CO 351 Network Flows (Winter 2010)
Assignment 2

Due: Thursday February 4th, in class.

Policy. All questions are worth 20 marks. For any question, if you simply write "I don't know" you will receive 20% of the marks for that question.

Question 1:

This exercise is based on formulating the following problem as a problem on dipaths. In the *change making problem* we are given a number k of coin denominations $a_1, a_2, \ldots, a_k$. The general problem is to determine whether or not a given nonnegative integer p can be "changed" into a collection of coins of the given denominations. In other words, do there exist nonnegative integers $x_1, x_2, \ldots, x_k$ such that $p = a_1x_1 + a_2x_2 + \cdots + a_kx_k$. For example, we may have $k = 3$ and $a_1 = 5$, $a_2 = 10$, $a_3 = 25$; then $p = 45$ can be changed, but $p = 17$ cannot be changed.

(1) Describe a method for determining all the numbers in a given range $[0, 1, 2, \ldots, u]$ that can be changed (u is a given positive integer).
HINT: One way is to construct a digraph with nodes labeled $0, 1, 2, \ldots, u$.

(2) Suppose that we can change a number p in the above range. Describe a method for finding a changing scheme that uses the minimum number of coins.

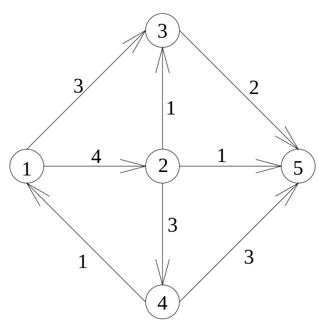
Question 2:

Let $D = (N, A)$ be a digraph with arc values $c \in \Re^A_+$ which we call *capacities*. Let $Q = v_1v_2...v_{k-1}v_k$ be an st-diwalk ($v_1 = s, v_k = t$). We define the *capacity* $c(Q)$ of Q to be the minimum over all arcs uv of Q of c_{uv}, i.e. $c(Q) = \min\{c_{uv} : uv \in Q\}$. Think of the arcs as pipes where the amount of liquid that can traverse the pipe is its capacity. The capacity of a sequence of pipes is the capacity of the smallest pipe (the bottleneck). Given an st-diwalk Q of maximum capacity, we would like to give a proof that it is indeed of maximum capacity. Call values $y \in \Re^N$ feasible potentials if

$$\min\{y_u, c_{uv}\} \leq y_v \quad \text{for all } uv \in A.$$

(1) Let y be feasible potentials and Q be an st-diwalk. Show that $\min\{y_s, c(Q)\} \leq y_t$.

(2) In this digraph, find a dipath from node 1 to node 5 of maximum capacity. Prove using (1) that it is indeed of maximum capacity.



(3) Consider now an arbitrary digraph. Prove that for every st-diwalk Q there is an st-dipath P of capacity $c(P) \geq c(Q)$.

(4) Can you always find feasible potentials (using the definition above)? Justify your answer.

Question 3:

Let $D = (N, A)$ be a digraph, with distinct nodes s, t_1, t_2 and $w \in \Re^A$. Throughout this exercise P_1 denotes an st_1-dipath, and P_2 denotes an st_2-dipaths. Consider the following linear program,

$$\begin{aligned} \min \sum_{uv \in A} w_{uv} x_{uv} & & \\ f_x(u) = \begin{cases} +1 & u = t_1 \text{ or } u = t_2 \\ -2 & u = s \\ 0 & \text{otherwise} \end{cases} & \qquad u \in N & \qquad \text{(P)} \\ x_{uv} \geq 0 & \qquad uv \in A & \end{aligned}$$

(1) Show that $x^{P_1} + x^{P_2}$ is a feasible solution for (P).

(2) Find the linear programming dual (D) of (P).

(3) Suppose we have a feasible solution y of (D) where $y_s = 0$. Using weak duality show that if $w(P_1) + w(P_2) = y_{t_1} + y_{t_2}$ then P_1 is a shortest st_1-dipath and P_2 is a shortest st_2-dipath.

(4) State the complementary slackness conditions for (P) and (D). Using the complementary slackness theorem, show that if y are feasible potentials and all arcs of P_1 and P_2 are equality arcs then P_1 is a shortest st_1-dipath and P_2 is a shortest st_2-dipath.

Question 4:

Let $D = (N, A)$ be a directed graph. Let $w \in \mathbb{R}^A$ be a vector of arc weights, with $w_a \geq 0$ for every arc $a \in A$. Let $s \in N$ be a node that can reach every other node.

Consider running Dijkstra's algorithm (as described in Section 2.3 of the Course Notes) on this digraph, with s as the starting node. Let S_i denote the set of nodes reachable from s in D' during the i^{th} iteration of Step 1. (The algorithm in the Course Notes simply calls this set S.) The proof of correctness (in Section 2.3.1) argues that

$$S_1 \subsetneq S_2 \subsetneq S_3 \subsetneq \cdots \subseteq N,$$

so the algorithm must terminate since eventually we'll have $S_j = N$. (The notation $A \subsetneq B$ means that A is a subset of B and $|A| < |B|$.)

Let A'_i be the set of equality arcs during the i^{th} iteration of Step 1. Is it also true that

$$A'_1 \subsetneq A'_2 \subsetneq A'_3 \subsetneq \cdots ?$$

If so, give a proof. If not, give a counterexample.

Question 5:

(1) Find a topological ordering of the nodes of the digraph below, using the algorithm described in class (Section 2.4.1 of the Course Notes). Indicate all steps.

(2) Using the algorithm for shortest dipaths in acyclic digraphs (Section 2.4.2 of the Course Notes), find for every node u the length of the longest su-dipath. Indicate all steps.