C&O 355
Lecture 8

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Outline

- Solvability of Linear Equalities & Inequalities
- Farkas' Lemma
- Fourier-Motzkin Elimination

Strong Duality

(for inequality form LP)

Primal LP: $\begin{aligned} \max \quad & c^{\mathsf{T}}x \\ \text{s.t.} \quad & Ax \le b \end{aligned}$

Dual LP: $\begin{aligned} \min \quad & b^{\mathsf{T}}y \\ \text{s.t.} \quad & A^{\mathsf{T}}y = c \\ & y \ge 0 \end{aligned}$

Strong Duality Theorem:

Primal has an opt. solution x $\Leftrightarrow$ Dual has an opt. solution y.

Furthermore, optimal values are same: $c^{\mathsf{T}}x = b^{\mathsf{T}}y$.

- Weak Duality implies $c^{\mathsf{T}}x \le b^{\mathsf{T}}y$. So strong duality says $c^{\mathsf{T}}x \ge b^{\mathsf{T}}y$.

  (for feasible x,y) (for optimal x,y)

- **Restatement of Theorem:**

  Primal has an optimal solution

  $\Leftrightarrow$ Dual has an optimal solution

  $\Leftrightarrow$ the following system is solvable:

$$Ax \le b \qquad A^{\mathsf{T}}y = c \qquad y \ge 0 \qquad c^{\mathsf{T}}x \ge b^{\mathsf{T}}y$$

"Solving an LP is equivalent to solving a system of inequalities"

- Can we characterize when systems of inequalities are solvable?

Systems of **Eq**ualities

- **Lemma:** Exactly one of the following holds:
  - There exists x satisfying $Ax=b$
  - There exists y satisfying $y^TA=0$ and $y^Tb<0$
- **Proof:**

  Simple consequence of Gaussian elimination working.

  Perform row eliminations on augmented matrix [A | b], so that A becomes upper-triangular

  If resulting system has i^{th} row of A equal to zero but b_i non-zero then no solution exists

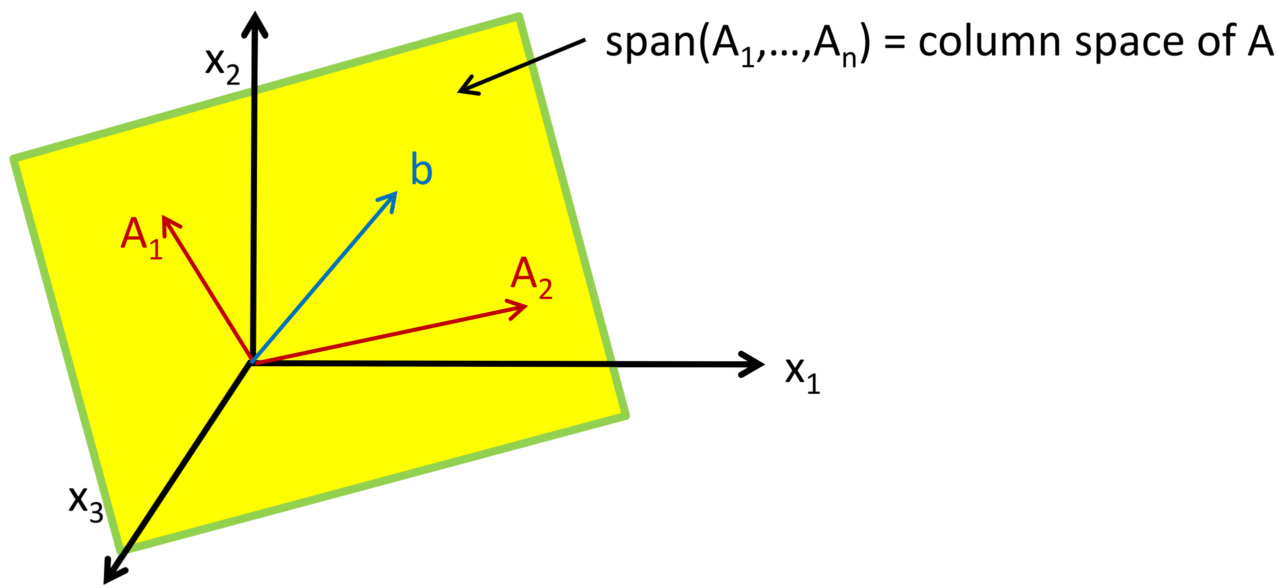
  - This can be expressed as $y^TA=0$ and $y^Tb<0$.

