# C&O 355
# Lecture 3

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# Outline

- Review Local-Search Algorithm
- Pitfall #1: Defining corner points
  - Polyhedra that don’t contain a line have corner points
- Pitfall #2: No corner points?
  - Equational form of LPs

# Local-Search Algorithm: Pitfalls & Details

**Algorithm**

Let x be any corner point

For each corner point y that is a neighbor of x

    If $c^T y > c^T x$ then set $x = y$

Halt

# Local-Search Algorithm: Pitfalls & Details

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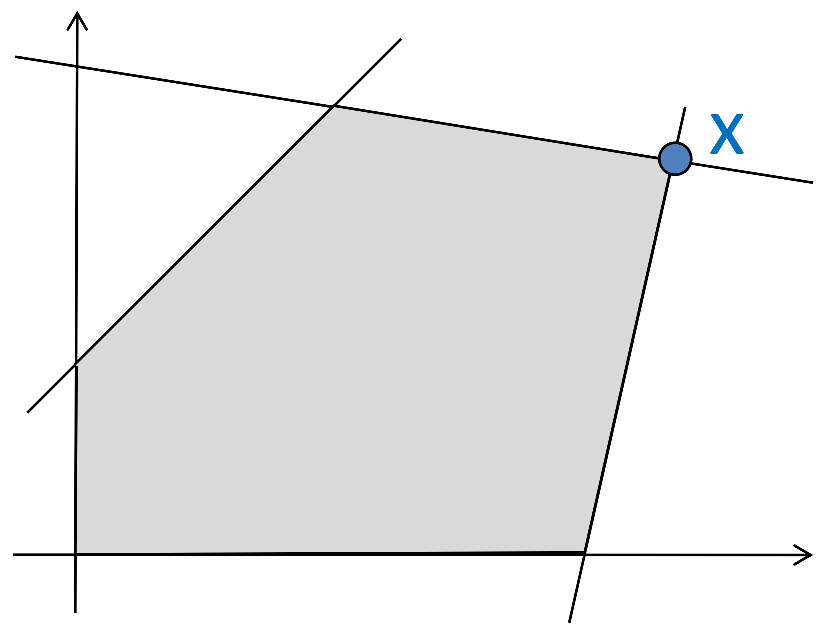
1. What is a corner point?
2. What if there are no corner points?
3. What are the "neighboring" corner points?
4. How to choose a neighboring point?
5. How can I find a starting corner point?
6. Does the algorithm terminate?
7. Does it produce the right answer?

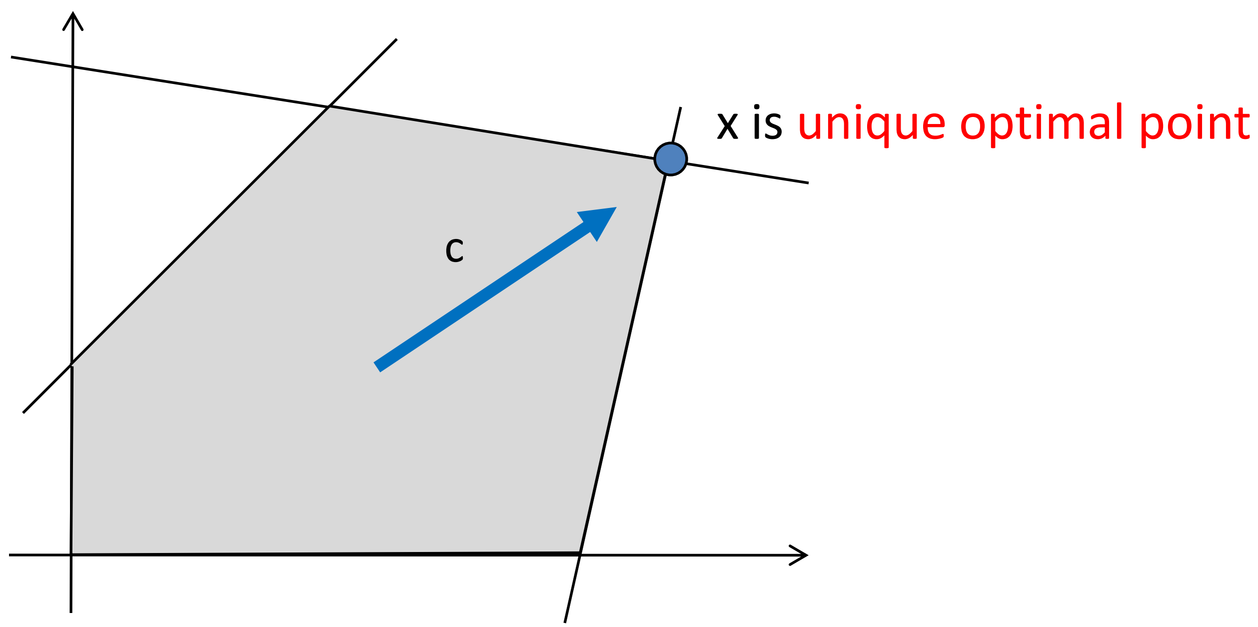
# Pitfall #1: What is a corner point?

