# C&O 355
# Lecture 5

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# Outline

- Neighboring Bases
- Finding Better Neighbors
- “Benefit” of a coordinate
- Quick optimality proof
- Alternative optimality proof
  - Generalized neighbors and generalized benefit

**Local-Search Algorithm**

Let B be a feasible basis (if none, infeasible)

For each feasible basis B’ that is a neighbor of B

  Compute BFS y defined by B’

  If $c^T y > c^T x$ then set x=y

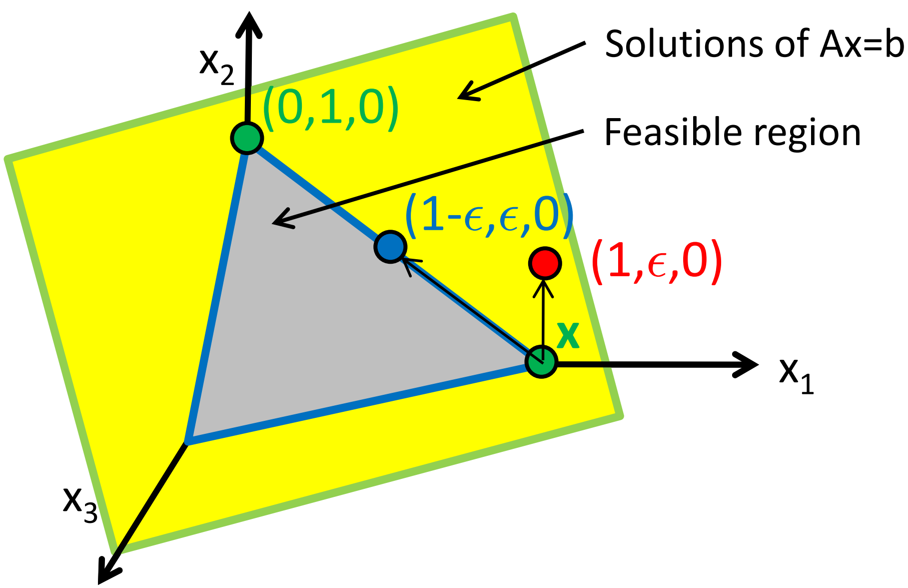
Halt

1. What is a corner point? (BFS and bases)
2. What if there are no corner points? (Infeasible)
3. What are the “neighboring” bases?
4. What if no neighbors are strictly better?
5. How can I find a starting feasible basis?
6. Does the algorithm terminate?
7. Does it produce the right answer?

# Neighboring Bases

- **Notation:** $A_k$ = $k^{th}$ column of A
- Suppose we have a feasible basis B (|B|=m, $A_B$ full rank)
  - It defines BFS x where $x_B = A_B^{-1} b \geq 0$ and $x_{\bar{B}} = 0$
- Can we find a basis "similar" to B but containing some $k \notin B$?
- Suppose we increase $x_k$ from 0 to $\epsilon$ for some $k \notin B$
  - We'll violate the constraints Ax=b unless we modify $x_B$

**Example:**




Just one constraint:

$A = [1, 1, 1]$, $b = [1]$

Feasible region:

$P = \{ x : x_1+x_2+x_3=1, x \geq 0 \}$

BFS **x**=(1,0,0), basis B={1}

Increase $x_2$ to $\epsilon$. Infeasible!

Modify $x_1$ to $1-\epsilon$. Feasible!

Increase $\epsilon$ to 1. Get BFS **y**=(0,1,0).

How did we decide this?

# Neighboring Bases

- **Notation:** $A_k$ = $k^{th}$ column of A
- Suppose we have a feasible basis B (|B|=m, $A_B$ full rank)
  - It defines BFS x where $x_B = A_B^{-1} b \geq 0$ and $x_{\bar{B}} = 0$
- Can we find a basis "similar" to B but containing some $k \notin B$?
- Suppose we increase $x_k$ from 0 to $\epsilon$ for some $k \notin B$
  - We'll violate the constraints Ax=b unless we modify $x_B$
  - Replace $x_B$ with $y_B$ satisfying $A_B y_B + \epsilon A_k = b$

$$y = [\underbrace{\quad y_B \quad}_{B}, \underset{\substack{\uparrow \\ k}}{\epsilon}, \underbrace{\quad 0 \quad}_{\overline{B \cup \{k\}}}]$$