  Or $y^Tb>0$, by negating y

  Otherwise, back-substitution yields a solution. ■

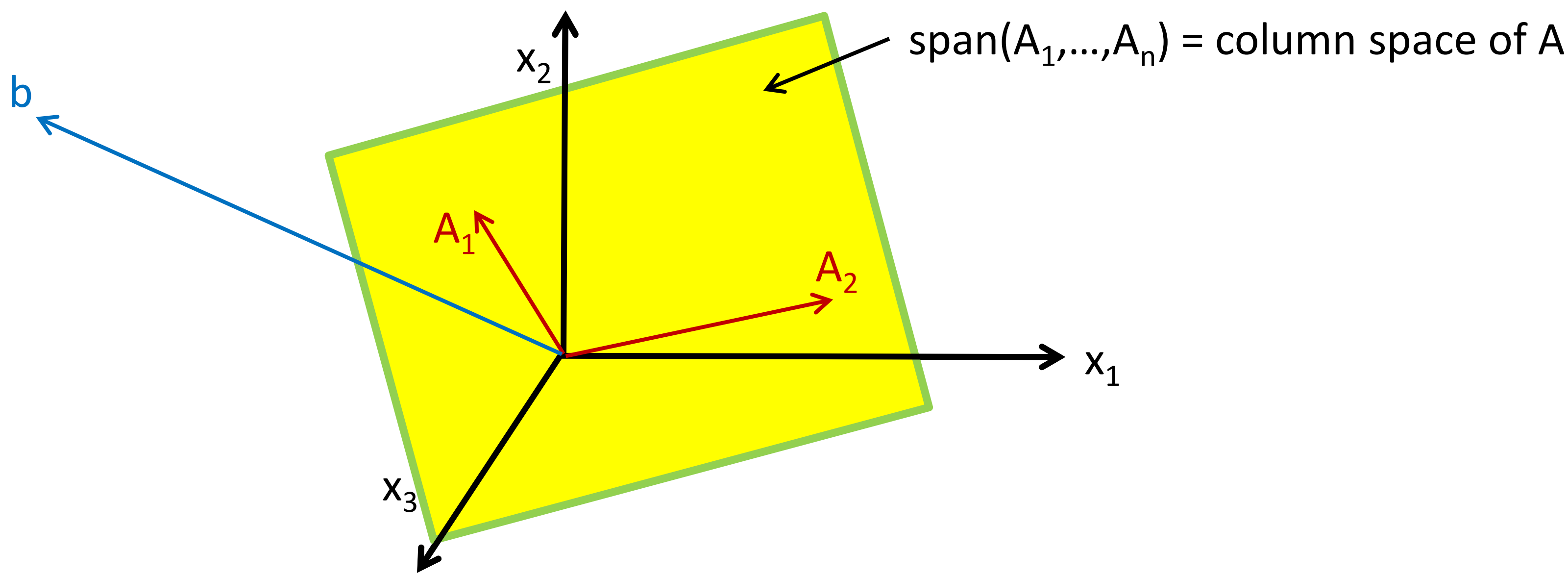
Systems of **Eq**ualities

- **Lemma:** Exactly one of the following holds:
  - There exists x satisfying $Ax=b$ (b is in column space of A)
  - There exists y satisfying $y^TA=0$ and $y^Tb<0$
- **Geometrically…**



Systems of **Eq**ualities

- **Lemma:** Exactly one of the following holds:
  - There exists x satisfying Ax=b (b is in column space of A)
  - There exists y satisfying $y^T A=0$ and $y^T b<0$ (or it is not)
- **Geometrically…**