- How should we define corner points?
- Under any reasonable definition, point x should be considered a corner point



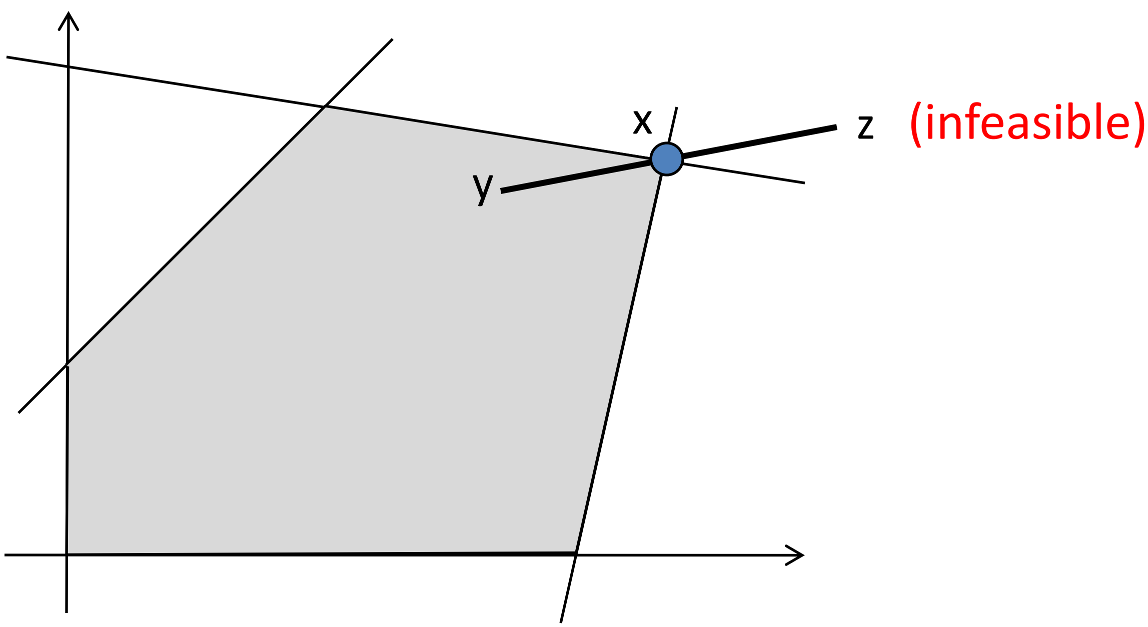
# Pitfall #1: What is a corner point?

- Attempt #1: "x is the 'farthest point' in some direction"
- Let P = { feasible region }
- There exists $c \in \mathbb{R}^n$ s.t. $c^T x > c^T y$ for all $y \in P \setminus \{x\}$
- "For some objective function, x is the unique optimal point when maximizing over P"
- Such a point x is called a "**vertex**"



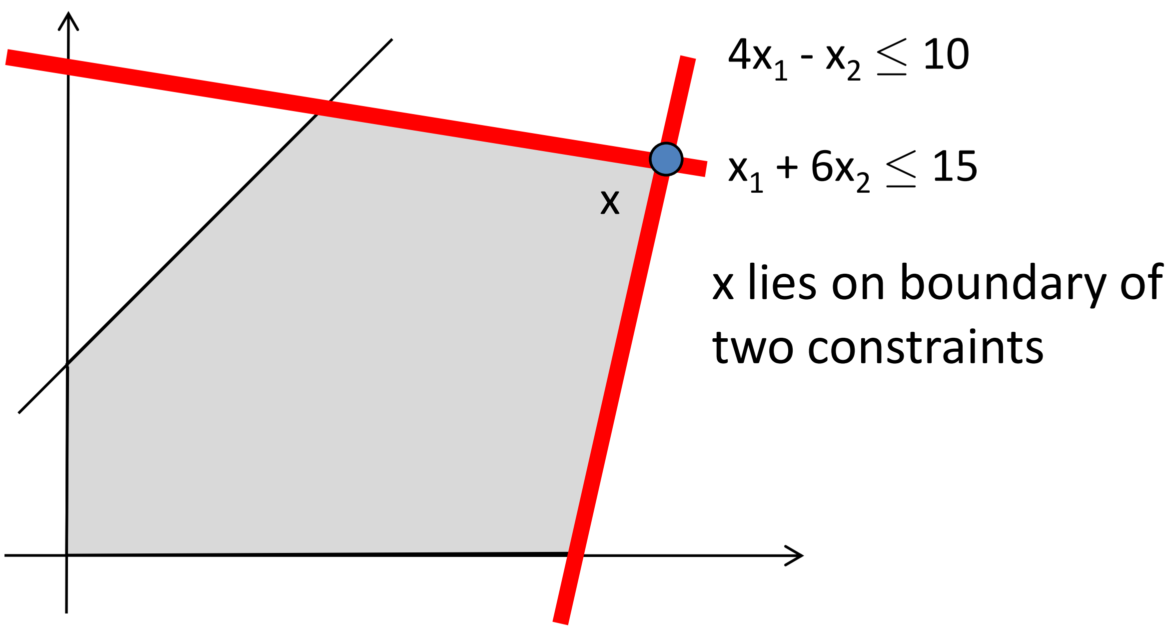
# Pitfall #1: What is a corner point?

