C&O 355
Lecture 2

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Outline

- LP definition & some equivalent forms
- Example in 2D
  - Possible outcomes
- Examples
  - Linear regression, bipartite matching, indep set
- Feasible Region, Convex Sets
- Corner solutions & certificates
- Local-Search Algorithm

Linear Program

- General definition
  - Parameters: $c, a_1,\ldots,a_m \in \mathbb{R}^n$, $b_1,\ldots,b_m \in \mathbb{R}$
  - Variables: $x \in \mathbb{R}^n$

$$\begin{array}{llll} \min & c^{\mathsf{T}} x & & \text{Objective function} \\ \text{s.t.} & a_i^{\mathsf{T}} x \leq b_i & \forall i = 1, \ldots, m & \text{Constraints} \end{array}$$

- Terminology
  - Feasible point: any x satisfying constraints
  - Optimal point: any feasible x that minimizes obj. func
  - Optimal value: value of obj. func for any optimal point

Linear Program

- General definition
  - Parameters: $c, a_1, \ldots, a_m \in \mathbb{R}^n$, $b_1, \ldots, b_m \in \mathbb{R}$
  - Variables: $x \in \mathbb{R}^n$

$$\begin{array}{lll} \min & c^\mathsf{T} x & \\ \text{s.t.} & a_i^\mathsf{T} x \leq b_i & \forall i = 1, \ldots, m \end{array}$$

- Matrix form

$$\begin{array}{ll} \min & c^\mathsf{T} x \\ \text{s.t.} & Ax \leq b \end{array}$$

- Parameters: $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$
- Variables: $x \in \mathbb{R}^n$

Simple LP Manipulations

- "max" instead of "min"

  $\max c^T x \quad \equiv \quad \min -c^T x$

- "$\geq$" instead of "$\leq$"

  $a^T x \geq b \quad \Leftrightarrow \quad -a^T x \leq -b$

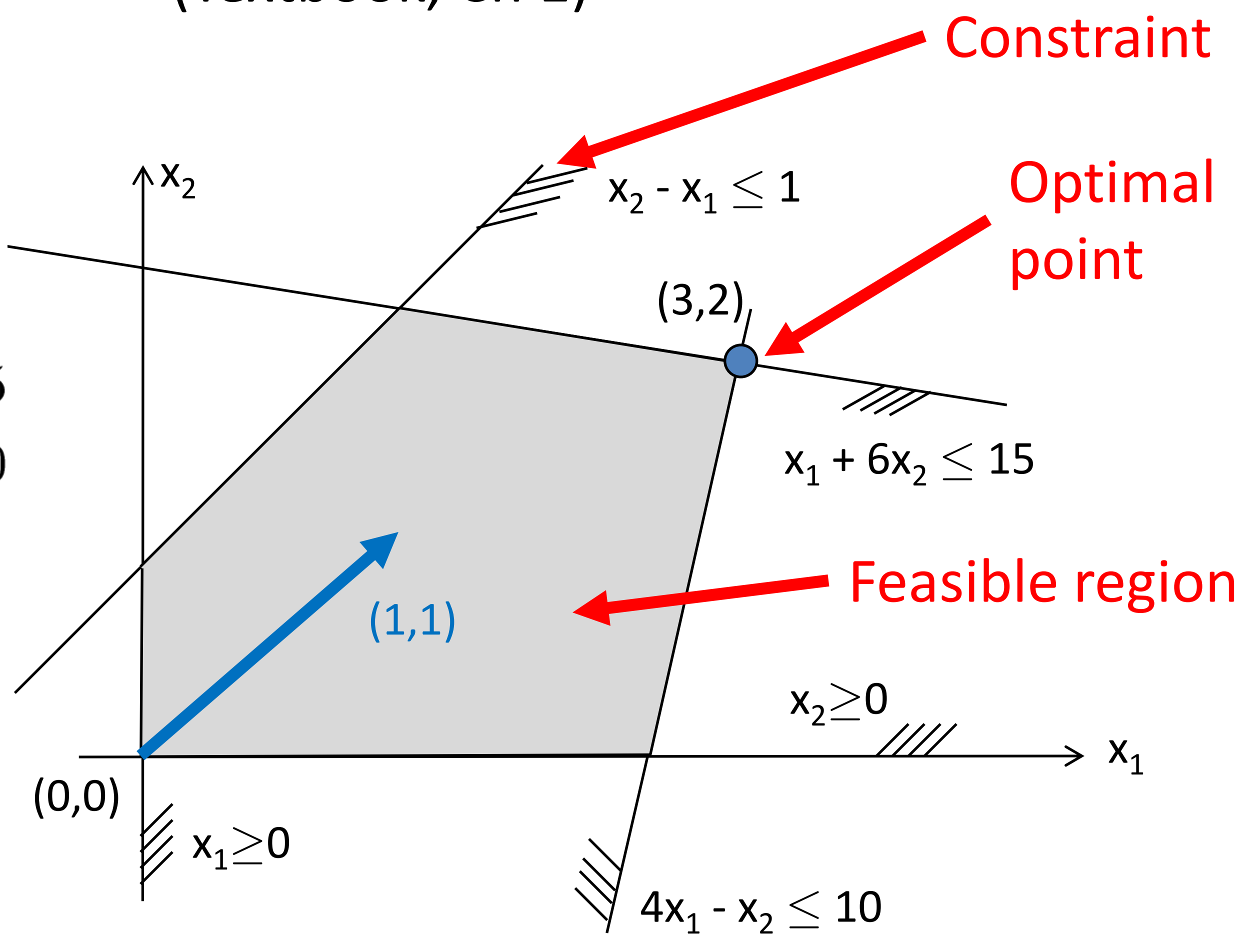
- "=" instead of "$\leq$"