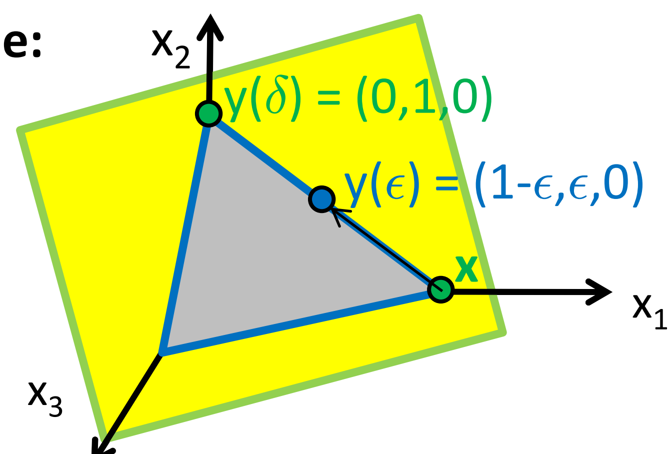
  - Given that $y_{\overline{B \cup \{k\}}} = 0$ and $y_k = \epsilon$, there is a **unique y** ensuring Ay=b

    $A_B y_B + \epsilon A_k = b \quad \Rightarrow \quad y_B = A_B^{-1}(b - \epsilon A_k) = x_B - \epsilon A_B^{-1} A_k$
  - So $y(\epsilon) = x + \epsilon d$ where: $d_B = -A_B^{-1} A_k$, $d_k = 1$, and $d_i = 0 \;\forall i \notin B \cup \{k\}$

# Neighboring Bases

- Suppose we increase $x_k$ from 0 to $\epsilon$ for some $k \notin B$
  - We'll violate the constraints $Ax=b$ unless we modify $x_B$
  - Replace $x_B$ with $y_B$ satisfying $A_B y_B + \epsilon A_k = b$
    $\Rightarrow y_B = A_B^{-1}(b - \epsilon A_k) = x_B - \epsilon A_B^{-1} A_k$
  - So $y(\epsilon) = x + \epsilon d$ where: $d_B = -A_B^{-1} A_k$, $d_k = 1$, and $d_i = 0 \ \forall i \notin B \cup \{k\}$
- $y(\epsilon)$ feasible if $y(\epsilon) \geq 0$, but not basic: it has (probably) m+1 non-zeros!
  - $y(\epsilon) \geq 0 \ \Leftrightarrow \ \forall i,\ y(\epsilon)_i = x_i + \epsilon d_i \geq 0 \ \Leftrightarrow \ \epsilon \leq \min\{ -x_i/d_i : i \text{ s.t. } d_i < 0 \}$
  - If $\min = \infty$, then feasible region is unbounded in direction d.
  - Otherwise, let h be an i minimizing min. Let $\delta = -x_h/d_h$. (Note: $h \in B$)
  - Then $y(\delta)_h = x_h + \delta d_h = 0$, so $y(\delta)$ has $\leq$m non-zeros

**Example:**




$A = [1, 1, 1]$, $b = [1]$
BFS **x**=(1,0,0), basis B={1}
$d = (-1, 1, 0)$
$\epsilon \leq -x_1/d_1 = 1$
Take $\delta$=1 and h=1

# Neighboring Bases

- Suppose we increase $x_k$ from 0 to $\epsilon$ for some $k \notin B$
  - We'll violate the constraints $Ax=b$ unless we modify $x_B$
  - Replace $x_B$ with $y_B$ satisfying $A_B y_B + \epsilon A_k = b$
    $\Rightarrow y_B = A_B^{-1}(b - \epsilon A_k) = x_B - \epsilon A_B^{-1} A_k$
  - So $y(\epsilon) = x + \epsilon d$ where: $\boxed{d_B = -A_B^{-1} A_k,}$ $d_k = 1$, and $d_i = 0 \;\forall i \notin B \cup \{k\}$
- $y(\epsilon)$ feasible if $y(\epsilon) \geq 0$, but not basic: it has (probably) m+1 non-zeros!
  - $y(\epsilon) \geq 0 \;\Leftrightarrow\; \forall i,\ y(\epsilon)_i = x_i + \epsilon d_i \geq 0 \;\Leftrightarrow\; \epsilon \leq \min\{ -x_i/d_i : i \text{ s.t. } d_i < 0 \}$
  - If $\min = \infty$, then feasible region is unbounded in direction d.
  - Otherwise, let h be an i minimizing min. Let $\delta = -x_h/d_h$. (Note: $h \in B$)
  - Then $y(\delta)_h = x_h + \delta d_h = 0$, so $y(\delta)$ has $\leq$m non-zeros
- **Claim:** Let $B' = B \setminus \{h\} \cup \{k\}$. Then B' is a basis.
- **Proof:** Suppose not. Then $A_k$ is a lin. comb. of vectors in $A_{B \setminus \{h\}}$.
  But $A_k$ is a **unique** lin. comb. of vectors in $A_B$, since B is a basis.
  So coefficient of h in this lin. comb. must be 0.
  The lin. comb. is $-d_B$, since $\boxed{-A_B d_B = A_k}$. But $-d_h \neq 0$. Contradiction! $\square$