- Attempt #2: "There is no feasible line-segment that goes through x in both directions"
- Whenever $x=\alpha y+(1-\alpha)z$ with $y,z\neq x$ and $\alpha\in(0,1)$, then either y or z must be infeasible.
- "If you write x as a convex combination of two feasible points y and z, the only possibility is x=y=z"
- Such a point x is called an "**extreme point**"



# Pitfall #1: What is a corner point?

- Attempt #3: "x lies on the boundary of many constraints"
- Note: This discussion differs from textbook



# Pitfall #1: What is a corner point?

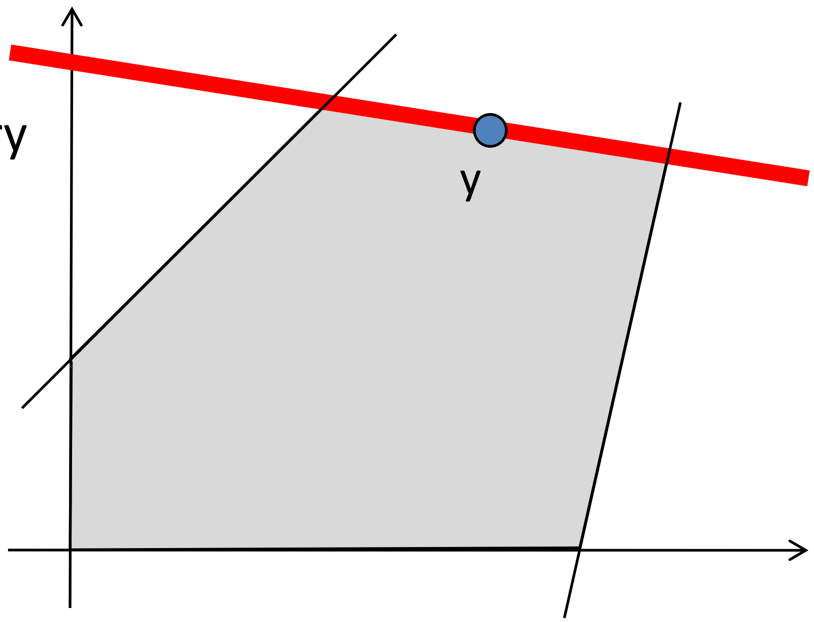
- Attempt #3: "x lies on the boundary of many constraints"
- Note: This discussion differs from textbook
- What if I introduce **redundant** constraints?

**Not the right condition**

$x_1 + 6x_2 \leq 15$

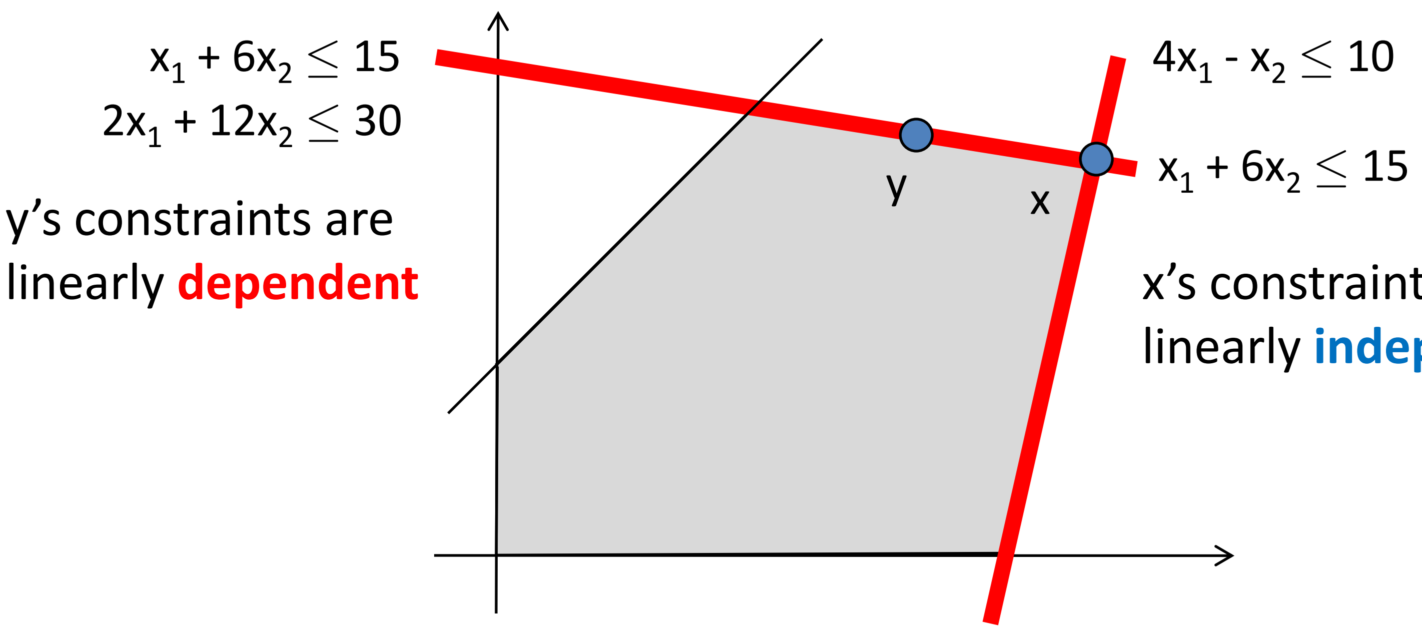
$2x_1 + 12x_2 \leq 30$

y also lies on boundary of two constraints



# Pitfall #1: What is a corner point?

- Revised Attempt #3: "x lies on the boundary of many **linearly independent** constraints"
- Feasible region: $P = \{ x : a_i^T x \leq b_i \ \forall\, i \} \subset \mathbb{R}^n$
- Let $\mathcal{I}_x = \{ i : a_i^T x = b_i \}$ and $\mathcal{A}_x = \{ a_i : i \in \mathcal{I}_x \}$. **("Tight constraints")**
- x is a **"basic feasible solution (BFS)"** if rank $\mathcal{A}_x = n$



**Lemma**: Let P be a polyhedron. The following are equivalent.

i. x is a vertex
ii. x is an extreme point
iii. x is a basic feasible solution (BFS)

**Lemma**: Let P be a polyhedron. The following are equivalent.

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**Proof** of (i)$\Rightarrow$(ii):

x is a vertex $\Rightarrow$ $\exists$ c s.t. x is unique maximizer of $c^T x$ over P

Suppose $x = \alpha y + (1-\alpha) z$ where $y,z \in P$ and $\alpha \in (0,1)$.

