CO 351 Network Flows (Winter 2010)
Assignment 4

Due: Thursday, March 11th in class.

Policy. For any question, if you simply write "I don't know" you will receive 20% of the marks for that question.

Question 1: (15 points)

Let $D = (N, A)$ be a digraph, with distinct nodes $s, t \in N$. Recall that an *st*-cut is a set of the form $\delta(S)$, where $s \in S$ and $t \notin S$.

For any set $F \subseteq A$, let $D \setminus F$ denote the digraph $(N, A \setminus F)$ obtained by deleting the arcs in F. A set $F \subseteq A$ is called an ***st*-disconnecting set** if there is no *st*-dipath in $D \setminus F$.

Prove that the minimum cardinality of an *st*-cut equals the minimum cardinality of an *st*-disconnecting set.

Question 2: (30 points)

Let $D = (N, A)$ be a digraph. For simplicity, assume that for every pair of nodes $u, v \in N$, at most one of uv or vu is an arc. (Informally, D does not contain any parallel-but-oppositely-directed pairs of arcs.)

Recall that a set $X \subseteq N$ is called a ***closure*** if $\delta(X) = \emptyset$, i.e., there is no arc uv with $u \in X$ and $v \in N \setminus X$. In class we discussed finding a closure which maximizes the weight of the nodes in X. In this problem, we want to find a closure that maximizes $|\delta(\bar{X})|$, i.e., the number of arcs uv with $u \in N \setminus X$ and $v \in X$.

Construct a digraph $\hat{D} = (\hat{N}, \hat{A})$ with nodes $s, t \in \hat{N}$ and arc-lengths w_a for all $a \in \hat{A}$ such that a minimum *st*-cut in $\hat{D}$ corresponds to a closure that maximizes $|\delta(\bar{X})|$.

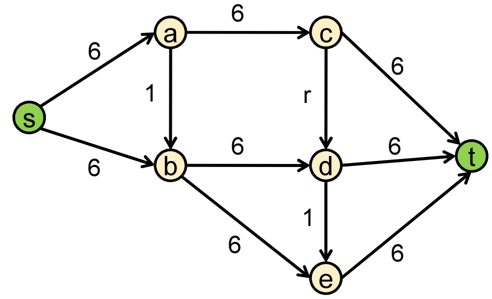
Question 3: (15 points)

Prove that the Ford-Fulkerson algorithm will terminate with an optimal solution, *regardless* of which incrementing path is chosen, for digraphs whose arc capacities are positive *rational numbers*.

Hint: In class, we proved the same statement for digraphs whose arc capacities are positive *integers*.

Question 4: (Part (a): 4 points, Parts (b)-(i): 2 points each, Part (j): 20 points)

Consider the digraph $D = (N, A)$ given in the following figure. The arcs are labeled by their capacities.



The capacity of the arc cd is $r = \frac{\sqrt{5}-1}{2} \approx 0.618$. (This is the inverse of the "golden ratio".) Two properties that this value r satisfies are

$$r^n = r^{n+1} + r^{n+2} \quad \forall n \in \mathbb{Z},\ n \geq 0 \tag{1}$$

$$1 + 2\sum_{i \geq 1} r^i < 5 \tag{2}$$

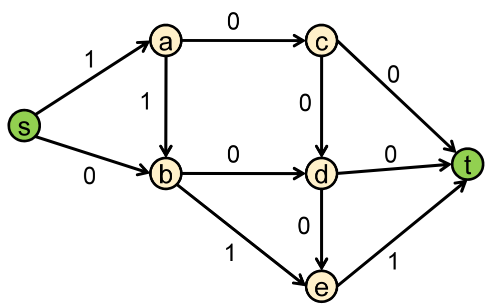
(a): By inspection, find a maximum st-flow and minimum st-cut in this digraph.

We will execute the Ford-Fulkerson algorithm on this digraph using very specific incrementing paths. Define:

$$\begin{aligned} P_0 &= sabet \\ P_1 &= sbacdet \\ P_2 &= sabdct \\ P_3 &= sabedt \end{aligned}$$

Note that P_1, P_2, P_3 are not dipaths in D, but they will be incrementing paths in the residual digraphs that we consider below.

The initial flow has zero flow on each arc. We use the path P_0 as the first incrementing path. It has $\gamma(P_0) = 1$. (Recall that $\gamma(P_0)$ denotes the maximum amount of additional flow that we can push on the path P_0.) The resulting flow is:



(b): Now we use the incrementing path P_1. What is $\gamma(P_1)$, as a function of r? For the resulting flow, give the flow on each arc, as a function of r.

(c): Now we use the incrementing path P_2. What is $\gamma(P_2)$, as a function of r? For the resulting flow, give the flow on each arc, as a function of r.

(d): Now we use the incrementing path P_1. What is $\gamma(P_1)$, as a function of r? For the resulting flow, give the flow on each arc, as a function of r. (You may be able to simplify using Eq. (1).)

(e): Now we use the incrementing path P_3. What is $\gamma(P_3)$, as a function of r? For the resulting flow, give the flow on each arc, as a function of r. What is the **objective value** of the current flow (i.e., the total amount of flow pushed so far), as a function of r?

Now let's consider a general situation where we start with a flow x such that, for some integer $n \geq 0$,

- the total flow value $f_x(t)$ is $1 + 2\sum_{i=1}^{n} r^i$. (This is less than 5, by Eq. (2).)
- $x_{ab} = 1$, $x_{cd} = r - r^{n+1}$, and $x_{de} = 1 - r^n$.

Note that the flow obtained above after using the first incrementing path P_0 satisfies these conditions with $n = 0$.

(f): Now we use the incrementing path P_1. What is $\gamma(P_1)$, as a function of r? Give the new flow values on arcs ab, cd, and de, as functions of r.

(g): Now we use the incrementing path P_2. What is $\gamma(P_2)$, as a function of r? Give the new flow values on arcs ab, cd, and de, as functions of r.

(h): Now we use the incrementing path P_1. What is $\gamma(P_1)$, as a function of r? Give the new flow values on arcs ab, cd, and de, as functions of r.

(i): Now we use the incrementing path P_3. What is $\gamma(P_3)$, as a function of r? Give the new flow values on arcs ab, cd, and de, as functions of r.

(j): Prove that the Ford-Fulkerson algorithm does not terminate on digraph D if it chooses incrementing paths in the order

$$P_0, \quad P_1, P_2, P_1, P_3, \quad P_1, P_2, P_1, P_3, \quad P_1, P_2, P_1, P_3, ...$$