C&O 355
Lecture 6

N. Harvey
http://www.math.uwaterloo.ca/~harvey/

Outline

- Proof of Optimality
- Does the algorithm terminate?
- Bland's Rule
- Corollaries

Local-Search Algorithm

Let B be a feasible basis (If none, Halt: LP is infeasible)
For each entering coordinate $k \notin B$
    If "benefit" of coordinate k is > 0
        Compute $y(\delta)$ (If $\delta=\infty$, Halt: LP is unbounded)
        Find leaving coordinate $h \in B$ ($y(\delta)_h=0$)
        Set $x=y(\delta)$ and $B'=B \setminus \{h\} \cup \{k\}$
Halt: return x

1. What is a corner point? (BFS and bases)
2. What if there are no corner points? (Infeasible)
3. What are the "neighboring" bases? (Increase one coordinate)
4. What if no neighbors are strictly better?
   Might move to a basis that isn't strictly better (if δ=0), but whenever x changes it's strictly better)
5. How can I find a starting feasible basis?
6. Does the algorithm terminate?
7. Does it produce the right answer?

The Benefits Vector

- **Recall:** Benefit of coordinate k is $c_k - c_B^T A_B^{-1} A_k$
- Let's define a benefits vector **r** to record all the benefits
- **Define:** $r = c^\mathsf{T} - c_B^\mathsf{T} A_B^{-1} A$
- **Note:** For $k \notin B$, r_k is benefit of coordinate k
  For $k \in B$, $r_k = 0$
- **Claim:** For any point z, we have $c^T z = c^T x + r_{\bar{B}}^T z_{\bar{B}}$
- **Proof:** $Az = b$

$$\implies A_B z_B + A_{\bar{B}} z_{\bar{B}} = b$$

$$\implies z_B = A_B^{-1} b - A_B^{-1} A_{\bar{B}} z_{\bar{B}}$$

$$\begin{aligned} c^\mathsf{T} z &= c_B^\mathsf{T} z_B + c_{\bar{B}}^\mathsf{T} z_{\bar{B}} \\ &= c_B^\mathsf{T} \left(A_B^{-1} b - A_B^{-1} A_{\bar{B}} z_{\bar{B}} \right) + c_{\bar{B}}^\mathsf{T} z_{\bar{B}} \\ &= c_B^\mathsf{T} x_B + \left(c_{\bar{B}}^\mathsf{T} - c_B^\mathsf{T} A_B^{-1} A_{\bar{B}} \right) z_{\bar{B}} \\ &= c^\mathsf{T} x + r_{\bar{B}} z_{\bar{B}} \end{aligned}$$

Quick Optimality Proof

- Suppose algorithm terminates with BFS x and basis B
- **Claim:** x is optimal.
- **Proof:** Loop terminates $\Rightarrow$ every $k \notin B$ has benefit ≤ 0 $\Rightarrow r \leq 0$
- For any feasible point z, we have:

$$c^T z = c^T x + r_{\overline{B}}^T z_{\overline{B}} \leq c^T x \quad \Rightarrow \quad x \text{ is optimal.} \quad \blacksquare$$

≤ 0 since loop terminated

≥ 0 since z feasible

Local-Search Algorithm

Let B be a feasible basis (If none, Halt: LP is infeasible)

For each entering coordinate $k \notin B$

  If "benefit" of coordinate k is > 0

    Compute $y(\delta)$ (If $\delta=\infty$, Halt: LP is unbounded)

    Find leaving coordinate $h \in B$ ($y(\delta)_h=0$)

    Set $x=y(\delta)$ and $B'=B \setminus \{h\} \cup \{k\}$

Halt: return x

What is a corner point? (BFS and bases)

What if there are no corner points? (Infeasible)

What are the "neighboring" bases? (Increase one coordinate)

What if no neighbors are strictly better?

Might move to a basis that isn't strictly better (if δ=0), but whenever x changes it's strictly better)

5. How can I find a starting feasible basis?

6. Does the algorithm terminate?

Does it produce the right answer? (Yes)

Does Algorithm Terminate?