Suppose $y \neq x$. Then

$$c^T x = \alpha \underbrace{c^T y}_{< c^T x} + (1-\alpha) \underbrace{c^T z}_{\leq c^T x}$$

(since $c^T x$ is optimal value)

(since x is **unique** optimizer)

$$\Rightarrow \quad c^T x < \alpha\, c^T x + (1-\alpha)\, c^T x = c^T x \qquad \textbf{Contradiction!}$$

So y=x. Symmetrically, z=x.

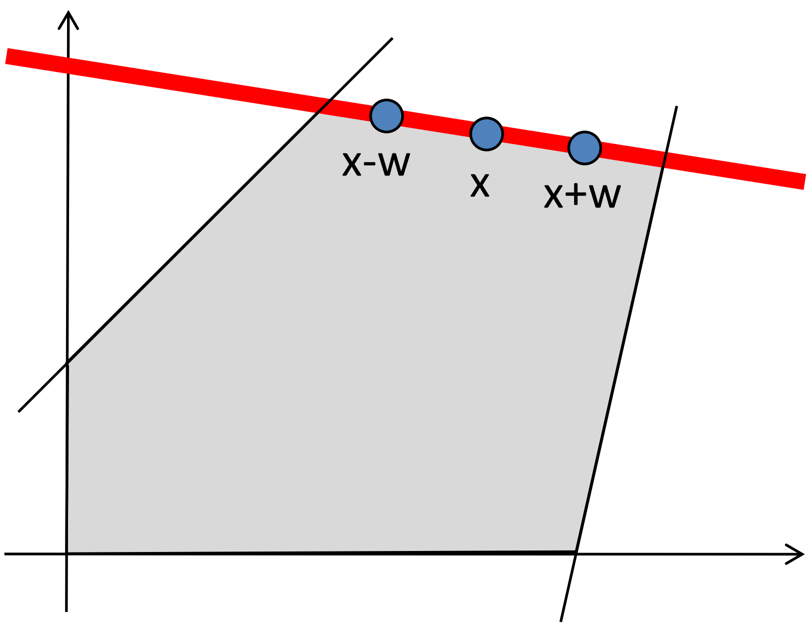
So x is an extreme point of P. ■

**Lemma**: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$. The following are equivalent.

i. x is a vertex
ii. x is an extreme point
iii. x is a basic feasible solution (BFS)

**Proof Idea** of (ii)$\Rightarrow$(iii):

x **not** a BFS $\Rightarrow$ rank $\mathcal{A}_x \leq$ n-1




- Each tight constraint removes one degree of freedom
- At least one degree of freedom remains
- So x can "wiggle" while staying on all the tight constraints
- Then x is a convex combination of two points obtained by "wiggling".
- So x is not an extreme point.

**Lemma**: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$. The following are equivalent.

i. x is a vertex
ii. x is an extreme point
iii. x is a basic feasible solution (BFS)

**Proof** of (ii)$\Rightarrow$(iii): We'll show contrapositive.

x **not** a BFS $\Rightarrow$ rank $\mathcal{A}_x<n$ (Recall $\mathcal{A}_x = \{ a_i : a_i^T x=b_i \}$)

**Claim:** $\exists w \in \mathbb{R}^n$, $w \neq 0$, s.t. $a_i^T w=0 \ \forall a_i \in \mathcal{A}_x$ (w orthogonal to all of $\mathcal{A}_x$)

**Proof:** Let M be matrix whose rows are the $a_i$'s in $\mathcal{A}_x$.

dim row-space(M) + dim null-space(M) = n

But dim row-space(M)<n $\Rightarrow \exists w \neq 0$ in the null space. ☐

**Lemma**: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$. The following are equivalent.

i. x is a vertex

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**Claim:** $\exists w \in \mathbb{R}^n$, $w \neq 0$, s.t. $a_i^T w=0 \ \forall a_i \in \mathcal{A}_x$ (w orthogonal to all of $\mathcal{A}_x$)

Let $y=x+\epsilon w$ and $z=x-\epsilon w$, where $\epsilon>0$.

**Claim:** If $\epsilon$ very small then $y,z \in P$.

**Proof:** First consider tight constraints at x. (i.e., those in $\mathcal{I}_x$)

$$a_i^T y = a_i^T x + \epsilon a_i^T w = b_i + 0$$

So y satisfies this constraint. Similarly for z.

Next consider the loose constraints at x. (i.e., those not in $\mathcal{I}_x$)

$$b_i - a_i^T y = \underbrace{b_i - a_i^T x}_{\text{Positive}} - \underbrace{\epsilon a_i^T w}_{\text{As small as we like}} \geq 0$$

So y satisfies these constraints. Similarly for z. $\square$

**Lemma**: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$. The following are equivalent.

i. x is a vertex
ii. x is an extreme point
iii. x is a basic feasible solution (BFS)

**Proof** of (ii)$\Rightarrow$(iii): We'll show contrapositive.

x **not** a BFS $\Rightarrow$ rank $\mathcal{A}_x < n$ (Recall $\mathcal{A}_x = \{ a_i : a_i^T x = b_i \}$)

**Claim:** $\exists w \in \mathbb{R}^n$, $w \neq 0$, s.t. $a_i^T w = 0 \ \forall a_i \in \mathcal{A}_x$ (w orthogonal to all of $\mathcal{A}_x$)

Let $y=x+\epsilon w$ and $z=x-\epsilon w$, where $\epsilon>0$.

