# C\&O 355 Lecture 7 

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## Outline

- Finding a starting point
- Two small issues
- Duality
- Geometric view
- Algebraic view
- Dual LP \& Weak Duality
- Primal vs Dual
- Strong Duality Theorem
- Certificates

```
Local-Search Algorithm
Let B be a feasible basis
(If none, Halt: LP is infeasible)
For each entering coordinate k}\not\in\textrm{B
    If "benefit" of coordinate k is > 0
        Compute y( }\delta\mathrm{ )
                                    (If }\delta=\infty, Halt: LP is unbounded
Find leaving coordinate }\textrm{h}\in\textrm{B
                                    (y (\delta) h=0)
Set }x=y(\delta)\mathrm{ and B'=B\{h}Ч{k}
Halt: return x
```

Nhat is a corner point?
Nhat if there are no corner points?
Nhat are the "neighboring" bases?
Nhat if no neighbors are strictly better?
Might move to a basis that isn't strictly better (if $\delta=0$ ), but whenever x changes it's strictly better)
ᄃ How can I find a starting feasible basis?
Joes the algorithm terminate?
(If Bland's rule used)
Joes it produce the right answer?

## Finding a starting point

- Consider LP max $\left\{c^{\top} x: x \in P\right\}$ where $P=\{x: A x=b, x \geq 0\}$
- How can we find a feasible point?
- Trick: Just solve a different LP!
- Note: c is irrelevant. We can introduce a new objective function
- WLOG, $b \geq 0$
(Can multiply constraints by -1)
- Allow "Ax=b" constraint to be violated via "artificial variables":

$$
Q=\{(x, y): A x+y=b, x \geq 0, y \geq 0\}
$$

- Note: $(x, 0) \in Q \Leftrightarrow x \in P$. Can we find such a point?
- Solve the new LP min $\left\{\Sigma_{i} y_{i}:(x, y) \in Q\right\}$
- If the optimal value is 0 , then $x \in P$. If not, $P$ is empty!
- How do we find feasible point for the new LP?
- $(x, y)=(0, b)$ is a trivial solution!

```
Local-Search Algorithm
Let B be a feasible basis
(If none, Halt: LP is infeasible)
For each entering coordinate k}\not\in
    If "benefit" of coordinate k is >0
        Compute y( }\delta\mathrm{ )
                                    (If }\delta=\infty, Halt: LP is unbounded
Find leaving coordinate h\inB
                                    (y}(\delta\mp@subsup{)}{\textrm{h}}{}=0
Set x=y(\delta) and B'=B\{h}\cup{k}
Halt: return x
```

Nhat is a corner point?
Nhat if there are no corner points?
Nhat are the "neighboring" bases?
Nhat if no neighbors are strictly better?
Might move to a basis that isn't strictly better (if $\delta=0$ ), but whenever x changes it's strictly better)
łow can I find a starting feasible basis?
(Solve an easier LP)
Joes the algorithm terminate?
(If Bland's rule used)
Joes it produce the right answer?

## Two Small Issues

- Our discussion of equational form LPs assumed that A has full row rank
- If A has contradictory constraints, LP is infeasible
- If A has redundant constraints, delete them
- See textbook p41
- To start the algorithm, we need a BFS. Our trick found a feasible point, but maybe not a BFS.
- Given any feasible point $x$, it is easy to find a BFS y with $c^{\top} y \geq c^{\top} x$ (unless $L P$ is unbounded).
- Very similar to our argument that polyhedra containing no line have an extreme point.
- See textbook p47-48


## Duality

"Another key visit took place in October 1947 at the Institute for Advanced Study (IAS) where Dantzig met with John von Neumann. Dantzig recalls, "l began by explaining the formulation of the linear programming model... I described it to him as I would to an ordinary mortal. 'Get to the point,' he snapped. In less than a minute, I slapped the geometric and algebraic versions of my problem on the blackboard. He stood up and said, 'Oh that.'" Just a few years earlier von Neumann had co-authored his landmark monograph on game theory. Dantzig goes on, "for the next hour and a half he proceeded to give me a lecture on the mathematical theory of linear programs." Dantzig credited von Neumann with edifying him on Farkas' lemma and the duality theorem (of linear programming)."