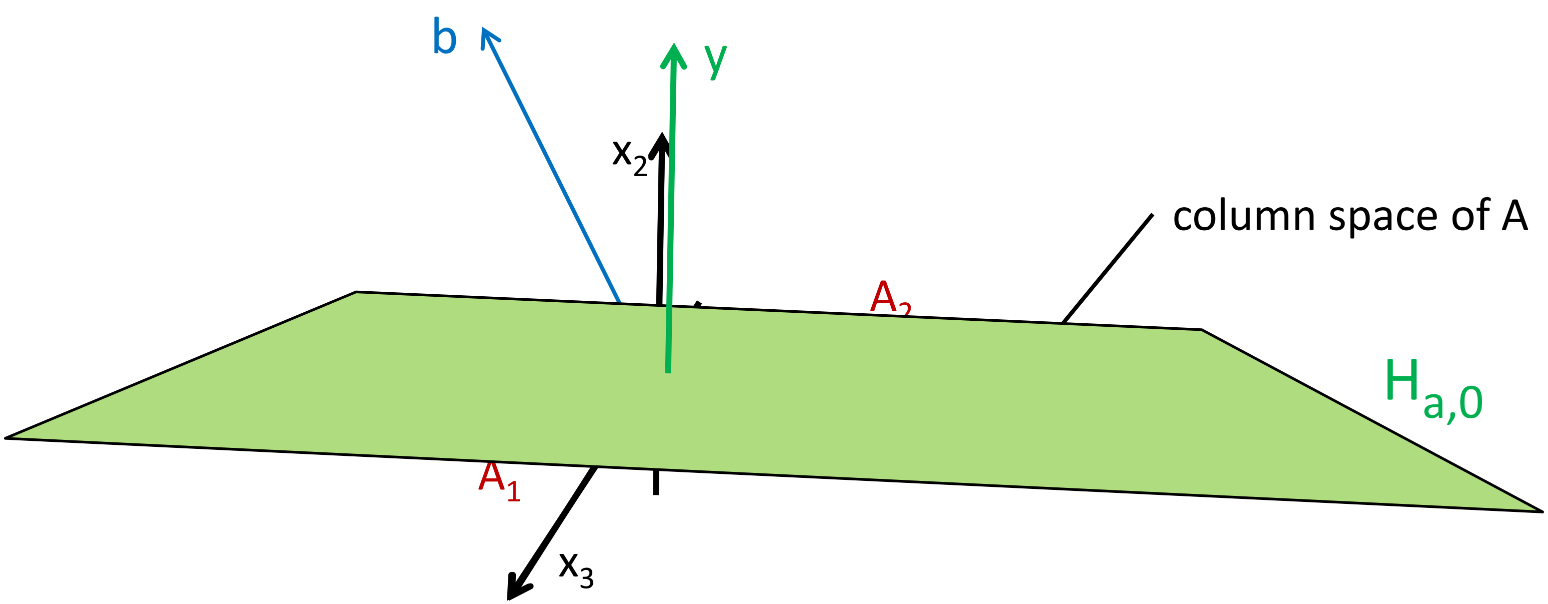


Systems of **Eq**ualities

- **Lemma:** Exactly one of the following holds:
  - There exists x satisfying $Ax=b$ (b is in column space of A)
  - There exists y satisfying $y^TA=0$ and $y^Tb>0$ (or it is not)
- **Geometrically…** col-space(A)$\subseteq H_{y,0}$ but $b \in H_{y,0}^{++}$

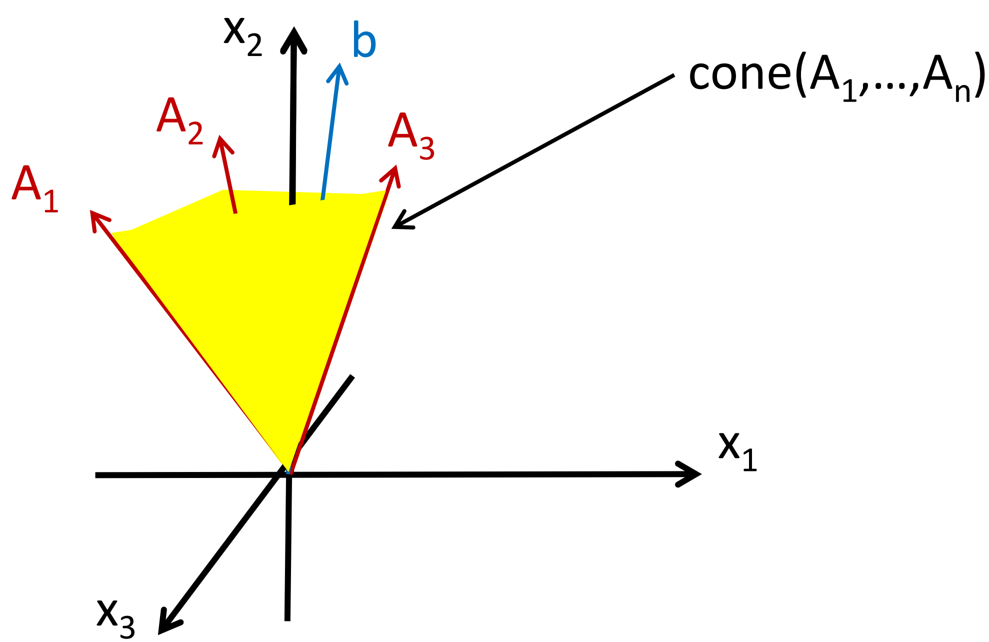
Hyperplane $H_{a,b} = \{ x \in \mathbb{R}^n : a^\mathsf{T} x = b \}$