- Depends on rule for choosing entering / leaving coordinates
- For some rules, examples exist where algorithm cycles forever [Hoffman 1951]
- How can this happen?
  - If coordinate k has benefit>0 but δ=0 then basis changes but BFS doesn't
  - So the BFS x is **degenerate** (it is defined by multiple bases)
  - Geometrically, there are **>m** tight constraints at x
  - "Unlikely coincidences" exist among the constraints
- Oldest rule to avoid cycling: "Perturbation method" [Dantzig, Orden, Wolfe 1954]
  - **Basic idea:** Add random noise to vector b. Then no unlikely coincidences will occur in constraints. If noise very small, optimal basis of noisy problem is also an optimal basis of original problem.
- Amazingly simple rule: "Smallest index rule" [Bland 1977]
  - Choose entering coordinate k to be smallest
    i.e., k = min{ i : coordinate i has positive benefit }
  - Choose leaving coordinate h to be the smallest
    i.e., h = min{ i : coordinate i has $y(\delta)_i$=0 }

Cycling

- Suppose algorithm cycles

If it enters during cycle, it must leave during cycle

- Let F = { i : i is an entering coordinate at some step on cycle }
- **Claim 1:** All BFS on cycle are the same BFS x, and $x_i=0$ $\forall i \in F$.
- **Proof:** If BFS changes then objective value increases. (Positive benefit)
  Obj. value never decreases, so we can't cycle back to worse BFS.
  So every step on cycle has $\delta=0 \Rightarrow y(\delta)=x$.
- A leaving coordinate h always has $y(\delta)_h=0$. But $y(\delta)=x$, so $x_h=0$. □

Bland's Rule Works

- **Claim:** Bland's rule never cycles.
- **Proof Idea:**
  - Suppose it cycles
  - Let F = { i : i is an entering coordinate at some step on cycle }
  - Let v = **maximum** i in F
  - Consider steps when v is entering or leaving
  - If v is entering/leaving, then **every other i∈F is ineligible**
    (because we always choose smallest i to enter/leave)
  - We can infer some strong properties about all other i∈F
  - Use those strong properties to get a contradiction
  - How? Non-obvious…
    - Define an auxiliary LP, show that it has an optimal solution, and it is also unbounded.

- Let B be a basis on cycle **just before** v enters
  Let r be the benefits vector at this step
  $\Rightarrow$ v has a positive benefit $\boxed{(r_v>0)}$
- No other $i \in F$ is eligible to enter
  $\Rightarrow$ every other $i \in F$ has non-positive benefit $\boxed{(r_i \leq 0 \ \forall i \in F \setminus \{v\})}$

- Let B' be a basis on cycle **just before** v leaves (some $u \in F$ enters)
  Let d be the direction we're trying to move in at this step
  Recall: $y(\delta) = x+\delta d$, where $\boxed{d_{B'} = -A_{B'}^{-1} A_u, \ \ d_u=1, \ \ d_i=0 \ \forall i \notin B' \cup \{u\}}$
- **Recall:** leaving coordinate is a minimizer of min$\{ -x_i/d_i : i \text{ s.t. } d_i<0 \}$
  $\Rightarrow \boxed{d_v<0}$
- **Recall:** $x_v=0$ (by Claim 1)
- No other $i \in F$ is eligible to leave
  $\Rightarrow$ every other $i \in F$ is **not** a minimizer of this
  But $x_i=0$ too $\Rightarrow$ They are not minimizers because $\boxed{d_i \geq 0 \ \forall i \in F \setminus \{v\}}$

Auxiliary LP

(AuxLP)

$$
\begin{aligned}
\max \quad & c^{\mathsf{T}} z \\
\text{s.t.} \quad & Az \quad = b \\
& z_i \quad \begin{cases} \leq 0 & (i = v) \\ \geq 0 & (i \in F \setminus v) \\ = 0 & (i \notin B \cup F) \\ \text{unrestricted} & (i \in B \setminus F) \end{cases}
\end{aligned}
$$

- Let x be the BFS (of original LP) for all bases in the cycle
- **Claim 2:** x is an optimal solution of AuxLP.
- **Claim 3:** We can increase $x_u \rightarrow \infty$ without ruining feasibility. So AuxLP is unbounded.

Claim 2: x is optimal for AuxLP

$$
\text{(AuxLP)} \qquad
\begin{array}{lll}
\max & c^{\mathsf{T}} z & \\
\text{s.t.} & Az & = b \\
& z_i & \begin{cases} \leq 0 & (i = v) \\ \geq 0 & (i \in F \setminus v) \\ = 0 & (i \notin B \cup F) \\ \text{unrestricted} & (i \in B \setminus F) \end{cases}
\end{array}
$$

- **Subclaim:** x is feasible for AuxLP
  - $x_i=0\ \forall i \notin B$, since B is a basis determining x
  - $x_i=0\ \forall i \in F$, by Claim 1
- For any z, we have $c^T z = c^T x + r_{\overline{B}}{}^T z_{\overline{B}}$
- **Subclaim:** For z feasible for AuxLP, $r_{\overline{B}}{}^T z_{\overline{B}} \leq 0$
  - For coordinate $i \notin B \cup F$, we have $z_i=0$
  - For coordinate $i \in F \setminus \{v\}$, we have $r_i \leq 0$ but $z_i \geq 0$
  - For coordinate v, we have $r_v>0$ but $z_v \leq 0$
- So $c^T z \leq c^T x \ \Rightarrow\ $ x is optimal ■