# Neighboring Bases

- Suppose we increase $x_k$ from 0 to $\epsilon$ for some $k \notin B$
  - We'll violate the constraints $Ax=b$ unless we modify $x_B$
  - Replace $x_B$ with $y_B$ satisfying $A_B y_B + \epsilon A_k = b$
    $\Rightarrow y_B = A_B^{-1}(b - \epsilon A_k) = x_B - \epsilon A_B^{-1} A_k$
  - So $y(\epsilon) = x + \epsilon d$ where: $d_B = -A_B^{-1} A_k$, $d_k = 1$, and $d_i = 0 \ \forall i \notin B \cup \{k\}$
- $y(\epsilon)$ feasible if $y(\epsilon) \geq 0$, but not basic: it has (probably) m+1 non-zeros!
  - $y(\epsilon) \geq 0 \iff \forall i,\ y(\epsilon)_i = x_i + \epsilon d_i \geq 0 \iff \epsilon \leq \min\{ -x_i/d_i : i \text{ s.t. } d_i < 0 \}$
  - If $\min = \infty$, then feasible region is unbounded in direction d.
  - Otherwise, let h be an i minimizing min. Let $\delta = -x_h/d_h$. (Note: $h \in B$)
  - Then $y(\delta)_h = x_h + \delta d_h = 0$, so $y(\delta)$ has $\leq m$ non-zeros
- **Claim:** Let $B' = B \setminus \{h\} \cup \{k\}$. Then B' is a basis.
- **Claim:** $y(\delta)$ is a BFS determined by B'.
- **Proof:** We enforced $Ay=b$. We showed $y(\delta) \geq 0$. (y is feasible)
  We showed B' is a basis. We ensured $y_i = 0 \ \forall i \notin B'$. □

# Neighboring Bases

- Suppose we increase $x_k$ from 0 to $\epsilon$ for some $k \notin B$
  - We'll violate the constraints $Ax=b$ unless we modify $x_B$
  - Replace $x_B$ with $y_B$ satisfying $A_B y_B + \epsilon A_k = b$
    $\Rightarrow y_B = A_B^{-1}(b - \epsilon A_k) = x_B - \epsilon A_B^{-1} A_k$
  - So $y(\epsilon) = x + \epsilon d$ where: $d_B = -A_B^{-1} A_k$, $d_k = 1$, and $d_i = 0 \; \forall i \notin B \cup \{k\}$
- $y(\epsilon)$ feasible if $y(\epsilon) \geq 0$, but not basic: it has (probably) m+1 non-zeros!
  - $y(\epsilon) \geq 0 \;\; \Leftrightarrow \;\; \forall i,\ y(\epsilon)_i = x_i + \epsilon d_i \geq 0 \;\; \Leftrightarrow \;\; \epsilon \leq \min\{ -x_i/d_i : i \text{ s.t. } d_i < 0 \}$
  - If min$=\infty$, then feasible region is unbounded in direction d.
  - Otherwise, let h be an i minimizing min. Let $\delta = -x_h/d_h$. (Note: $h \in B$)
  - Then $y(\delta)_h = x_h + \delta d_h = 0$, so $y(\delta)$ has $\leq m$ non-zeros
- **Claim:** Let $B' = B \setminus \{h\} \cup \{k\}$. Then B' is a basis.
- **Claim:** $y(\delta)$ is a BFS determined by B'.
- B' is a **neighboring basis** of B, and $y(\delta)$ is a **neighboring BFS** of x.