  $a^T x = b \quad \Leftrightarrow \quad a^T x \leq b$ and $a^T x \geq b$

2D Example

(Textbook, Ch 1)

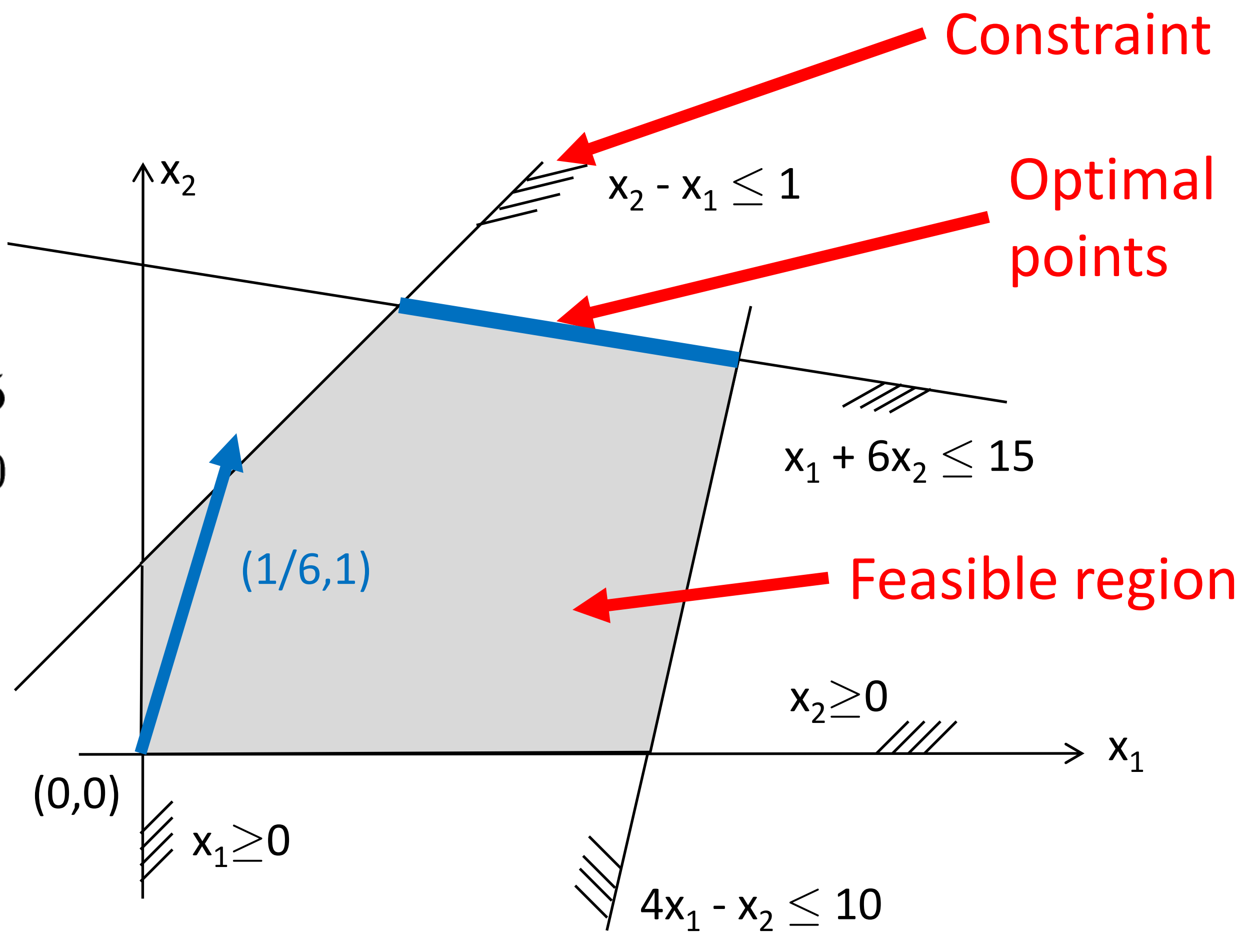
$$
\begin{array}{lll}
\max & x_1 + x_2 & \\
\text{s.t.} & x_2 - x_1 & \leq 1 \\
& x_1 + 6x_2 & \leq 15 \\
& 4x_1 - x_2 & \leq 10 \\
& x_1 & \geq 0 \\
& x_2 & \geq 0
\end{array}
$$



Unique optimal solution exists

2D Example

$$\begin{array}{lll} \max & x_1/6 + x_2 & \\ \text{s.t.} & x_2 - x_1 & \leq 1 \\ & x_1 + 6x_2 & \leq 15 \\ & 4x_1 - x_2 & \leq 10 \\ & x_1 & \geq 0 \\ & x_2 & \geq 0 \end{array}$$