## Duality: Geometric View

- If $c=[-1,1]$ then $c^{\top} x=-x_{1}+x_{2} \leq 1$
- $x$ is feasible and constraint is tight $\Rightarrow x$ is optimal



## Duality: Geometric View

- If $c=[1,6]$ then $c^{\top} x=x_{1}+6 x_{2} \leq 15$
- $x$ is feasible and constraint is tight $\Rightarrow x$ is optimal



## Duality: Geometric View

- If $\mathrm{c}=\alpha \cdot[1,6]$ and $\alpha>0$ then $\mathrm{c}^{\top} \mathrm{x}=\alpha \cdot\left(\mathrm{x}_{1}+6 \mathrm{x}_{2}\right) \leq 15 \alpha$
- x is feasible and constraint is tight $\Rightarrow \mathrm{x}$ is optimal



## Duality: Geometric View

- What if c does not align with any constraint?
- Can we "generate" a new constraint aligned with c?



## Duality: Geometric View

- Can we "generate" a new constraint aligned with c?
- One way is to "average" the tight constraints
- Maybe $u+v=c$ ?
- Then $c^{\top} x=(u+v)^{\top} x=\left(-x_{1}+x_{2}\right)+\left(x_{1}+6 x_{2}\right) \leq 1+15=16$
- x is feasible and both constraints tight $\Rightarrow \mathrm{x}$ is optimal



## Duality: Geometric View

- Can we "generate" a new constraint aligned with c?
- One way is to "average" the tight constraints
- Maybe $\alpha \mathrm{u}+\beta \mathrm{v}=\mathrm{c}$ for $\alpha, \beta>0$ ?
- Then $\mathrm{c}^{\top} \mathrm{x}=(\alpha \mathrm{u}+\beta \mathrm{v})^{\top} \mathrm{x}=\alpha\left(-\mathrm{x}_{1}+\mathrm{x}_{2}\right)+\beta\left(\mathrm{x}_{1}+6 \mathrm{x}_{2}\right) \leq \alpha+15 \beta$
- x is feasible and both constraints tight $\Rightarrow \mathrm{x}$ is optimal



## Duality: Geometric View

- Can we "generate" a new constraint aligned with c?
- One way is to "average" the tight constraints
- Definition: cone $(u, v)=\{\alpha u+\beta v: \alpha, \beta \geq 0\}$
"cone generated by u \& v"
- Conclusion: For any c in cone(u,v), $x$ is optimal



## Duality: Algebraic View

$$
\begin{array}{ll}
\max & c^{\top} x \\
\text { s.t. } & a_{i}^{\top} x \leq b_{i} \quad \forall i=1, \ldots, m
\end{array}
$$

Definition: A new constraint $a^{\top} x \leq b$ is valid if it is satisfied by all feasible points
x feasible

$$
\begin{aligned}
& \Rightarrow \mathrm{a}_{1}^{\top} \mathrm{x} \leq \mathrm{b}_{1} \text { and } \mathrm{a}_{2}^{\top} \mathrm{x} \leq \mathrm{b}_{2} \\
& \Rightarrow\left(\mathrm{a}_{1}+\mathrm{a}_{2}\right)^{\top} \mathrm{x} \leq \mathrm{b}_{1}+\mathrm{b}_{2}
\end{aligned}
$$

More generally, for any $\lambda_{1}, \ldots, \lambda_{m} \geq 0$
x feasible $\quad \Rightarrow\left(\sum_{i} \lambda_{i} \mathrm{a}_{\mathrm{i}}\right)^{\top} \mathrm{x} \leq \sum_{i} \lambda_{i} \mathrm{~b}_{\mathrm{i}}$
(new valid constraint)

## "Any non-negative linear combination of the constraints gives a new valid constraint"

To get upper bound on objective function $\mathrm{c}^{\top} \mathrm{x}$, need $\left(\Sigma_{i} \lambda_{i} \mathrm{a}_{i}\right)=\mathrm{c}$ Want best upper bound $\Rightarrow$ want to minimize $\Sigma_{i} \lambda_{i} \mathrm{~b}_{i}$