**Claim:** If $\epsilon$ very small then $y,z \in P$.

Then $x=\alpha y+(1-\alpha)z$, where $y,z \in P$, $y,z \neq x$, and $\alpha=1/2$.

So x is **not** an extreme point. ■

**Lemma**: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$. The following are equivalent.

i. x is a vertex
ii. x is an extreme point
iii. x is a basic feasible solution (BFS)

**Proof** of (iii)$\Rightarrow$(i):

Let x be a BFS $\Rightarrow$ rank $\mathcal{A}_x$=n (Recall $\mathcal{A}_x = \{ a_i : a_i^T x = b_i \}$)

Let $c = \sum_{i \in \mathcal{I}_x} a_i$.

**Claim:** $c^T x = \sum_{i \in \mathcal{I}_x} b_i$

**Proof:** $c^T x = \sum_{i \in \mathcal{I}_x} a_i^T x = \sum_{i \in \mathcal{I}_x} b_i$. $\square$

**Lemma**: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$. The following are equivalent.

i. x is a vertex
ii. x is an extreme point
iii. x is a basic feasible solution (BFS)

**Proof** of (iii)$\Rightarrow$(i):

Let x be a BFS $\Rightarrow$ rank $\mathcal{A}_x = n$ (Recall $\mathcal{A}_x = \{ a_i : a_i^T x = b_i \}$)

Let $c = \sum_{i \in \mathcal{I}_x} a_i$.

**Claim:** $c^T x = \sum_{i \in \mathcal{I}_x} b_i$

**Claim:** x is an optimal point of $\max \{ c^T x : x \in P \}$.

**Proof:** $y \in P \Rightarrow a_i^T y \leq b_i$ for all i

$\Rightarrow c^T y = \sum_{i \in \mathcal{I}_x} a_i^T y \leq \sum_{i \in \mathcal{I}_x} b_i = c^T x.$ ☐

If one of these is strict, then this is strict.

**Claim:** x is the **unique** optimal point of $\max \{ c^T x : x \in P \}$.

**Proof:** If for any $i \in \mathcal{I}_x$ we have $a_i^T y < b_i$ then $c^T y < c^T x$.
So every optimal point y has $a_i^T y = b_i$ for all $i \in \mathcal{I}_x$.
Since rank $\mathcal{A}_x = n$, there is only one solution: y=x! ☐

**Lemma**: Let $P=\{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$. The following are equivalent.

i. x is a vertex

ii. x is an extreme point

iii. x is a basic feasible solution (BFS)

**Proof** of (iii)$\Rightarrow$(i):

Let x be a BFS $\Rightarrow$ rank $\mathcal{A}_x$=n (Recall $\mathcal{A}_x = \{ a_i : a_i^T x = b_i \}$)

Let $c = \Sigma_{i \in \mathcal{I}_x} a_i$.

**Claim:** $c^T x = \Sigma_{i \in \mathcal{I}_x} b_i$

**Claim:** x is an optimal point of $\max \{ c^T x : x \in P \}$.

**Claim:** x is the **unique** optimal point of $\max \{ c^T x : x \in P \}$.

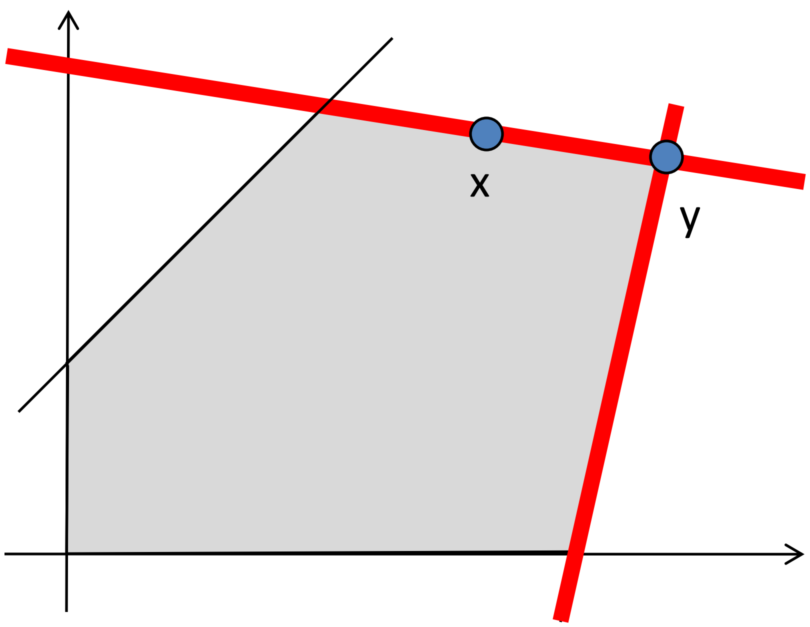
So x is a vertex. ■

# More on corner points

**Definition**: A **line** is a set $L=\{ r+\lambda s : \lambda \in \mathbb{R} \}$ where $r,s \in \mathbb{R}^n$ and $s \neq 0$.