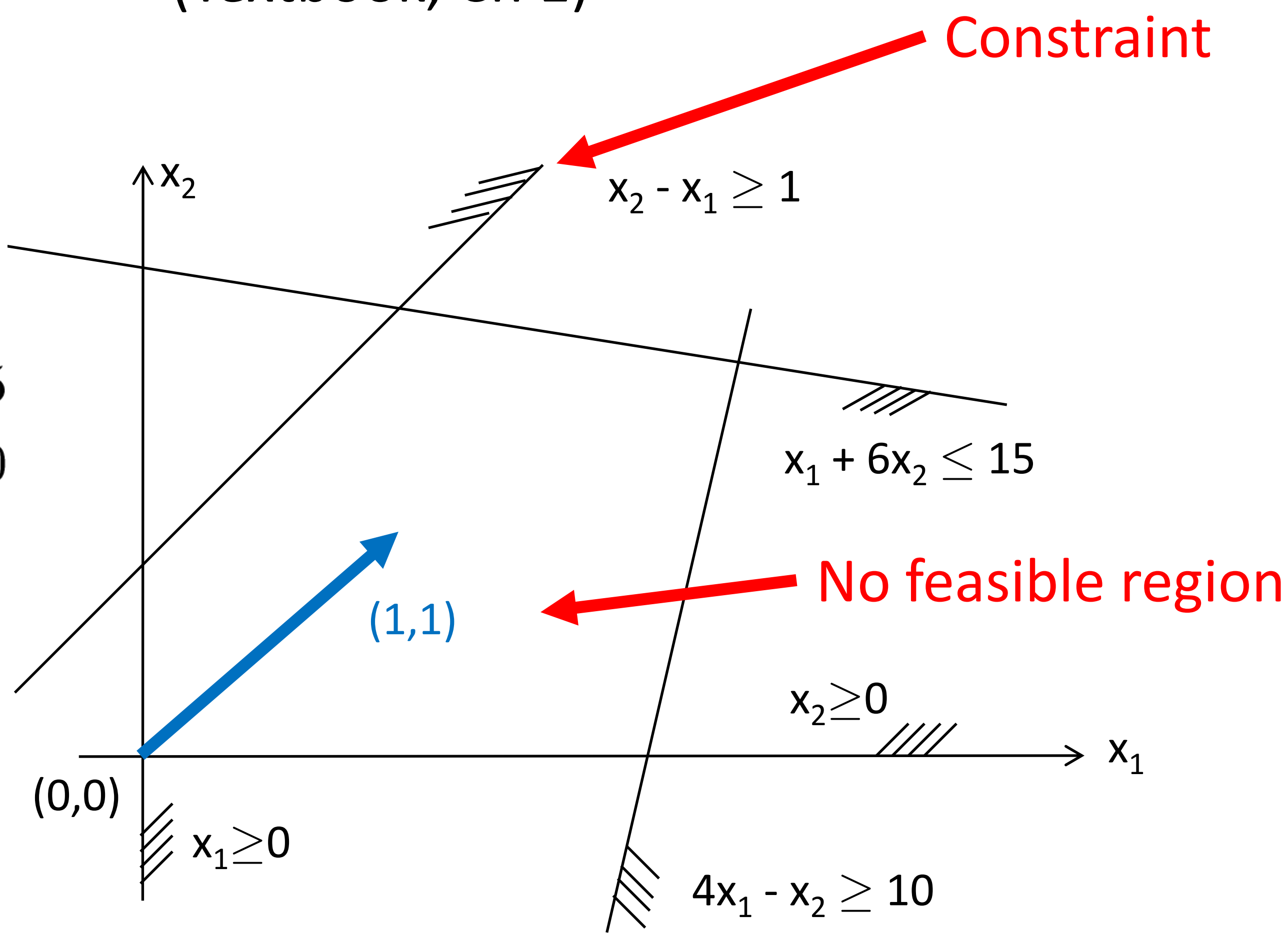


Optimal solutions exist: Infinitely many!

2D Example

(Textbook, Ch 1)

$$\begin{array}{lll} \max & x_1 + x_2 & \\ \text{s.t.} & x_2 - x_1 & \geq 1 \\ & x_1 + 6x_2 & \leq 15 \\ & 4x_1 - x_2 & \geq 10 \\ & x_1 & \geq 0 \\ & x_2 & \geq 0 \end{array}$$



