{B}_n}{\partial t} \Delta l = (\vec{v} \times \vec{B})_2 - (\vec{v} \times \vec{B})_1$$

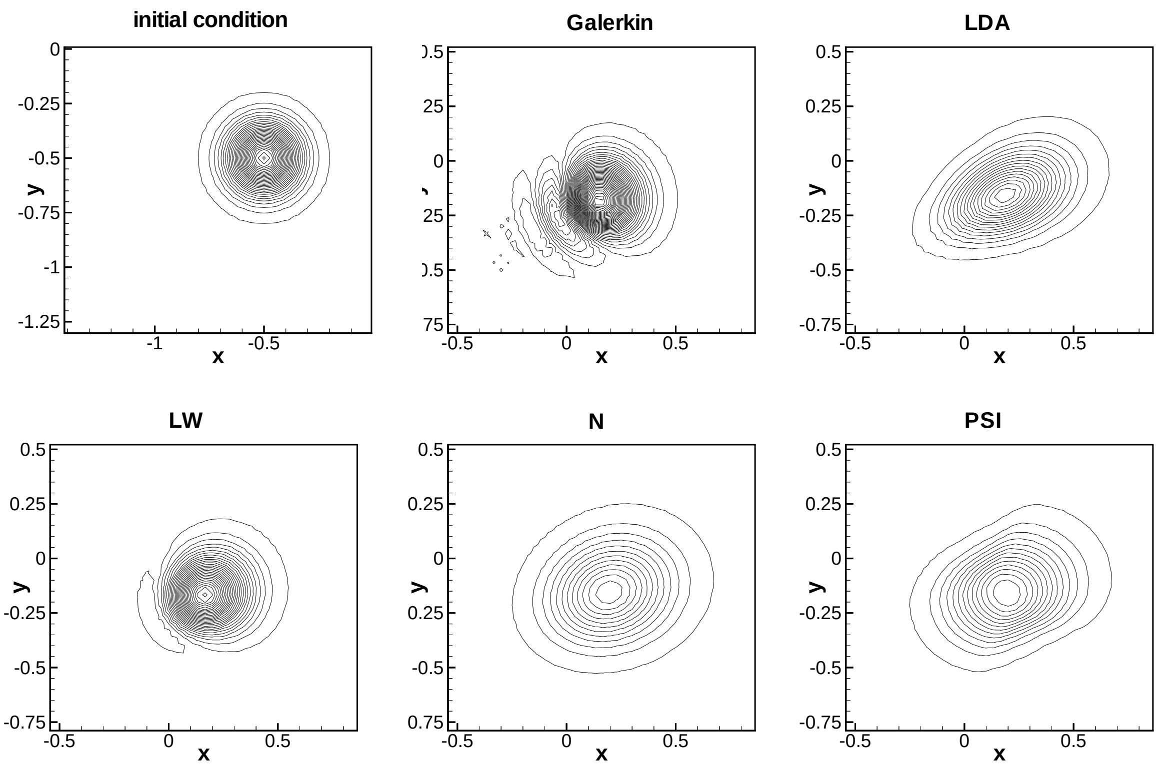
$$\vec{B}_{cell} = \sum_{j=1}^{3} \vec{P}_j \, B_{n,j}$$

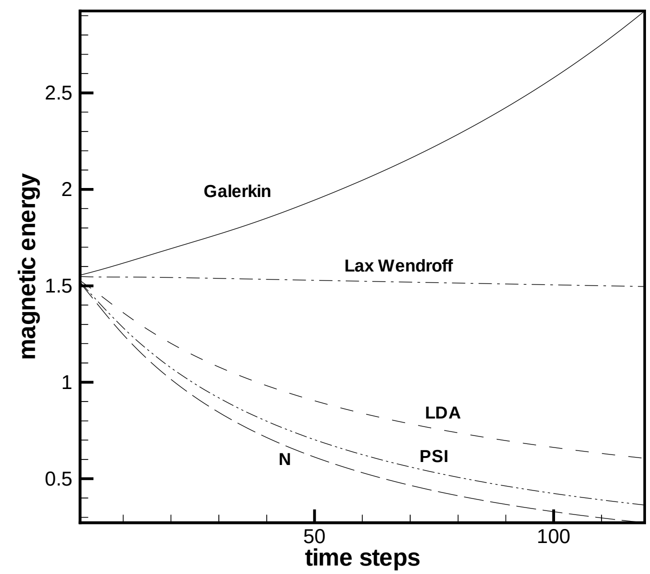
$$\vec{B}_i \left(\sum_{cells j} \beta_j S_{T,j} \right) = \sum_{cells j} \vec{B}_{cell,j} \beta_j S_{T,j}$$



