C&O 355: Lecture 21 Notes

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1 Minimum s-t Cuts

In Lecture 20, we looked at the minimum s-t cut problem. Let's now look at a variant of this problem which introduces capacities on the arcs.

Problem 1.1. Let $G = (V, A)$ be a directed graph. Let s and t be particular vertices in V. For each arc $a \in A$, there is a capacity $c_a \geq 0$. A ***cut*** (or s-t cut) is a set $F \subseteq A$ such that there is no s-t path in $G \setminus F$. The goal is to find a cut F that minimizes its capacity, namely $\sum_{a \in F} c_a$.

Let $\mathcal{P}$ be the set of all simple directed paths from s to t. An integer program formulation of this problem is:

$$
\text{(IP-MC)} \qquad \begin{array}{lll} \min & c^{\mathsf{T}} y & \\ \text{s.t.} & \displaystyle\sum_{a \in p} y_a \geq 1 & \forall p \in \mathcal{P} \\ & y \in \{0,1\} & \end{array}
$$

A linear program relaxation is:

$$
\text{(LP-MC)} \qquad \begin{array}{lll} \min & c^{\mathsf{T}} y & \\ \text{s.t.} & \displaystyle\sum_{a \in p} y_a \geq 1 & \forall p \in \mathcal{P} \\ & y \geq 0 & \end{array}
$$

Last time we proved the following theorem.

Theorem 1.2. Assume $c_a = 1$ for all $a \in A$. Then there is an optimal solution z such that, for some $U \subseteq V$, we have

$$
z_a = \begin{cases} 1 & (\text{if } a \in \delta^+(U)) \\ 0 & (\text{otherwise}). \end{cases}
$$

That is, z is the characteristic vector of the set $\delta^+(U)$.

The previous theorem is actually true for any capacities $c \geq 0$, and the same proof works. Our proof used the dual of LP-MC, complementary slackness, and Strong LP Duality. In this lecture we will prove a theorem very similar to Theorem 1.2, using a very different approach. We will again consider the dual of LP-MC, which is:

$$
\text{(LP-PF)} \qquad \begin{array}{lll} \max & \sum_{p \in \mathcal{P}} x_p & \\ \text{s.t.} & \sum_{p\,:\,a \in p} x_p \ \le c_a & \forall a \in A \\ & x \ \ge 0 & \end{array}
$$

We will prove the following theorem.

Theorem 1.3. For any non-negative, integral arc capacities c, there is an integral optimal solution to LP-PF.

2 Network Flows

We will think of the solutions to LP-PF as specifying ***path-flows***. For every directed s-t path, it specifies a "flow value" x_p, such that every arc a has at most c_a units of flow passing through it. These objects are studied in the area known as "network flows", and the course C&O 351 focuses on this area.

However, LP-MF is not the usual linear program to consider. It is more common to look at what I'll call ***arc-flows***, which are feasible solutions to the linear program LP-AF, defined next.

For any $u \in V$, define

$$
\begin{aligned}
\delta^+(u) &= \{\ (u,v)\ :\ (u,v) \in A\ \} \\
\delta^-(u) &= \{\ (v,u)\ :\ (v,u) \in A\ \}
\end{aligned}
$$

So $\delta^+(u)$ is the set of outgoing arcs from u, and $\delta^-(u)$ is the set of incoming arcs to u. The arc-flow linear program is:

$$
\text{(LP-AF)} \qquad \begin{array}{lll} \max & \sum_{a \in \delta^+(s)} z_a - \sum_{a \in \delta^-(s)} z_a & \\ \text{s.t.} & \sum_{a \in \delta^+(v)} z_a - \sum_{a \in \delta^-(v)} z_a \ = 0 & \forall v \in V \setminus \{s,t\} \\ & 0 \le z_a \le c_a & \forall a \in A \end{array}
$$

The equality constraint in this LP is called the ***balance constraint***, or sometimes the "flow conservation constraint". We will show that the path-flow LP and the arc-flow LP are equivalent, in a certain sense.

Claim 2.1. Let x be a feasible solution to LP-PF. For each arc $a \in A$, define $z_a = \sum_{p\,:\,a \in p} x_p$. Then z is feasible for LP-AF. Furthermore, the objective value of LP-PF at x equals the objective value of LP-AF at z.

Proof. Since $x_p \ge 0$ for all p, we have $z_a \ge 0$ for all a. The constraints of LP-PF are equivalent to $z_a \le c_a$. For every path $p \in \mathcal{P}$ and for every vertex $v \in V \setminus \{s,t\}$, the path p enters and leaves v the same number of times, so z satisfies the balance constraint.

No path $p \in \mathcal{P}$ contains any arc incoming to s so $\sum_{a \in \delta^-(s)} z_a = 0$. So the objective value at z is $\sum_{a \in \delta^+(s)} z_a$, which equals $\sum_{p \in \mathcal{P}} x_p$. ■

We can also translate solutions of LP-AF to solutions of LP-PF.

Claim 2.2. Let z be a feasible solution of LP-AF with non-negative objective value. Then we can construct a feasible solution x of LP-PF with the same objective value. Furthermore, if z is integral, then our construction will yield an integral x.

Proof. We describe a procedure for constructing x.

Suppose there is a directed cycle C all of whose arcs have positive z-value. Let $\epsilon = \min_{a \in C} z_a$. Construct a new solution z by subtracting ϵ from all arcs on C. This does not affect feasibility of z, or change its objective value. So we may assume that z has no directed cycles.

If the objective value of z is zero then we may take x to be the zero vector.