Positive open halfspace $H_{a,b}^{++} = \{ x \in \mathbb{R}^n : a^\mathsf{T} x > b \}$



Systems of **Ineq**ualities

- **Lemma:** Exactly one of the following holds:
  - There exists $x \geq 0$ satisfying $Ax=b$ (b is in cone($A_1,\ldots,A_n$))
  - There exists y satisfying $y^T A \geq 0$ and $y^T b < 0$
- **Geometrically…**
  Let cone($A_1,\ldots,A_n$) = $\{ \Sigma_i \lambda_i A_i : \lambda \geq 0 \}$ "cone generated by $A_1,\ldots,A_n$"

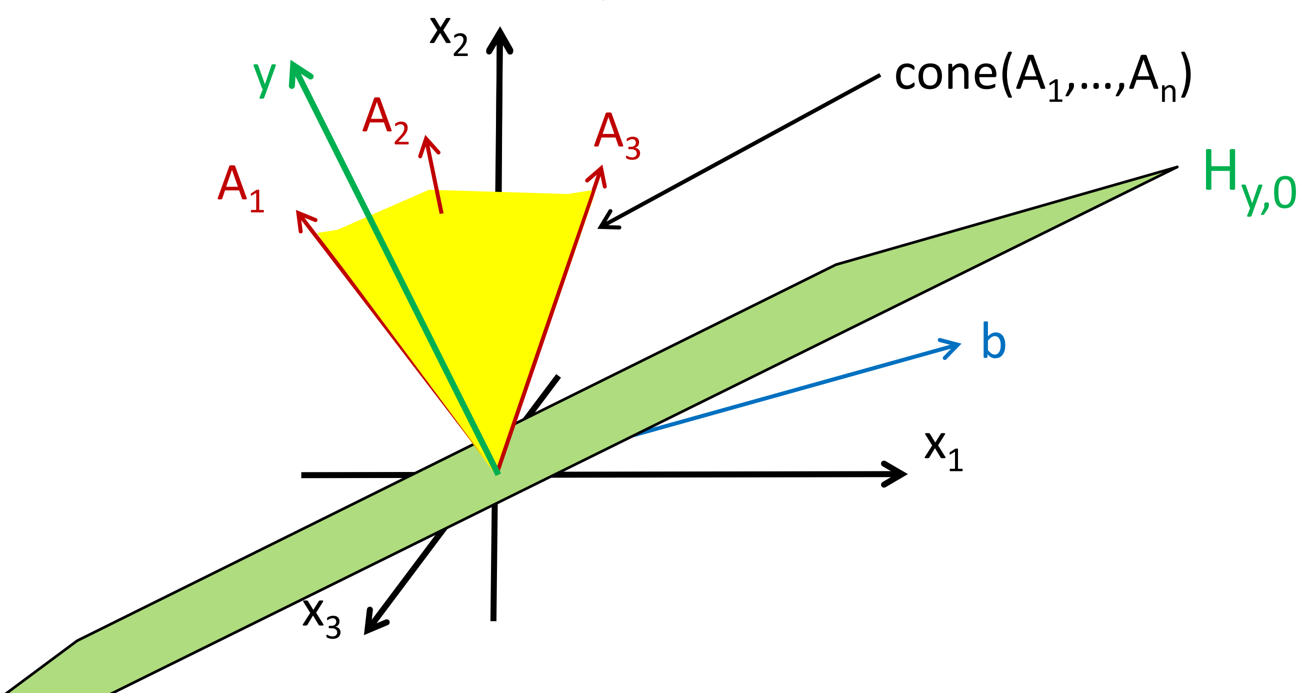


Systems of **Inequalities**

- **Lemma:** Exactly one of the following holds:
  - There exists $x \geq 0$ satisfying $Ax=b$ (b is in cone($A_1,\ldots,A_n$))
  - There exists y satisfying $y^T A \geq 0$ and $y^T b < 0$ (y gives a "separating hyperplane")
- **Geometrically…** $\text{cone}(A_1,\ldots,A_n) \in H^{+}_{y,0}$ but $b \in H^{--}_{y,0}$