## Duality: Algebraic View

$$
\begin{array}{|lll|}
\hline \max & c^{\top} x & \\
\text { s.t. } & a_{i}^{\top} x \quad \leq b_{i} \quad \forall i=1, \ldots, m
\end{array} \quad \text { Primal LP }
$$

To get upper bound on objective function $c^{\top} x$, need $\left(\Sigma_{i} \lambda_{i} a_{i}\right)=c$ Want best upper bound $\Rightarrow$ want to minimize $\Sigma_{i} \lambda_{i} b_{i}$

## We can write this as an LP too!

$$
\begin{array}{ll}
\min & \sum_{i} \lambda_{i} b_{i} \\
\text { s.t. } & \sum_{i} \lambda_{i} a_{i}=c \quad \equiv \quad \begin{array}{ll}
\min & b^{\top} \lambda \\
\text { s.t. } & A^{\top} \lambda=c \\
& \lambda \geq 0
\end{array} \quad \begin{array}{l}
\text { Dual LP }
\end{array} \\
\hline
\end{array}
$$

Theorem: "Weak Duality Theorem"
If $x$ feasible for Primal and $\lambda$ feasible for Dual then $c^{\top} x \leq b^{\top} \lambda$.
Proof: $c^{\top} x=\left(A^{\top} \lambda\right)^{\top} x=\lambda^{\top} A x \leq \lambda^{\top} b$. $\square$

## Dual of Dual

## Dual LP

$\min b^{\top} \lambda$
s.t. $A^{\top} \lambda=c$ $\lambda \geq 0$

Primal
$\max c^{\top} x$
s.t. $\quad A x \leq b$

Let $\mathrm{x}=\mathrm{v}$-u
w is a "slack variable"

## Inequality Form


$u, v, w \geq 0$
Dual of Dual

Conclusion: "Dual of Dual is Primal!"

## Dual of Equality Form LP



| $\max$ | $c^{\top} x$ |
| :--- | :--- |
| s.t. | $A x=b$ |
|  | $x \geq 0$ |$\quad$ Primal LP

For any $\lambda \in \mathbb{R}^{m}$, x feasible $\Rightarrow \lambda^{\top} \mathrm{Ax}=\lambda^{\top} \mathrm{b} \quad$ (new valid constraint)
"Any linear combination of the constraints gives a new valid constraint"
Get upper bound on objective function $c^{\top} x$ if $\lambda^{\top} A \geq c^{\top}$
Want best upper bound $\Rightarrow$ want to minimize $\lambda^{\top} b$

$$
\begin{array}{ll}
\hline \min & b^{\top} \lambda \\
\text { s.t. } & A^{\top} \lambda \geq c
\end{array} \quad \text { Dual LP }
$$

Similar arguments show:

- "Weak Duality Theorem"

If $x$ feasible for Primal and $\lambda$ feasible for Dual then $c^{\top} x \leq b^{\top} \lambda$.

- "Dual of Dual is Primal"


## Rules for Duals

|  | Primal | Dual |
| :---: | :---: | :---: |
| Objective | max $c^{\top} x$ | $\min b^{\top} y$ |
| Variables | $\mathrm{x}_{1}, \ldots, \mathrm{x}_{\mathrm{n}}$ | $\mathrm{y}_{1}, \ldots, \mathrm{y}_{\mathrm{m}}$ |
| Constraint matrix | A | $\mathrm{A}^{\top}$ |
| Right-hand vector | b | c |
| Constraints <br> versus <br> Variables | $\mathrm{ith}^{\text {th }}$ constraint | $y_{i} \geq 0$ |
|  | $\mathrm{i}^{\text {th }}$ constraint: $\geq$ <br> $\mathrm{i}^{\text {th }}$ constraint: $=$ | $\begin{aligned} & y_{i} \leq 0 \\ & y_{i} \text { unrestricted } \end{aligned}$ |
|  | $\begin{aligned} & x_{i} \geq 0 \\ & x_{i} \leq 0 \\ & x_{i} \text { unrestricted } \end{aligned}$ | $\begin{aligned} & \mathrm{j}^{\text {th }} \text { constraint }: \geq \\ & \mathrm{j}^{\text {th }} \text { constraint }: \leq \\ & \mathrm{j}^{\text {th }} \text { constraint }: \end{aligned}$ |

