CO 351 Network Flows (Winter 2010)
Assignment 3

Due: Thursday February 11th, in class.

Policy. All questions are worth 20 marks. For any question, if you simply write "I don't know" you will receive 20% of the marks for that question.

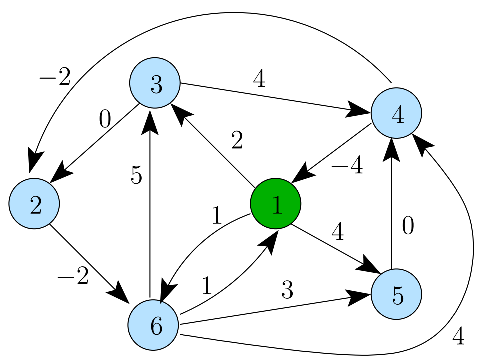
Let $D = (N, A)$ be a digraph with arc-lengths $w \in \mathbb{R}^A$, and let $s \in N$ be such that every node in N is reachable from s. Let $n = |N|$. The Bellman-Ford Algorithm, as described in class, was as follows:

- Set $y_s^{(0)} = 0$ and set $y_v^{(0)} = \infty$ for all $v \in N \setminus \{s\}$.
- For $i = 1, ..., n-1$ do
  - For each $v \in N$ do
    - Set $y_v^{(i)} = \min\left\{ y_v^{(i-1)}, \min\left\{ \; y_u^{(i-1)} + w_{u,v} \; : \; \text{arc } uv \in A \; \right\}\right\}$.

Question 1: *Bellman-Ford Example*

Consider the instance of the shortest-path problem shown below. The number labeling an arc indicates the length of the arc. The table below lists the $y_v^{(i)}$ values obtained by running the first two iterations of the Bellman-Ford algorithm with $s = 1$.

Continue the algorithm from this point to compute $y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}$. (Strictly speaking the algorithm as written above would only compute up to $y^{(5)}$, but for this problem we want $y^{(6)}$ too.) Does the final vector $y^{(6)}$ give feasible potentials?



	$y_v^{(i)}$					
i	1	2	3	4	5	6
0	0	∞	∞	∞	∞	∞
1	0	∞	2	∞	4	1
2	0	2	2	4	4	1

Question 2: *Termination in Bellman-Ford*

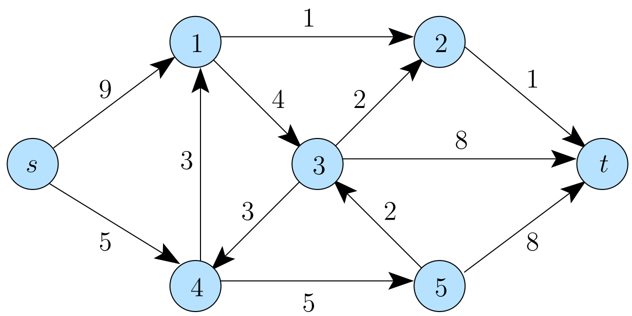
Suppose we modify the Bellman-Ford algorithm slightly by running it for n iterations instead of $n-1$, i.e., we compute the vectors $y^{(0)}, y^{(1)}, ..., y^{(n-1)}, y^{(n)}$.

(1) Suppose that $y_v^{(n)} = y_v^{(n-1)}$ for all $v \in N$. Prove that: (i) the values $y_v^{(n)}$ give feasible potentials, (ii) D does not contain a negative dicycle, and (iii) D contains a spanning tree rooted at s, whose arcs are all equality arcs (with respect to potentials $y^{(n)}$).
Hint for (iii): Use the fact that $y_v^{(n-1)}$ gives the length of the shortest sv-diwalk using at most $n-1$ arcs. (This was shown in class.)

(2) Suppose $y^{(n)} \neq y^{(n-1)}$, i.e., for some node $v \in N$, $y_v^{(n)} < y_v^{(n-1)}$. Show that D contains a negative dicycle.
Hint: There must be an sv-diwalk that traverses n arcs, which is more arcs than any dipath can traverse.

Question 3: *Ford-Fulkerson algorithm*

The following figure shows an instance of the maximum st-flow problem. The number labeling an arc gives the capacity of the arc.



Run the Ford-Fulkerson algorithm on the above instance to compute a maximum st-flow. Give the following information with each iteration of the algorithm: the value of the current flow, the residual graph with respect to the current flow, the augmenting path used to augment the flow, and the amount by which the flow is augmented. What is the maximum value of an st-flow?