**Lemma:** Let $P=\{ x : a_i^T x \leq b_i \ \forall i \}$. Suppose P is non-empty and P does not contain any line. Then P has a corner point.

**Proof Idea:** Pick any $x \in P$. Suppose x not a BFS.




- At least one degree of freedom remains at x
- So x can "wiggle" while staying on all the tight constraints
- x cannot wiggle off to infinity in both directions because P contains no line
- So when x wiggles, it hits a constraint
- When it hits first constraint, it is still feasible.
- So we have found a point y which has a new tight constraint.

**Definition**: A **line** is a set $L=\{\ r+\lambda s : \lambda\in\mathbb{R}\ \}$ where $r,s\in\mathbb{R}^n$ and $s\neq 0$.

**Lemma:** Let $P=\{\ x : a_i^T x \leq b_i\ \forall i\ \}$. Suppose P is non-empty and P does not contain any line. Then P has a corner point.

**Proof:** Pick $x\in P$. Suppose x not a BFS.

**Claim:** $\exists w\in\mathbb{R}^n$, $w\neq 0$, s.t. $a_i^T w=0\ \ \forall i\in\mathcal{I}_x$ (We saw this before)

Let $y(\epsilon)=x+\epsilon w$. Note $y(0)=x\in P$.

**Claim:** $\exists\epsilon$ s.t. $y(\epsilon)\notin P$. WLOG $\epsilon>0$. (Otherwise P contains a line)

**Lemma:** Let $P=\{ x : a_i^T x \leq b_i \ \forall i \}$. Suppose P is non-empty and P does not contain any line. Then P has a corner point.

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Let $y(\epsilon)=x+\epsilon w$. Note $y(0)=x \in P$.

**Claim:** $\exists \epsilon$ s.t. $y(\epsilon) \notin P$. WLOG $\epsilon>0$. (Otherwise P contains a line)

So set $\delta=0$ and gradually increase $\delta$. What is largest $\delta$ s.t. $x \in P$?

$$
\begin{aligned}
y(\delta) \in P \ &\Leftrightarrow \ a_i^T y(\delta) \leq b_i \quad \forall i \\
&\Leftrightarrow \ a_i^T x + \delta a_i^T w \leq b_i \quad \forall i \qquad \text{(Always satisfied if } a_i^T w \leq 0\text{)} \\
&\Leftrightarrow \ \delta \leq \underbrace{(b_i - a_i^T x)/a_i^T w \quad \forall i \text{ s.t. } a_i^T w > 0}
\end{aligned}
$$

Let h be the i that minimizes this. Then $\delta=(b_h-a_h^T x)/a_h^T w$.

**Claim:** $\mathcal{I}_x \subseteq \mathcal{I}_{y(\delta)}$.

**Proof:** If $i \in \mathcal{I}_x$ then $a_i^T x = b_i$. But $a_i^T w = 0$, so $a_i^T y(\delta) = b_i$ too. $\square$

**Lemma:** Let $P=\{ x : a_i^T x \leq b_i \ \forall i \}$. Suppose P is non-empty and P does not contain any line. Then P has a corner point.

**Proof:** Pick $x \in P$. Suppose x not a BFS.

**Claim:** $\exists w \in \mathbb{R}^n$, $w \neq 0$, s.t. $a_i^T w = 0 \ \ \forall i \in \mathcal{I}_x$ (We saw this before)

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Let h be the i that minimizes this. Then $\delta=(b_h-a_h^T x)/a_h^T w$.

**Claim:** $\mathcal{I}_x \subseteq \mathcal{I}_{y(\delta)}$.

**Claim:** $h \in \mathcal{I}_{y(\delta)} \setminus \mathcal{I}_x$. ($y(\delta)$ has at least one more tight constraint)

**Proof:** By definition $a_h^T w>0$, so $h \notin \mathcal{I}_x$.

But $a_h^T y(\delta) = a_h^T x + \delta a_h^T w = a_h^T x + \big((b_h - a_h^T x)/a_h^T w\big) a_h^T w = b_h \Rightarrow h \in \mathcal{I}_{y(\delta)}$. $\square$

**Lemma:** Let $P=\{\, x : a_i^T x \leq b_i \;\forall i \,\}$. Suppose P is non-empty and P does not contain any line. Then P has a corner point.

**Proof:** Pick $x \in P$. Suppose x not a BFS.

**Claim:** $\exists w \in \mathbb{R}^n$, $w \neq 0$, s.t. $a_i^T w = 0 \;\; \forall i \in \mathcal{I}_x$ (We saw this before)

Let $y(\epsilon)=x+\epsilon w$. Note $y(0)=x \in P$.

**Claim:** $\exists \epsilon$ s.t. $y(\epsilon) \notin P$. WLOG $\epsilon>0$. (Otherwise P contains a line)

So set $\delta=0$ and gradually increase $\delta$. What is largest $\delta$ s.t. $x \in P$?

$$y(\delta) \in P \;\Leftrightarrow\; a_i^T y(\delta) \leq b_i \;\; \forall i$$

$$\Leftrightarrow\; a_i^T x + \delta a_i^T w \leq b_i \;\; \forall i \qquad \text{(Always satisfied if } a_i^T w \leq 0\text{)}$$

$$\Leftrightarrow\; \delta \leq \underbrace{(b_i - a_i^T x)/a_i^T w \;\; \forall i \text{ s.t. } a_i^T w > 0}$$

Let h be the i that minimizes this. Then $\delta=(b_h - a_h^T x)/a_h^T w$.