Vector potential MUCT schemes

$$\frac{\partial A_z}{\partial t} = -\vec{v} \cdot \nabla A_z$$





3D MUCT schemes

- interpolation MUCT schemes: need residual distribution to the edges

  $\vec{t}_i$ are the six vectors connecting the middles of the six edges

  of the tetrahedron to the centroid of the tetrahedron

  $$\Rightarrow \sum_{i=1}^{6} \vec{t}_i = 0$$

  $\Rightarrow$ define the upwind parameters k_i as $k_i = \vec{\lambda}^T \cdot \vec{t}_i$, with $i = 1 \ldots 6$

  (not positive because no N scheme)

- vector potential MUCT schemes

  need a positive system N scheme for the system HJ equation

  (distribution to the nodes or to the edges)

  $$\frac{\partial \vec{A}}{\partial t} = \vec{v} \times (\nabla \times \vec{A})$$

(4) MUCT Schemes for ‘Shallow Water’ MHD

‘Shallow Water’ MHD

(Gilman 2000, De Sterck 2001)

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\,\vec{v}) = 0$$
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} - (\vec{B} \cdot \nabla)\vec{B} + g\,\nabla h = 0$$
$$\frac{\partial \vec{B}}{\partial t} + (\vec{v} \cdot \nabla)\vec{B} - (\vec{B} \cdot \nabla)\vec{v} = 0$$

$$\nabla \cdot (h\,\vec{B}) = 0$$

- from MHD: incompressible, 2D variation, magnetohydrostatic equilibrium
- 4 wave modes: 2 magneto-gravity waves (nonlinear), 2 Alfvén waves (linear)
- one spurious ‘div(B)’-wave (MHD!)

(4) MUCT Schemes for 'Shallow Water' MHD

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ h\,\vec{v} \\ h\vec{B} \end{bmatrix} + \nabla\cdot\begin{bmatrix} h\,\vec{v} \\ h\,\vec{v}\vec{v} - h\,\vec{B}\vec{B} + \mathsf{I}\left(gh^2\,/2\right) \\ h\vec{v}\vec{B} - h\vec{B}\vec{v} \end{bmatrix} = 0$$