Positive closed halfspace $H^{+}_{a,b} = \{\, x \in \mathbb{R}^n \,:\, a^{\mathsf{T}} x \geq b \,\}$

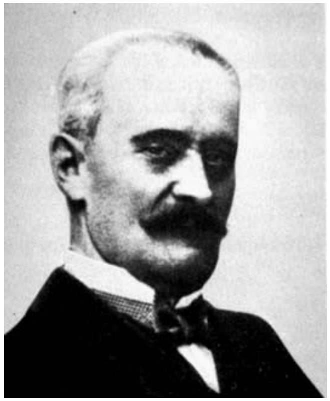
Negative open halfspace $H^{--}_{a,b} = \{\, x \in \mathbb{R}^n \,:\, a^{\mathsf{T}} x < b \,\}$



Systems of **Inequalities**

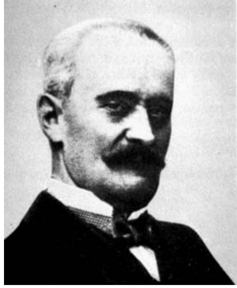
- **Lemma:** Exactly one of the following holds:
  - There exists $x \geq 0$ satisfying $Ax=b$ (b is in cone($A_1,\ldots,A_n$))
  - There exists y satisfying $y^T A \geq 0$ and $y^T b<0$ (y gives a "separating hyperplane")

- This is called **"Farkas' Lemma"**
  - It has many interesting proofs. (see Ch 6 of text)
  - It is "equivalent" to strong duality for LP.
  - There are several "equivalent" versions of it.



Gyula Farkas

Variants of Farkas' Lemma



Gyula Farkas

The System	$Ax \leq b$	$Ax = b$
has **no** solution $x \geq \mathbf{0}$ iff	$\exists y \geq 0$, $A^T y \geq \mathbf{0}$, $b^T y < 0$	$\exists y \in \mathbb{R}^{\mathbf{n}}$, $A^T y \geq \mathbf{0}$, $b^T y < 0$
has **no** solution $x \in \mathbb{R}^{\mathbf{n}}$ iff	$\exists y \geq 0$, $A^T y = \mathbf{0}$, $b^T y < 0$	$\exists y \in \mathbb{R}^{\mathbf{n}}$, $A^T y = \mathbf{0}$, $b^T y < 0$

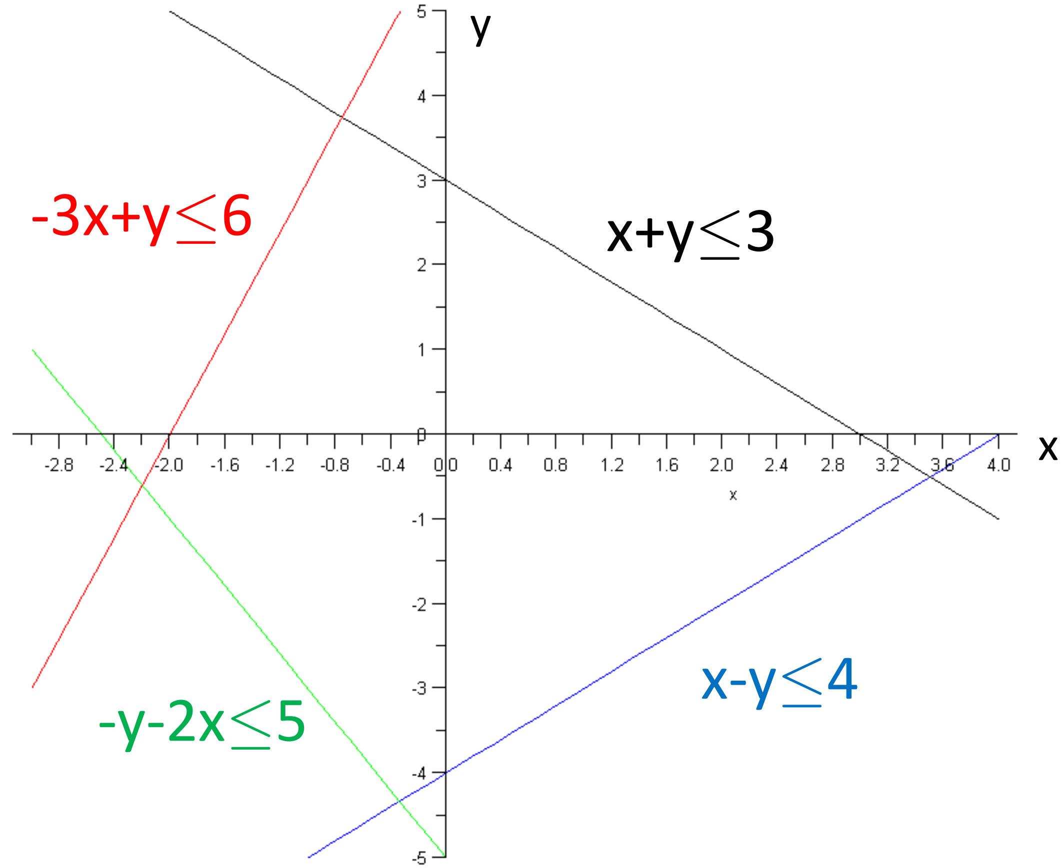
These are all "equivalent"