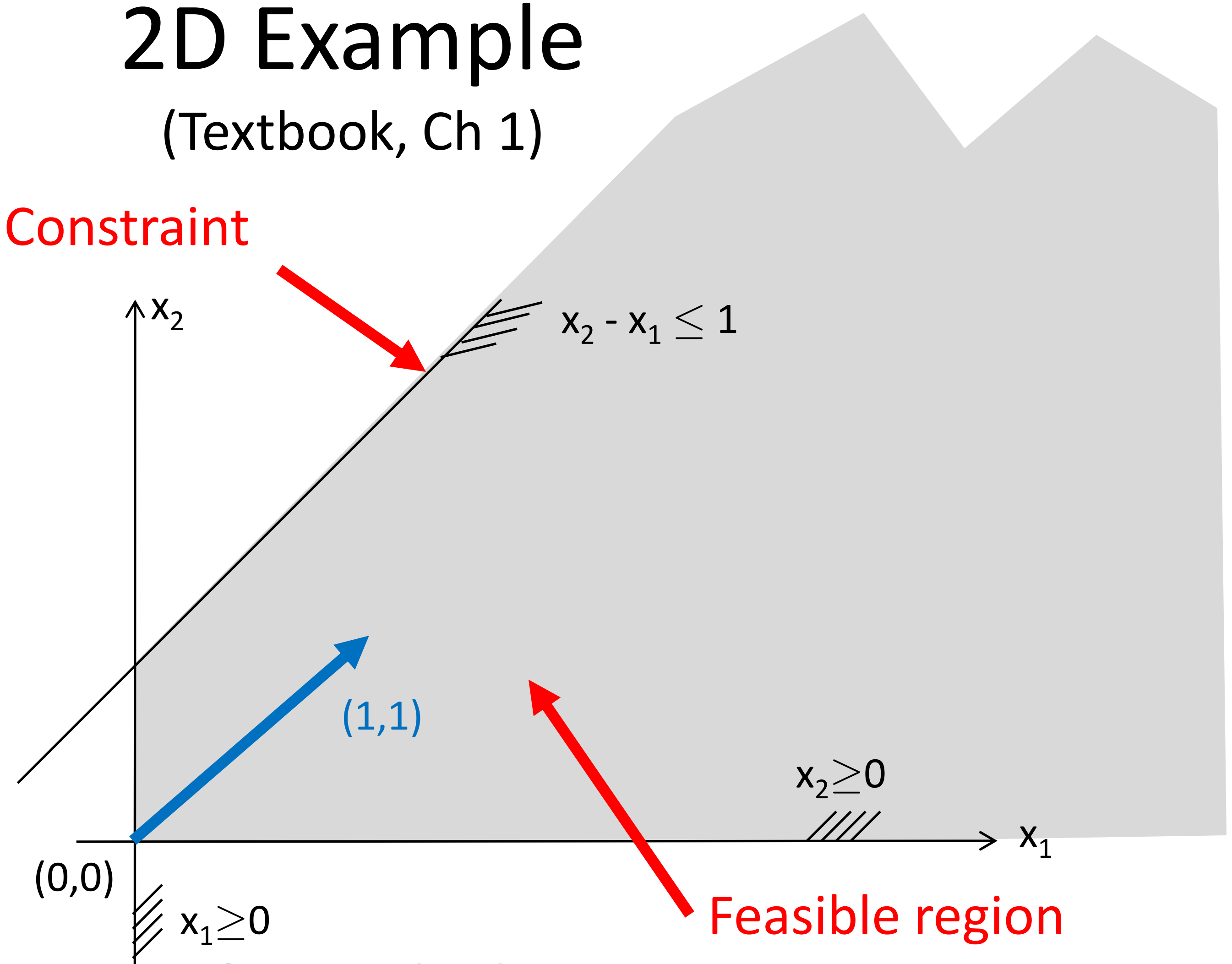
Infeasible

No feasible solutions

2D Example

(Textbook, Ch 1)

$$\begin{array}{lll} \max & x_1 + x_2 & \\ \text{s.t.} & x_2 - x_1 & \leq 1 \\ & x_1 & \geq 0 \\ & x_2 & \geq 0 \end{array}$$



Unbounded

Feasible solutions, but no optimal solution

(Optimal value = ∞)

“Fundamental Theorem” of LP

- **Theorem**: For any LP, the outcome is either:
  - Optimal solution (unique or infinitely many)
  - Infeasible
  - Unbounded (optimal value is ∞ or $-\infty$)

- **Proof**: Later in the course!

Example: Linear regression

- Given data $(x_1, y_1), \ldots, (x_n, y_n)$ in $\mathbb{R}^2$
- Find a line $y = ax + b$ that fits the points

Usual setup

$$\min \textstyle\sum_{i=1}^{n} (ax_i + b - y_i)^2$$

Easy: differentiate, set to zero

Our setup

$$\min \textstyle\sum_{i=1}^{n} |ax_i + b - y_i|$$

Not differentiable!