So suppose the objective value of z is strictly positive. Then some arc $(s, v_1) \in \delta^+(s)$ has $z_{(s,v_1)} > 0$. By the balance constraint at v_1, some arc (v_1, v_2) has $z_{(v_1,v_2)} > 0$. We may repeat this process until we either find a directed cycle, which is a contradiction, or we arrive at vertex t. In the latter case, we have a found a directed path p from s to t. Let $\epsilon = \min_{a \in p} z_a$. Decrease all z_a on this path by ϵ and increase x_p by ϵ. This decreases the objective value at z by ϵ, and increases the objective value at x by ϵ.

Note that if z is integral then the value of ϵ is always an integer, and hence x is integral. ■

3 Integral Arc-Flows and Path-Flows

Suppose the capacities c_a are integers.

Claim 3.1. Every optimal BFS z for LP-AF is an integral vector.

We will prove this by a combinatorial argument for finding fractional cycles, much like our proof of integrality for the perfect matching LP in Lecture 19.

Proof. Let z be a BFS. Let us call any arc a with $z_a \notin \mathbb{Z}$ a "fractional arc". Which vertices could have exactly one incident fractional arc? This is only possible for the vertices s and t: if any vertex $v \in V \setminus \{s, t\}$ has an incoming fractional arc, it must either have an outgoing fractional arc, or another incoming fractional arc, by the balance constraint at v.

So starting at any fractional arc, we can either find a cycle consisting of fractional arcs, or an s-t path consisting of fractional arcs. However it is not necessarily a directed cycle or a directed s-t path; the directions of the arcs along these paths might be inconsistent. Let us call the arcs on this cycle (or s-t path) either a "forward arc" or a "backward arc", depending on whether its direction is the same as the direction of the cycle (or path).

Suppose we find a cycle. Consider adding ϵ to the forward arcs and subtracting ϵ from the backward arcs. Is the resulting arc-flow feasible? This change cannot violate the balance constraints. If $|\epsilon|$ is sufficiently small, the constraints $0 \leq z_a \leq c_a$ are also not violated. But then z is a convex combination of two feasible solutions obtained by picking ϵ positive and ϵ negative. (Just like our proof in Lecture 19.) So z is not a BFS.

Suppose we find an s-t path. Consider adding $\epsilon > 0$ to the forward arcs and subtracting ϵ from the backward arcs. This increases the objective value by ϵ. As above, if ϵ is sufficiently small ϵ, then feasibility is not violated. So z is not optimal. ■

Now we are able to prove Theorem 1.3.

Proof (of Theorem 1.3). Let x be an optimal solution to LP-PF. Let z be the corresponding solution to LP-AF obtained by Claim 2.1. If z is not integral then, by the argument of Claim 3.1, we may obtain an *integral* z' with objective value at least as large. By Claim 2.2, we can find an *integral* path-flow x' with the same objective value as z'. ■

To summarize, we've shown that, for every choice of integer capacities c, the path-flow LP has an integral optimal solution, and hence the optimal value is an integer.

Amazingly, this is enough to prove that every BFS of LP-MC is an integral vector! (This is not quite the same statement as Theorem 1.3, but very similar.) To prove this fact, we will need a general result about integral polyhedra.

4 Integral Polyhedra

In the next lecture, we will prove this theorem.

Theorem 4.1. Let $P = \{ \, x \, : \, Ax \leq b \, \} \subset \mathbb{R}^n$ be a non-empty polyhedron that does not contain a line, and such that the entries of A and b are integers. Then the following are equivalent.

- **(1):** For all vectors $c \in \mathbb{Z}^n$, if the LP $\min \{ \, c^{\mathsf{T}} x \, : \, x \in P \, \}$ has finite optimal value, then the optimal value is an integer.
- **(2):** Every extreme point of the polyhedron P is an integral vector.

Actually, our hypotheses of this theorem are unnecessarily strong. This theorem also holds if the entries of A and b are arbitrary real numbers. Furthermore, it can be generalized to polyhedra that do contain a line.

5 More on Minimum s-t Cuts

Theorem 5.1. Every BFS of LP-MC is an integral vector.

Proof. Let c be an arbitrary vector of integer arc capacities. If $c_a < 0$ for some arc a then LP-PF is infeasible, so LP-MC is unbounded (since it is clearly feasible). If $c_a \geq 0$ then, by Theorem 1.3, LP-PF has an integral optimal solution, and hence its optimal value is an integer. By Strong LP Duality, the optimal value of LP-MC is also an integer. So we have shown the property (1) of Theorem 4.1 is satisfied. So property (2) also holds, and every BFS of LP-MC is an integral vector. ■

This theorem is in some ways stronger than Theorem 1.2 (since it tells us that *all* BFS are integral), and in some ways it is weaker, since it does not provide a solution of the form $\delta^+(U)$.

6 Max-Flow Min-Cut Theorem

The following theorem is quite famous and useful. It is known as the "max-flow min-cut theorem".

Theorem 6.1. Let $G = (V, A)$ be a directed graph with arc capacities $c_a \geq 0$, and two distinguished vertices $s, t \in V$. The maximum amount of flow that can be sent from s to t without exceeding the arc capacities equals the minimum capacity of any s-t cut. Furthermore, if c is an integral vector then there is a maximum flow that is integral.

This theorem follows directly from our previous results.

Proof. The maximum amount of flow that can be sent from s to t is the optimal value of LP-PF. By strong LP duality, this equals the optimal value of LP-MC. By Theorem 1.2 or Theorem 5.1, there is an integral optimal solution of LP-MC. That is, there is an s-t cut achieving the same objective value. If the arc capacities are integers, then by Theorem 1.3 we may find an integral path-flow achieving the same objective value. ■