Claim 3: AuxLP is unbounded

(AuxLP)

$$\begin{array}{lll} \max & c^{\mathsf{T}} z & \\ \text{s.t.} & Az & = b \\ & z_i & \begin{cases} \leq 0 & (i = v) \\ \geq 0 & (i \in F \setminus v) \\ = 0 & (i \notin B \cup F) \\ \text{unrestricted} & (i \in B \setminus F) \end{cases} \end{array}$$

- **Claim:** B' is a basis for AuxLP and it determines BFS x.
- **Claim:** coordinate u has positive benefit.
  (Definition of benefit does not depend on sign constraints)
- If we increase x_u to ϵ, must adjust $x_{B'}$ to preserve "Az=b" constraint
  - New point is $z(\epsilon)=x+\epsilon d$.
  - The d vector is the same as before: it does not depend on sign constraints
- $z(\epsilon)$ feasible $\Leftrightarrow x_i + \epsilon d_i \begin{cases} \leq 0 & (i = v) \\ \geq 0 & (i \in F \setminus v) \\ = 0 & (i \notin B \cup F) \end{cases}$. We know: $d_i \begin{cases} \leq 0 & (i = v) \\ \geq 0 & (i \in F \setminus v) \\ = 0 & (i \notin B' \cup \{u\}) \end{cases}$
- But $i \notin B \cup F \Rightarrow i \notin B' \cup \{u\}$. So $z(\epsilon)$ feasible $\forall \epsilon \geq 0$. ■

Local-Search Algorithm

Let B be a feasible basis (If none, Halt: LP is infeasible)

For each entering coordinate $k \notin B$

  If "benefit" of coordinate k is > 0

    Compute $y(\delta)$ (If $\delta = \infty$, Halt: LP is unbounded)

    Find leaving coordinate $h \in B$ ($y(\delta)_h = 0$)

    Set $x = y(\delta)$ and $B' = B \setminus \{h\} \cup \{k\}$

Halt: return x

- What is a corner point? (BFS and bases)
- What if there are no corner points? (Infeasible)
- What are the "neighboring" bases? (Increase one coordinate)
- What if no neighbors are strictly better?
  Might move to a basis that isn't strictly better (if δ=0), but whenever x changes it's strictly better)
- How can I find a starting feasible basis?
- Does the algorithm terminate? (If Bland's rule used)
- Does it produce the right answer? (Yes)

Corollaries

- **Corollary:** "Fundamental Theorem of LP"
  For any LP, the outcome is either:
  - An optimal solution exists
  - Problem is Infeasible
  - Problem is Unbounded

- **Corollary:** For any equational form LP, if an optimal solution exists, then a BFS is optimal.

- More coming soon…

Finding a starting basis

- Consider LP max { $c^T x$: $x \in P$ } where P={ x : Ax=b, $x \geq 0$ }
- How can we find a feasible point?
- **Trick:** Just solve a different LP!
  - Note: c is irrelevant. We can introduce a new objective function
  - WLOG, $b \geq 0$ (Can multiply constraints by -1)
  - Get "relaxation" of P by introducing more "slack variables":

    $$Q = \{ (x,y) : Ax+y=b,\ x \geq 0,\ y \geq 0 \}$$
  - Note: $(x,0) \in Q \Leftrightarrow x \in P$. Can we find such a point?
  - Solve the new LP min { $\Sigma_i\, y_i$: $(x,y) \in Q$ }
  - If the optimal value is 0, then $x \in P$. If not, P is empty!
  - How do we find feasible point for the new LP?
  - (x,y)=(0,b) is a trivial solution!

Local-Search Algorithm

Let B be a feasible basis (If none, Halt: LP is infeasible)
For each entering coordinate $k \notin B$
    If "benefit" of coordinate k is > 0
        Compute $y(\delta)$ (If $\delta=\infty$, Halt: LP is unbounded)
        Find leaving coordinate $h \in B$ ($y(\delta)_h=0$)
        Set $x=y(\delta)$ and $B'=B \setminus \{h\} \cup \{k\}$
Halt: return x

- What is a corner point? (BFS and bases)
- What if there are no corner points? (Infeasible)
- What are the "neighboring" bases? (Increase one coordinate)
- What if no neighbors are strictly better?
  Might move to a basis that isn't strictly better (if δ=0), but whenever x changes it's strictly better)
- How can I find a starting feasible basis? (Solve an easier LP)
- Does the algorithm terminate? (If Bland's rule used)
- Does it produce the right answer? (Yes)