$h\vec{B}$: store normal components B_n *on edges*

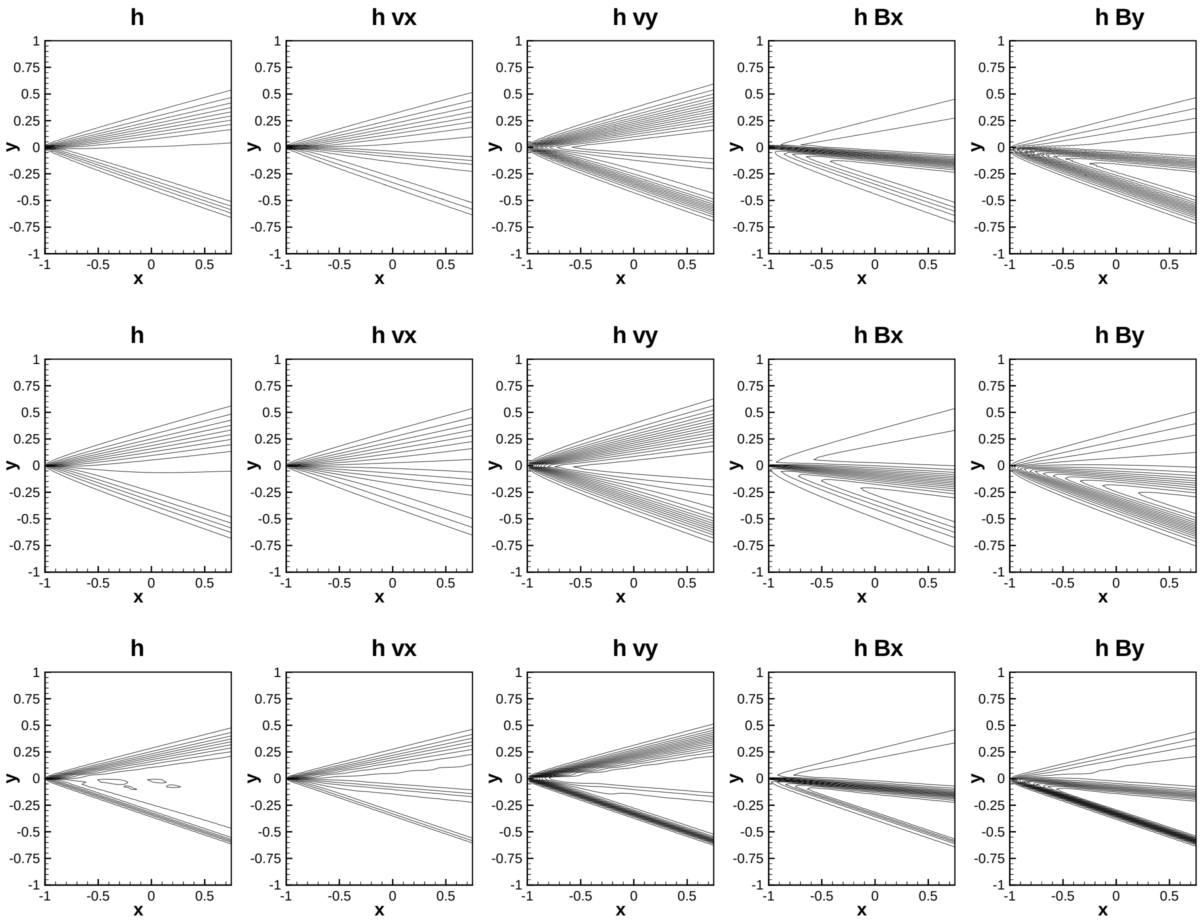
- MUCT (upwind interpolation or upwind scheme for vector potential)
- Hamilton-Jacobi equation discretized fully upwind, positive
- *only* need to use $\vec{v}$ for upwinding!! (use $\vec{v}_{cell}$ averaged from nodes)

$h, h\vec{v}$: store *nodally*

- regular system N scheme
- fully upwind using all 5 SMHD wave modes, positive
- need nodal values, also for $h\vec{B}$ (Galerkin interpolated from $\vec{B}_{cell}$)

$\Rightarrow$ system N MUCT scheme

Steady Riemann problem

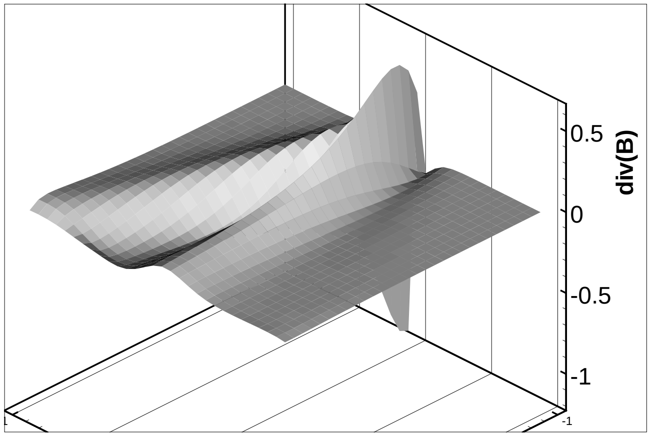


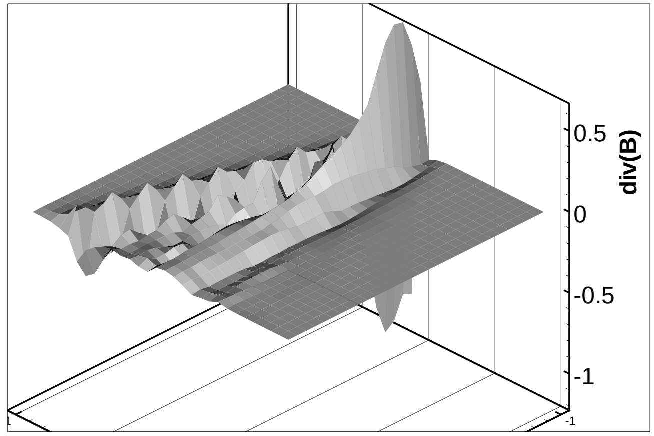
System N MUCT solution of the steady Riemann problem on a grid of 91×91 nodes. (No oscillations!)