(each can be proved using another)

This is the simple lemma on systems of **eq**ualities

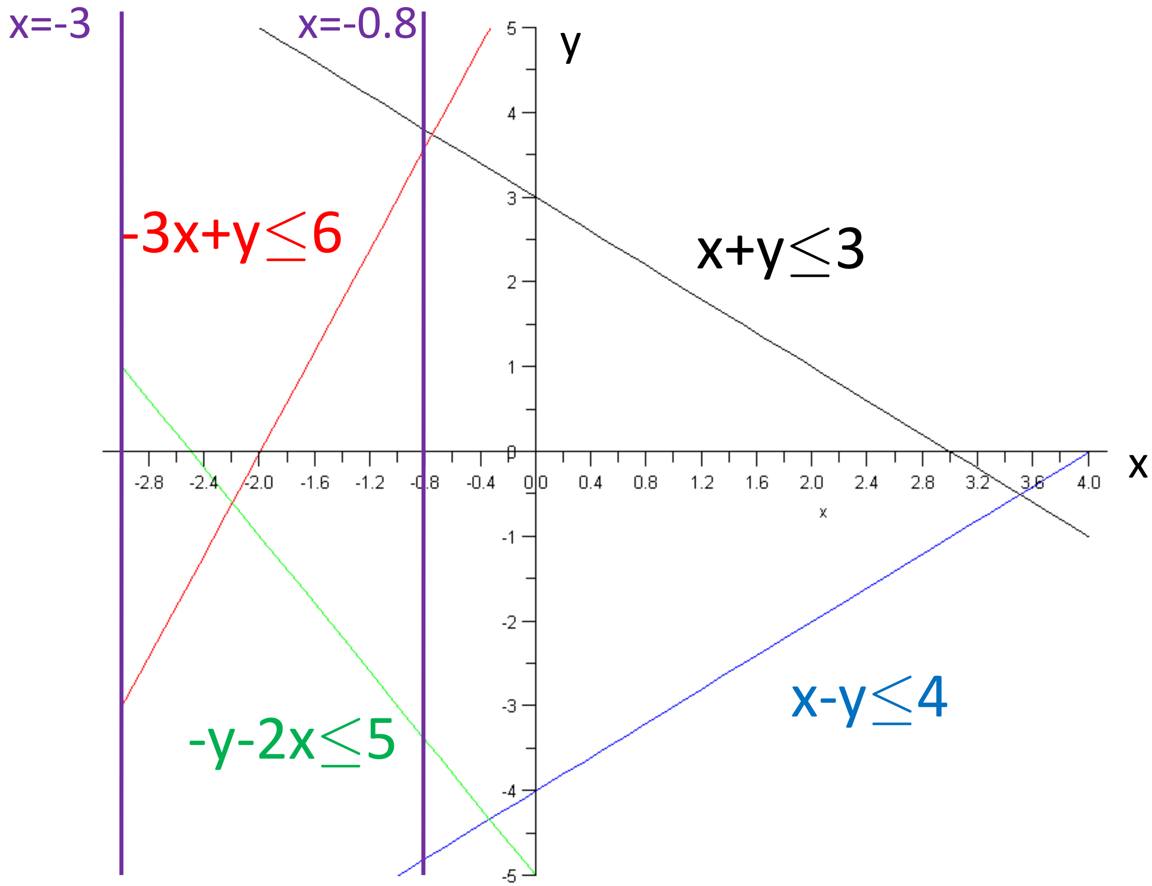
2D System of Inequalities

Consider the polyhedron
$Q = \{ (x,y) : -3x+y \leq 6,$
$x+y \leq 3,$
$-y-2x \leq 5,$
$x-y \leq 4 \}$



- **Given x, for what values of y is (x,y) feasible?**
  - Need: $y \leq x+6$, $y \leq -x+3$, $y \geq -2x-5$, and $y \geq x-4$

2D System of Inequalities



Consider the polyhedron

$Q = \{ (x,y) : -3x+y \leq 6,$
$x+y \leq 3,$
$-y-2x \leq 5,$
$x-y \leq 4 \}$

- **Given x, for what values of y is (x,y) feasible?**
  - i.e., $y \leq \min\{3x+6, -x+3\}$ and $y \geq \max\{ -2x-5, x-4 \}$
  - For x=-0.8, (x,y) feasible if $y \leq \min\{\mathbf{3.6}, 3.8\}$ and $y \geq \max\{\mathbf{-3.4}, -4.8\}$
  - For x=-3, (x,y) feasible if $y \leq \min\{\mathbf{-3}, 6\}$ and $y \geq \max\{\mathbf{1}, -7\}$ **Impossible!**

2D System of Inequalities

- Consider the polyhedron
  $Q = \{ (x,y) : -3x+y \leq 6,\ x+y \leq 3,\ -y-2x \leq 5,\ x-y \leq 4 \}$
- Given x, for what values of y is (x,y) feasible?
  - i.e., $y \leq \min\{3x+6, -x+3\}$ and $y \geq \max\{ -2x-5, x-4 \}$
  - Such a y exists $\Leftrightarrow$ $\max\{-2x-5, x-4\} \leq \min\{3x+6, -x+3\}$

    $\Leftrightarrow$ the following inequalities are solvable

Every "lower" constraint is $\leq$ every "upper" constraint