(Unless $\delta=0$, in which case $y(\delta)=x$.)

# Neighboring Bases: Summary

- Suppose we have a feasible basis B (|B|=m, $A_B$ full rank)
  - It defines BFS x where $x_B=A_B^{-1}b\geq 0$ and $x_B=0$
- Pick a coordinate $k\notin B$
- Compute $y(\delta)$
  - $y(\epsilon)=x+\epsilon d$ where: $d_B=-A_B^{-1}A_k$, $d_k=1$, and $d_i=0\ \forall i\notin B\cup\{k\}$
  - $\delta = \max\{\ \epsilon : y(\epsilon)$ feasible $\}$
- If $\delta<\infty$ then $y(\delta)$ is a BFS (This is completely determined by B and k)
- Pick any $h\in B$ with $y(\delta)_h=0$ (Might be several possible h)
- $y(\delta)$ is determined by $B'=B\setminus\{h\}\cup\{k\}$
- $y(\delta)$ is a **neighboring BFS** of x, and B' is a **neighboring basis** of B

Unless $\delta=0 \Rightarrow y(\delta)=x$. B' is a neighboring basis but, $y(\delta)$ is the **same BFS**.

or $\delta=\infty \Rightarrow$ feasible region unbounded in direction d.

k is called "entering coordinate"
h is called "leaving coordinate"

**Local-Search Algorithm**

Let B be a feasible basis (if none, infeasible)
For each neighboring basis B’ of B
    Compute BFS y defined by B’
    If $c^T y > c^T x$ then set x=y
Halt

What is a corner point? (BFS and bases)
What if there are no corner points? (Infeasible)
What are the “neighboring” bases? (Increase one coordinate)
4. What if no neighbors are strictly better?
5. How can I find a starting feasible basis?
6. Does the algorithm terminate?
7. Does it produce the right answer?

# Finding better neighbors

- Consider LP $\max \{ c^T x : Ax=b, x \geq 0 \}$
- We have BFS x determined by basis B
- Find a neighbor: pick $k \notin B$, compute $y(\delta)$ ($\delta = \max\{ \epsilon : y(\epsilon) \text{ feasible} \}$)
- Is $y(\delta)$ better? $\Leftrightarrow c^T y(\delta) > c^T x$

  $\Leftrightarrow c^T (x+\delta d) > c^T x \Leftrightarrow c^T d > 0$ (Assuming $\delta>0$)
- $c^T d$ is the **benefit** of increasing coordinate k (Might be negative)
- If $\delta=0$ we have **same BFS** $\Rightarrow$ same objective value ($y(\delta)=x$)
- Suppose $c^T d>0$
  - If $\delta=\infty$, then **LP is unbounded** ($y(\delta)$ feasible and $c^T y(\delta)=\infty$)
  - If $0<\delta<\infty$, then $y(\delta)$ is a **strictly better BFS** ($c^T y(\delta)>c^T x$)
- Concise expression for benefit
  - Recall: $d_B=-A_B^{-1}A_k$, $d_k=1$, and $d_i=0 \; \forall i \notin B \cup \{k\}$

  $\Rightarrow$ $\boxed{\text{Benefit of coordinate k is } c^T d = c_k - c_B^T A_B^{-1} A_k}$

## Local-Search Algorithm

Let B be a feasible basis (If none, Halt: LP is infeasible)

For each $k \notin B$

    If "benefit" of coordinate k is > 0

        Compute $y(\delta)$ (If $\delta=\infty$, Halt: LP is unbounded)

        Find leaving variable $h \in B$ ($y(\delta)_h=0$)

        Set $x=y(\delta)$ and $B'=B \setminus \{h\} \cup \{k\}$

Halt: return x

1. What is a corner point? (BFS and bases)
2. What if there are no corner points? (Infeasible)
3. What are the "neighboring" bases? (Increase one coordinate)
4. What if no neighbors are strictly better?
   Might move to a basis that isn't strictly better (if $\delta=0$), but whenever x changes it's strictly better)
5. How can I find a starting feasible basis?
6. Does the algorithm terminate?
7. Does it produce the right answer?

# Example

See http://www.math.uwaterloo.ca/~harvey/F09/Lecture5Example.pdf