First order Lax-Friedrichs finite volume solution of the steady Riemann problem on a grid of 90×90 finite volumes.

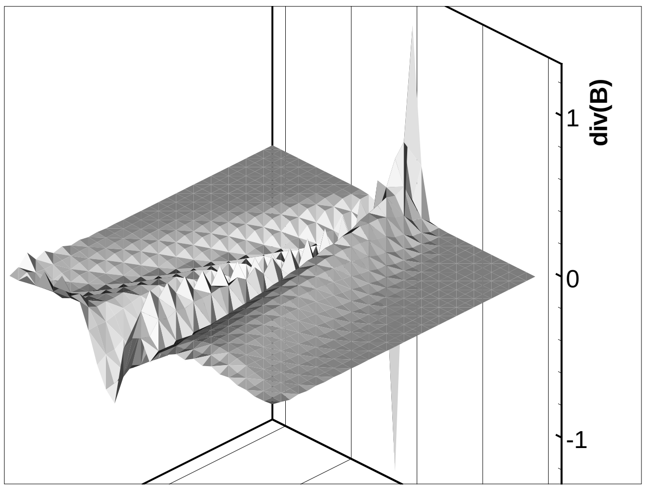
Second order Lax-Friedrichs finite volume solution of the steady Riemann problem on a grid of 90×90 finite volumes.

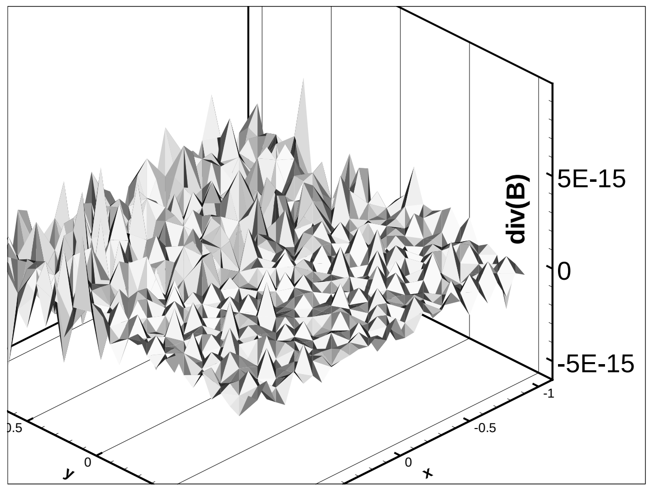
(4) MUCT Schemes for ‘Shallow Water’ MHD





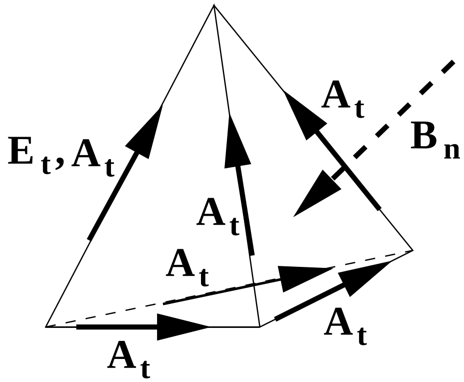
$\nabla \cdot \vec{B}$ for the first order (left) and second order (right) Lax-Friedrichs simulation of the steady Riemann problem on a grid of 30×30 finite volumes.





$\nabla \cdot \vec{B}$ for the full system N (left) and system N MUCT (right) simulation of the steady Riemann problem on a grid of 31×31 nodes.

Conclusions



- represent $\vec{B}$ by $\bar{B}_n$: normal component on surfaces

  or represent $\vec{A}$ by $\bar{A}_t$: tangential component along edges

- on unstructured grids, $\vec{B}$ and $\vec{A}$ can be reconstructed everywhere in the domain using vector basis functions (face elements for $\vec{B}$ and edge elements for $\vec{A}$)
- update $\bar{B}_n$ or $\bar{A}_t$ using MU schemes (via MU interpolation of the reconstructed fields)
- this conserves the $\nabla \cdot \vec{B} = 0$ constraint at the discrete level up to machine accuracy
- this has been tested for Faraday, Shallow Water MHD (system MUCT scheme)
- easy extensions: 2nd order (blended scheme), MHD, 3D, . . .

= generalization of CT to multi-dimensional methods on unstructured grids

(De Sterck, AIAA paper 2001-2623) (http://amath.colorado.edu/faculty/desterck)