- Absolute value trick:

$$|w| \quad \equiv \quad \begin{array}{lll} \min & e & \\ \text{s.t.} & e & \geq w \\ & e & \geq -w \end{array}$$

- Our setup can be written as an LP

$$\begin{array}{llll} \min & \displaystyle\sum_{i=1}^{n} e_i & & \\ \text{s.t.} & e_i & \geq ax_i + b - y_i & \forall i \\ & e_i & \geq -(ax_i + b - y_i) & \forall i \end{array}$$

Example: Bipartite Matching

- Given bipartite graph G=(V, E)
- Find a maximum size matching
  - A set M $\subseteq$ E s.t. every vertex has at most one incident edge in M
- Write an integer program

(IP)
$$\begin{array}{llll} \max & \sum_{e \in E} x_e & & \\ \text{s.t.} & \sum_{e \text{ incident to } v} x_e & \leq 1 & \forall v \in V \\ & x_e & \in \{0,1\} & \forall e \in E \end{array}$$

- But we don't know how to solve IPs. Try an LP instead.

(LP)
$$\begin{array}{llll} \max & \sum_{e \in E} x_e & & \\ \text{s.t.} & \sum_{e \text{ incident to } v} x_e & \leq 1 & \forall v \in V \\ & x_e & \geq 0 & \forall e \in E \end{array}$$

- **Theorem**: (IP) and (LP) have the same solution!
- **Proof**: Later in the course!
- **Corollary**: Bipartite matching can be solved efficiently (it's in P).

Example: Independent Set

- Given graph G=(V, E)
- Find a maximum size independent set
  - A set U$\subseteq$V s.t. {u,v} $\notin$ E for every distinct u,v $\in$ U
- Write an integer program

(IP)
$$\begin{array}{llll} \max & \sum_{v \in V} x_v & & \\ \text{s.t.} & x_u + x_v & \leq 1 & \forall \{u,v\} \in E \\ & x_v & \in \{0,1\} & \forall v \in V \end{array}$$

- But we don't know how to solve IPs. Try an LP instead.

(LP)
$$\begin{array}{llll} \max & \sum_{v \in V} x_v & & \\ \text{s.t.} & x_u + x_v & \leq 1 & \forall \{u,v\} \in E \\ & x_v & \geq 0 & \forall v \in V \end{array}$$

- Unfortunately (IP) and (LP) are extremely different.
- **Fact**: There are graphs for which $\mathrm{OPT}_{\mathrm{LP}}$ / $\mathrm{OPT}_{\mathrm{IP}} \geq |V|/2$.

Feasible Region

- For any a$\in\mathbb{R}^n$, b$\in\mathbb{R}$, define

$$\begin{aligned} H_{a,b} &= \{ x \in \mathbb{R}^n : a^\mathsf{T} x = b \} \\ H^+_{a,b} &= \{ x \in \mathbb{R}^n : a^\mathsf{T} x \geq b \} \\ H^-_{a,b} &= \{ x \in \mathbb{R}^n : a^\mathsf{T} x \leq b \} \end{aligned}$$

Hyperplane

Halfspaces

- So feasible region of

$$\boxed{\begin{array}{lll} \min & c^\mathsf{T} x & \\ \text{s.t.} & a_i^\mathsf{T} x \leq b_i & \forall i = 1, ..., m \end{array}}$$

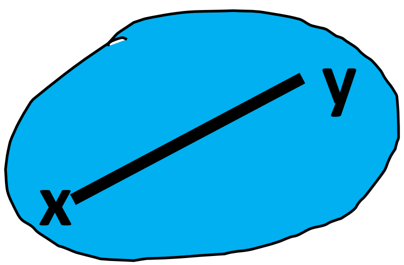
is $P = \bigcap_{i=1}^{m} H^-_{a_i,b_i}$

- Intersection of finitely many halfspaces is **polyhedron**
- A **bounded** polyhedron is a **polytope**, i.e., P $\subseteq$ { x : -M $\leq$ x_i $\leq$ M $\forall$i } for some M

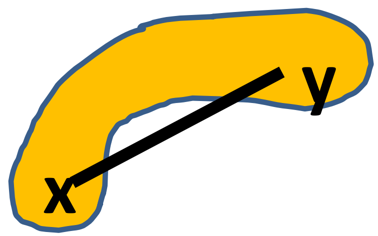
Convex Sets

- $C \subseteq \mathbb{R}^n$ is **convex** if for every $x,y \in C$,
  C contains **line segment** between x and y.
  i.e., $\forall \alpha \in [0,1]$, we have $\underbrace{\alpha x + (1-\alpha) y}$ $\in C$.