**Claim:** $\mathcal{I}_x \subseteq \mathcal{I}_{y(\delta)}$.

**Claim:** $h \in \mathcal{I}_{y(\delta)} \setminus \mathcal{I}_x$. ($y(\delta)$ has at least one more tight constraint)

**Claim:** $a_h \notin \operatorname{span}(\mathcal{A}_x)$.

**Proof:** $a_h^T x < b_h$ but $a_h^T y(\delta) = b_h \;\Rightarrow\; 0 \neq a_h^T(\, y(\delta) - x \,) = \epsilon\, a_h^T w$.

But, $a_i^T w = 0 \;\; \forall a_i \in \mathcal{A}_x \;\Rightarrow\; a_i^T w = 0 \;\; \forall a_i \in \operatorname{span}(\mathcal{A}_x) \;\Rightarrow\; a_h \notin \operatorname{span}(\mathcal{A}_x)$. $\square$

**Lemma:** Let $P=\{ x : a_i^T x \leq b_i \ \forall i \}$. Suppose P is non-empty and P does not contain any line. Then P has a corner point.

**Proof:** Pick $x \in P$. Suppose x not a BFS.

**Claim:** $\exists w \in \mathbb{R}^n$, $w \neq 0$, s.t. $a_i^T w = 0 \ \ \forall i \in \mathcal{I}_x$ (We saw this before)

Let $y(\epsilon)=x+\epsilon w$. Note $y(0)=x \in P$.

**Claim:** $\exists \epsilon$ s.t. $y(\epsilon) \notin P$. WLOG $\epsilon>0$. (Otherwise P contains a line)

So set $\delta=0$ and gradually increase $\delta$. What is largest $\delta$ s.t. $x \in P$?

$$\begin{aligned} y(\delta) \in P \ &\Leftrightarrow \ a_i^T y(\delta) \leq b_i \ \ \forall i \\ &\Leftrightarrow \ a_i^T x + \delta a_i^T w \leq b_i \ \ \forall i \qquad \text{(Always satisfied if } a_i^T w \leq 0\text{)} \\ &\Leftrightarrow \ \delta \leq \underbrace{(b_i - a_i^T x)/a_i^T w \ \ \forall i \text{ s.t. } a_i^T w > 0} \end{aligned}$$

Let h be the i that minimizes this. Then $\delta=(b_h-a_h^T x)/a_h^T w$.

**Claim:** $\mathcal{I}_x \subseteq \mathcal{I}_{y(\delta)}$.

**Claim:** $h \in \mathcal{I}_{y(\delta)} \setminus \mathcal{I}_x$. ($y(\delta)$ has at least one more tight constraint)

**Claim:** $a_h \notin \text{span}(\mathcal{A}_x)$.

So rank $\mathcal{A}_{y(\delta)}$ > rank $\mathcal{A}_x$. Repeat this argument with $y(\delta)$ instead of x.

Eventually find z with rank $\mathcal{A}_z = n \ \Rightarrow$ z is a BFS. ■

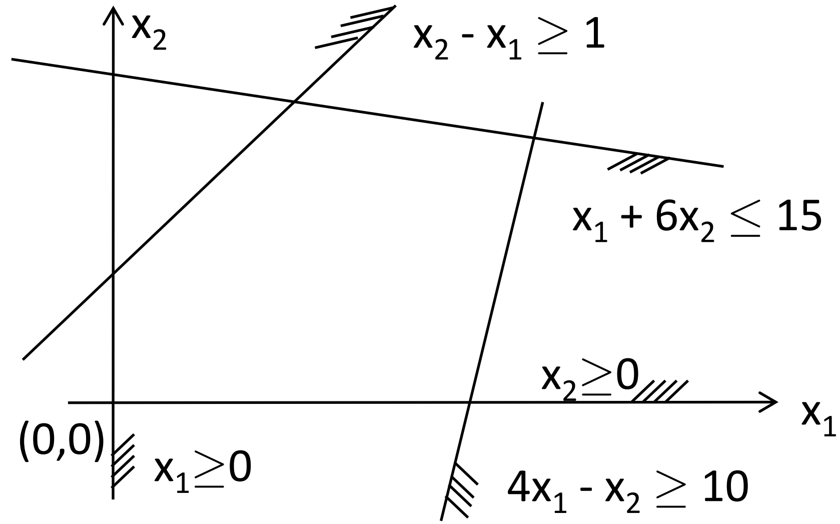
# Local-Search Algorithm: Pitfalls & Details

**Algorithm**
Let x be any corner point
For each corner point y that is a neighbor of x
    If $c^T y > c^T x$ then set x=y
Halt

1. What is a corner point?
2. What if there are no corner points?
3. What are the “neighboring” corner points?
4. What if there are no neighboring corner points?
5. How can I find a starting corner point?
6. Does the algorithm terminate?
7. Does it produce the right answer?

