CO 351 Network Flows (Winter 2010)
Assignment 5

Due: Tuesday, March 30th in class.

Policy. All questions are worth 30 marks. For any question, if you simply write "I don't know" you will receive 20% of the marks for that question.

Question 1: Let $D = (N, A)$ be a digraph, let $c \in \mathbb{Z}^A$ be integer arc capacities, and let $s, t \in N$. Let x^* be a maximum s-t flow in this instance.

(a): Let $\delta(Z_1)$ and $\delta(Z_2)$ be two distinct minimum s-t cuts. Show that $\delta(Z_1 \cap Z_2)$ and $\delta(Z_1 \cup Z_2)$ are also minimum s-t cuts.
(HINT: Look at the residual digraph D'_{x^*}.)

(b): Argue, using part (a), or directly, that there exists a "*shallowest*" minimum s-t cut $\delta(Z_{\min})$ and a "*deepest*" minimum s-t cut $\delta(Z_{\max})$. By "shallowest" and "deepest" we mean that if $\delta(Z)$ is any minimum s-t cut then $Z_{\min} \subseteq Z \subseteq Z_{\max}$.

(c): Assume that all capacities are integers. Show that increasing the capacity of an arc (u, v) by 1 increases the value of the maximum s-t flow if and only if $u \in Z_{\min}$ and $v \notin Z_{\max}$.

Question 2: Let $D = (N, A)$ be a digraph with arc-capacities $c \in \mathbb{R}^A$, and lower bounds $\ell \in \mathbb{R}^A$ such that $\ell_{uv} \leq c_{uv}$ for all $uv \in A$. Recall from the lectures that a feasible circulation is any $x \in \mathbb{R}^A$ that satisfies the flow conservation and capacity/lower bound conditions:

$$f_x(v) = 0 \qquad \forall v \in N$$
$$\ell_{uv} \leq x_{uv} \leq c_{uv} \qquad \forall uv \in A$$

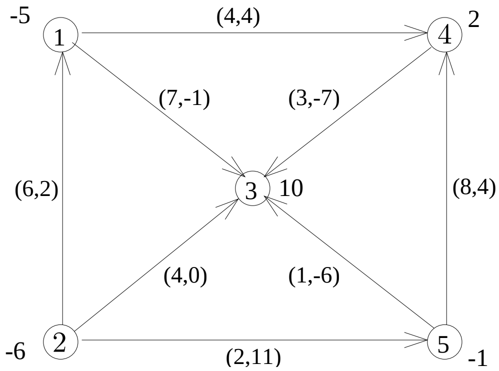
Prove that a feasible circulation exists iff for all $S \subseteq N$, we have

$$\ell(\delta(S)) \leq c(\delta(\bar{S})).$$

Question 3: You are in charge of processing the garbage of the city of Waterloo for the next 100 days. For each day i, d_i units of garbage are generated by the citizens of Waterloo. You can deal with each unit of garbage in one of three different ways: (1) you can incinerate the garbage at a cost of α_i dollar per unit, (2) you can ship the garbage to Michigan at a cost of β_i dollar per unit, (3) you can place it into temporary storage — this is free, but the storage facility has a maximum capacity of 50 units. At the start of the 100 days the storage facility is empty and it must be empty again at the end. The problem is to process all the garbage while minimizing the incineration and shipping costs.

Formulate this problem as a minimum cost flow problem by giving a digraph with arc capacities, arc costs, and node demands. A feasible flow in your flow problem should correspond to a plan for dealing with the garbage, and the cost of the flow should correspond to the cost of the plan.

Question 4: Consider the minimum cost flow problem given below. Arc labels are (c, w), node labels are b.



(1) Find a flow by constructing an auxiliary st-flow problem and by finding a maximum st-flow. You do not need to provide details on **how** you find the maximum st-flow.

(2) Use the algorithm described on page 68 to find a minimum cost flow. Indicate every negative dicycle that you find and the new flow obtained from it. You do not need to provide details on **how** you find the negative dicycles.

(3) At the end D' will have no negative dicycles. Find feasible potentials y for D'. You do not need to provide details on **how** you find the feasible potentials.

(4) Using (3) show that the flow that you obtained in (2) is a minimum cost flow by showing that the optimality conditions are satisfied.