$$\begin{array}{r} -2x-5 \leq 3x+6 \\ x-4 \leq 3x+6 \\ -2x-5 \leq -x+3 \\ x-4 \leq -x+3 \end{array} \equiv Q' = \left\{ x : \begin{array}{c} -5x \leq 11 \\ -2x \leq 10 \\ -x \leq 8 \\ 2x \leq 7 \end{array} \right\} = \left\{ x : \begin{array}{l} x \geq -11/5 \\ x \geq -5 \\ x \geq -8 \\ x \leq 7/2 \end{array} \right\}$$

- **Conclusion:** Q is non-empty $\Leftrightarrow$ Q' is non-empty.
- This is easy to decide because Q' involves only 1 variable!

Joseph Fourier

Fourier-Motzkin Elimination

Theodore Motzkin

- **Generalization:** given a polyhedron $Q = \{ (x_1,\ldots,x_n) : Ax \leq b \}$, we want to find polyhedron $Q' = \{ (x_1,\ldots,x_{n-1}) : A'x' \leq b' \}$ satisfying

$$(x_1,\ldots,x_{n-1}) \in Q' \quad \Leftrightarrow \quad \exists x_n \text{ s.t. } (x_1,\ldots,x_{n-1},x_n) \in Q$$

- Q' is called a **projection** of Q
- Fourier-Motzkin Elimination is a procedure for producing Q' from Q
- **Consequences:**
  - An (inefficient!) algorithm for solving systems of inequalities, and hence for solving LPs too
  - A way of proving Farkas' Lemma by induction

- **Lemma:** Let $Q = \{ (x_1,\ldots,x_n) : Ax \leq b \}$. There is a polyhedron $Q' = \{ (x_1,\ldots,x_{n-1}) : A'x' \leq b' \}$ satisfying
  1) Q is non-empty $\Leftrightarrow$ Q' is non-empty
  2) Every inequality defining Q' is a non-negative linear combination of the inequalities defining Q.
- **Proof:** Divide inequalities of Q into three groups:

  $Z = \{ i : a_{i,n} = 0 \}$ $\quad$ $P = \{ j : a_{j,n} > 0 \}$ $\quad$ $N = \{ k : a_{k,n} < 0 \}$
- WLOG, $a_{j,n} = 1 \ \forall j \in P$ and $a_{k,n} = -1 \ \forall k \in N$
- For any $x \in \mathbb{R}^n$, let $x' \in \mathbb{R}^{n-1}$ be vector obtained by deleting coordinate x_n
- The constraints defining Q' are:
  - $a_i'x' \leq b_i \quad \forall i \in Z$
  - $a_j'x' + a_k'x' \leq b_j + b_k \quad \forall j \in P, \forall k \in N$

This says every "lower" constraint is $\leq$ every "upper" constraint

This is sum of j^{th} and k^{th} constraints of Q, because n^{th} coordinate of $a_j + a_k$ is zero!

- This proves (2).
- (2) implies: every $x \in Q$ satisfies all inequalities defining Q'.
  So Q is non-empty $\Rightarrow$ Q' is non-empty.

- **Lemma:** Let $Q = \{ (x_1,\ldots,x_n) : Ax \leq b \}$. There is a polyhedron $Q' = \{ (x_1,\ldots,x_{n-1}) : A'x' \leq b' \}$ satisfying
  1) Q is non-empty $\Leftrightarrow$ Q' is non-empty
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- **Proof:** Divide inequalities of Q into three groups:

  $Z=\{ i : a_{i,n}=0 \}$ $\quad P=\{ j : a_{j,n}=1 \}$ $\quad N=\{ k : a_{k,n}=-1 \}$
- The constraints defining Q' are:
  - $a_i'x' \leq b_i \quad \forall i \in Z$
  - $a_j'x'+a_k'x' \leq b_j+b_k \quad \forall j \in P, \forall k \in N$
- **Claim:** Let $x' \in Q'$. There exists x_n such that $(x'_1,\ldots,x'_{n-1},x_n) \in Q$.
- **Proof:** Note that $a_k'x'-b_k \leq b_j-a_j'x' \quad \forall j \in P, \forall k \in N$.

$$\Rightarrow \underbrace{\max_{k \in N} \{ a_k'x'-b_k \}} \leq \min_{j \in P} \{ b_j-a_j'x' \}$$

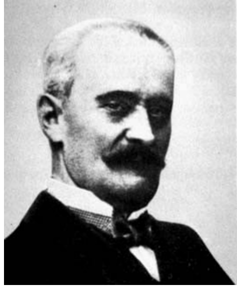
Let x_n be this value, and let $x = (x'_1,\ldots,x'_{n-1},x_n)$.

Then:

$$\left.\begin{aligned} a_kx-b_k &= a_k'x'-x_n-b_k \leq 0 \quad \forall k \in N \\ b_j-a_jx &= b_j-a_j'x'-x_n \geq 0 \quad \forall j \in P \\ a_ix &= a_i'x' \leq b_i \quad \forall i \in Z \end{aligned}\right\} \Rightarrow x \in Q$$

$\blacksquare$

Variants of Farkas' Lemma



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We'll prove this one

- **Lemma:** Exactly one of the following holds:
  - There exists $x \in \mathbb{R}^n$ satisfying $Ax \leq b$
  - There exists $y \geq 0$ satisfying $y^T A = 0$ and $y^T b < 0$
- **Proof:** By induction. Trivial for n=0, so let $n \geq 1$.

  Suppose no solution x exists. Use Fourier-Motzkin Elimination.

  Get an **equivalent** system $A'x' \leq b'$ where

$$(A' | 0) = MA \qquad b' = Mb$$

  for some **non-negative matrix** M.

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  Get an equivalent system $A'x' \leq b'$ where

$$(A' \mid 0) = MA \qquad b' = Mb$$

  for some non-negative matrix M.

  $A'x' \leq b'$ not solvable $\Rightarrow$ $\exists y'$ s.t. $y'^T A' = 0$ and $y'^T b' < 0$.

  Let $y = M^T y'$.

  Then: $y \geq 0$, because $y' \geq 0$ and M non-negative

$$y^T A = y'^T MA = y'^T (A' \mid 0) = 0$$

$$y^T b = y'^T Mb = y'^T b' < 0 \qquad \blacksquare$$