# Pitfall #2: No corner points?

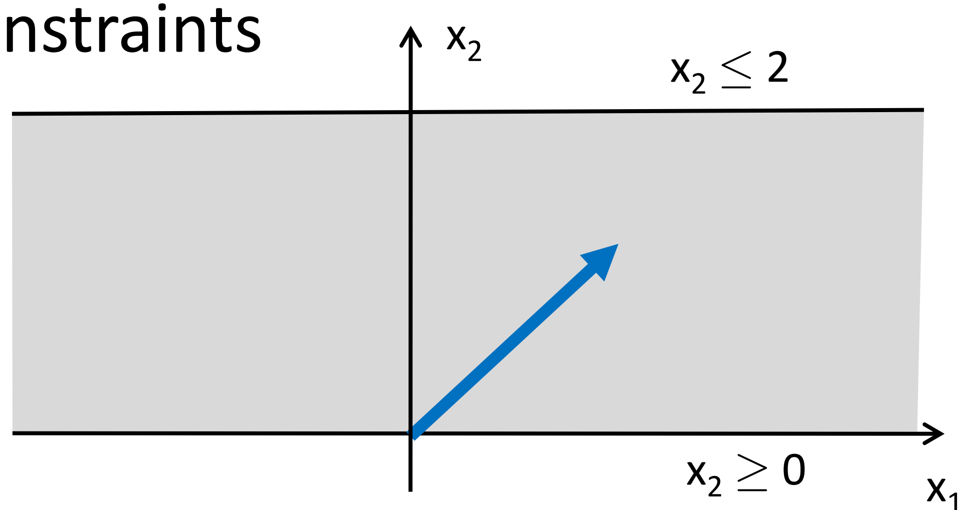
- This is possible
  - Case 1: LP infeasible




This is unavoidable.
Algorithm must detect this case.

  - Case 2: Not enough constraints

$$\begin{array}{lll} \max & x_1 + x_2 & \\ \text{s.t.} & x_2 & \leq 2 \\ & x_2 & \geq 0 \end{array}$$




**A Fix!**

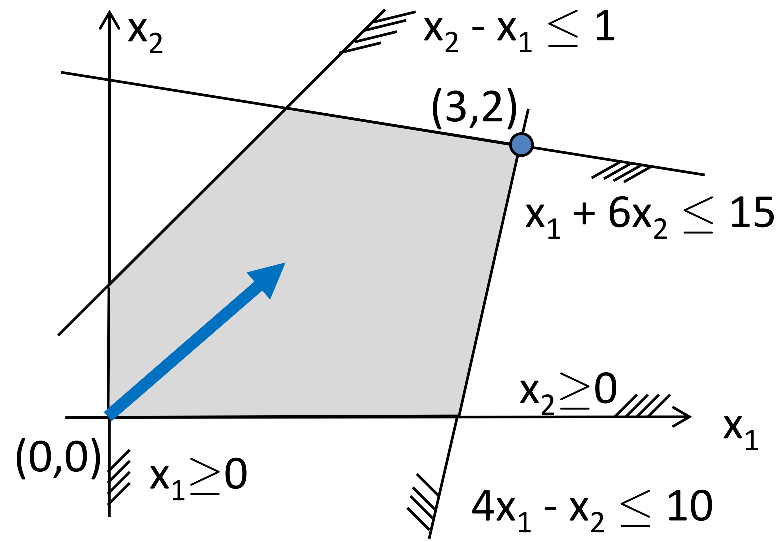
We avoid this case by manipulating the LP a bit…

# Converting to Equational Form

- General form of an LP

$$\begin{aligned} \max \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \le b \end{aligned}$$

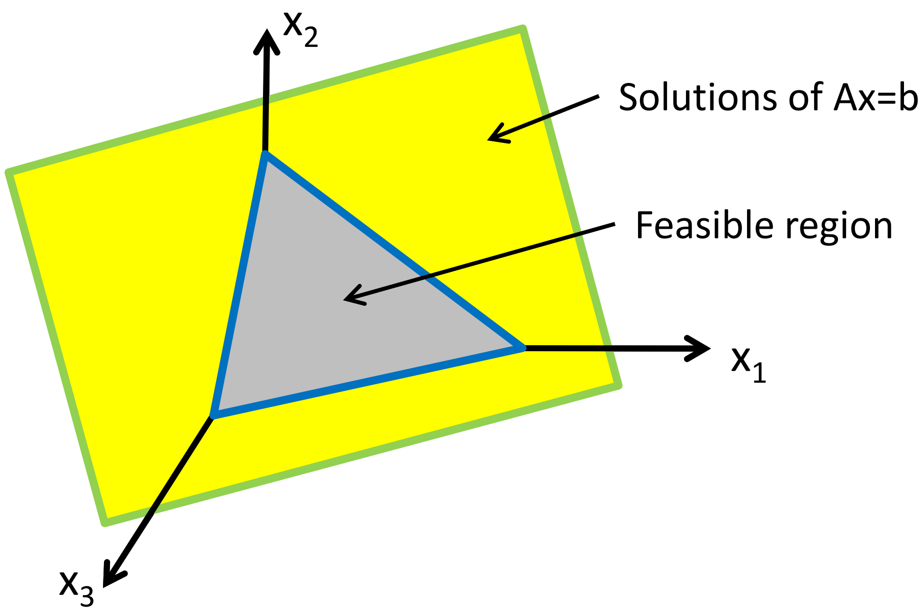
“Intersection of finitely many half-spaces”




- Another form of an LP

$$\begin{aligned} \max \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax = b \\ & x \ge 0 \end{aligned}$$

“Intersection of an affine space with the non-negative orthant”



# Converting to Equational Form

- General form of an LP

$$\begin{array}{lll} \max & c^{\mathsf{T}} x & \\ \text{s.t.} & Ax & \leq b \end{array}$$

"Inequality form"
or "Canonical form"

- Another form of an LP

$$\begin{array}{lll} \max & c^{\mathsf{T}} x & \\ \text{s.t.} & Ax & = b \\ & x & \geq 0 \end{array}$$

"Equational form"
or "Standard form"

- **Claim**: These two forms of LPs are equivalent.