## Useful Mnemonic

"Natural" bound on a variable is $\geq 0$
"Natural" constraint for a maximization problem is $\leq 0$
"Natural" constraint for a minimization problem is $\geq 0$

## Example: Bipartite Matching

(from Lecture 2)

- Given bipartite graph $\mathrm{G}=(\mathrm{V}, \mathrm{E})$
- Find a maximum size matching
- A set $M \subseteq E$ s.t. every vertex has at most one incident edge in $M$
- Write an integer program
(IP)

$$
\begin{array}{llll}
\max & \sum_{e \in E} x_{e} & & \\
\text { s.t. } & \sum_{e \text { incident to } v} x_{e} \leq 1 & \forall v \in V \\
& x_{e} & \in\{0,1\} & \forall e \in E
\end{array}
$$

- But we don't know how to solve IPs. Try an LP instead.

$$
\begin{array}{llll}
\max & \sum_{e \in E} x_{e} & & \\
\text { s.t. } & \sum_{e \text { incident to } v} x_{e} & \leq 1 & \forall v \in V  \tag{LP}\\
& x_{e} & \geq 0 & \forall e \in E
\end{array}
$$

## Example: Bipartite Matching

(from Lecture 2)

- The LP formulation is:
$\begin{array}{lllll} & \max & \sum_{e \in E} x_{e} & & \\ \text { (Primal) } & \sum_{e \text { incident to } v} x_{e} & \leq 1 & \forall v \in V \\ & \text { s.t. } & \geq 0 & \forall e \in E\end{array}$
- Using "Rules for Duals":
$\min \quad \sum_{v \in V} y_{v}$
(Dual)

$$
\begin{array}{lll}
\text { s.t. } & y_{u}+y_{v} & \geq 1 \\
& y_{v} & \geq 0
\end{array} \quad \forall v \in\{u, v\} \in E
$$

## Primal vs Dual

Fundamental Theorem of LP: For any LP, the outcome is either: Infeasible, Unbounded, Optimum Point Exists.

## Weak Duality Theorem:

If $x$ feasible for Primal and $\lambda$ feasible for Dual then $c^{\top} x \leq b^{\top} \lambda$.


## Strong Duality Theorem:

If Primal has an opt. solution $x$, then Dual has an opt. solution $\lambda$. Furthermore, optimal values are same: $c^{\top} x=b^{\top} \lambda$.

## Strong Duality

(for equational form LP)

Primal LP: s.t. $\quad A x=b$
Dual LP:

$$
\begin{array}{ll}
\min & b^{\top} \lambda \\
\text { s.t. } & A^{\top} \lambda \geq c
\end{array}
$$

Suppose LP is not infeasible and not unbounded.
Algorithm terminates with BFS x defined by basis B.
Define: $\lambda=\left(A_{B}^{-1}\right)^{\top} c_{B}$.
Claim: $\lambda$ feasible for Dual LP.
Proof: alg terminates $\Rightarrow$ benefits vector $r=c^{\top}-c_{B}^{\top} A_{B}^{-1} A \leq 0$
$\Longrightarrow \quad A^{\top}\left(A_{B}^{-1}\right)^{\top} c_{B} \geq c$. So $\lambda$ feasible. $\square \quad$ since $B$ defines $x$
Claim: $c^{\top} x=b^{\top} \lambda$.
Proof: $b^{\top} \lambda=\lambda^{\top} b=c_{B}^{\top} A_{B}^{-1} b=c_{B}^{\top} x_{B}=c^{\top} x . \square$
So $x$ and $\lambda$ are both optimal!

## Certificates

For any LP, I can convince you that optimal value is..

- $\geq k$ : by giving a primal feasible $x$ with obj. value $\geq k$.
- $\leq \mathrm{k}$ : by giving a dual feasible $\lambda$ with obj. value $\leq \mathrm{k}$.

Theorem: Such certificates always exists. (stated in Lecture 2)
Proof: Immediate from strong duality theorem.
Theorems like this are very strong and useful.
Other famous examples:

- Konig / Hall's Theorem
- Max-flow Min-cut Theorem
- Hilbert's Nullstellensatz
(Graph theory)
(Network flow theory)
(Algebraic geometry)