Such a point is called a **convex combination** of x and y



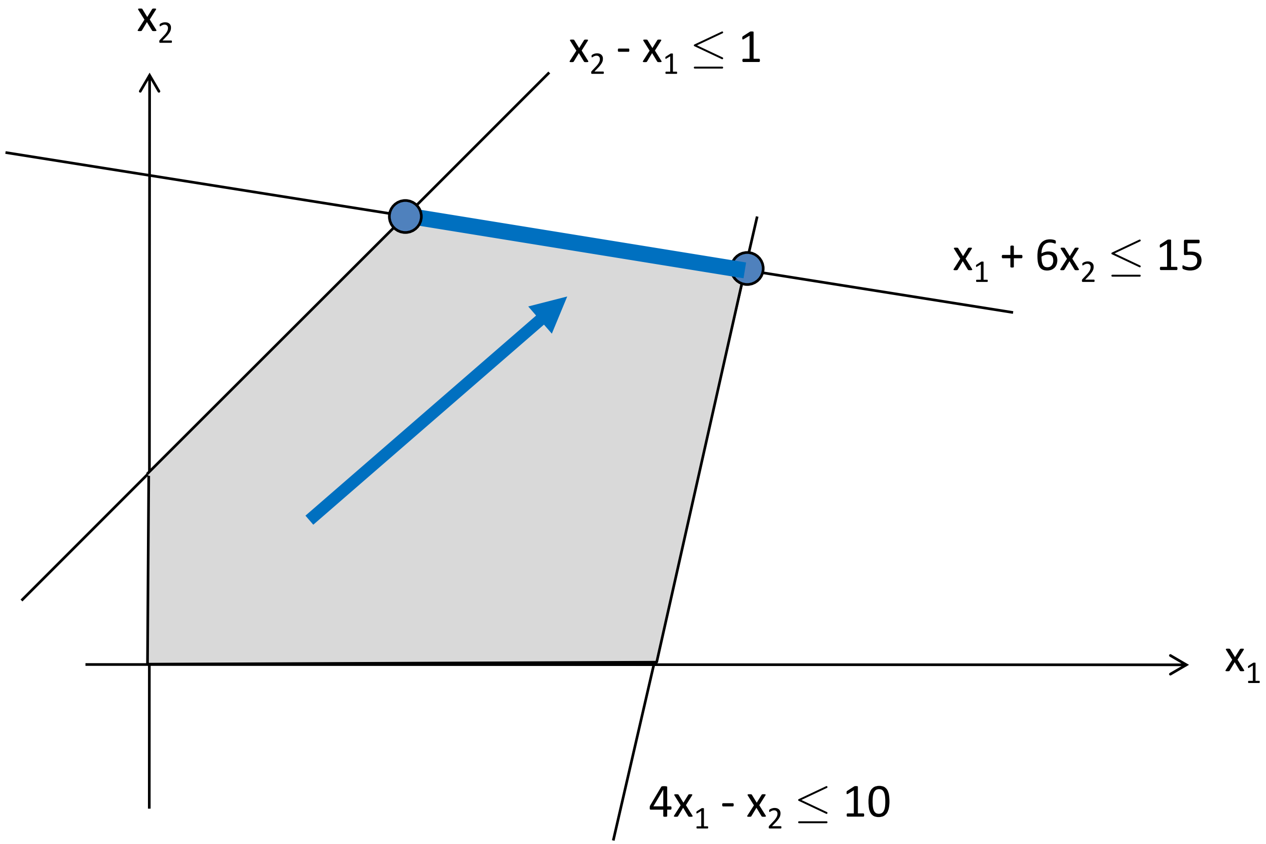
Convex



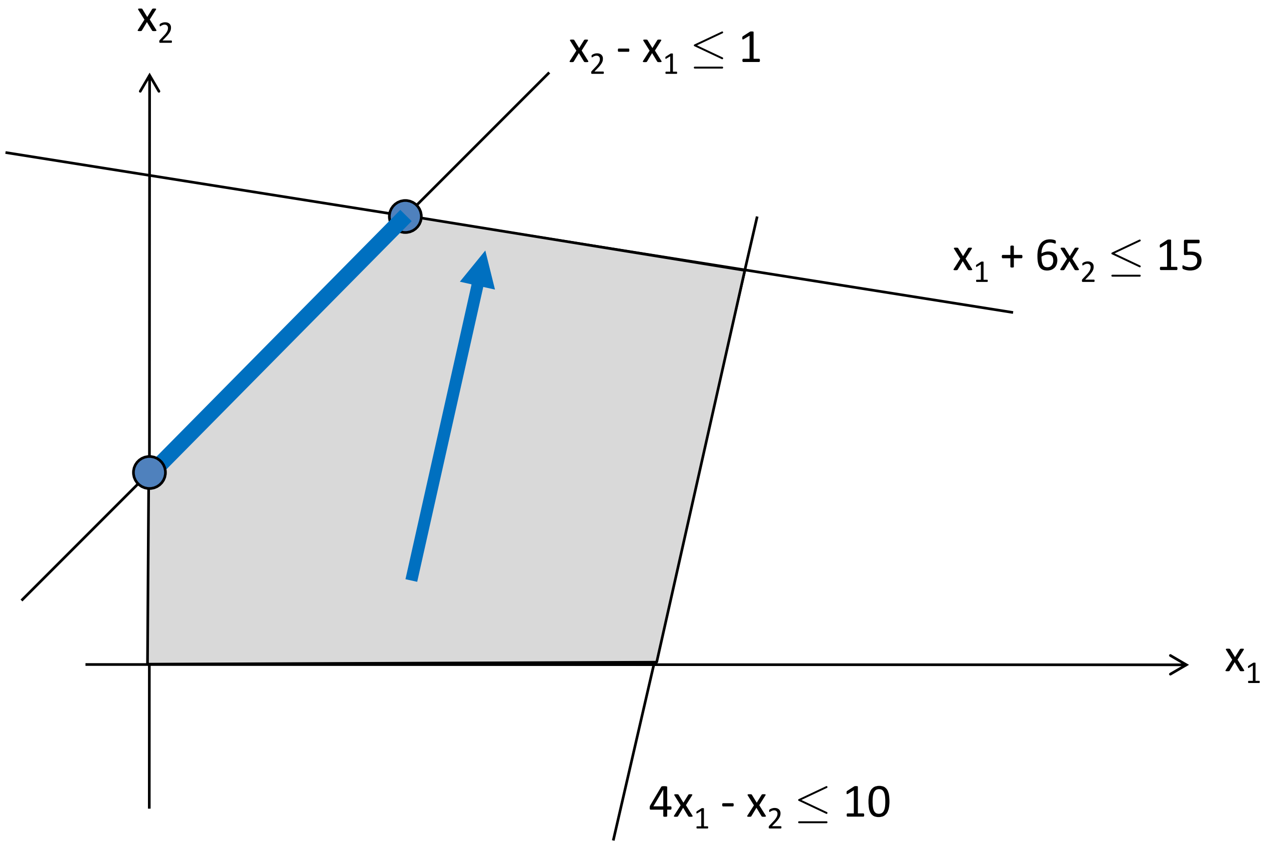
Not convex

- **Claim 1**: Any halfspace is convex.
- **Claim 2**: The intersection of any number of convex sets is convex.
- **Corollary**: Any polyhedron is convex.

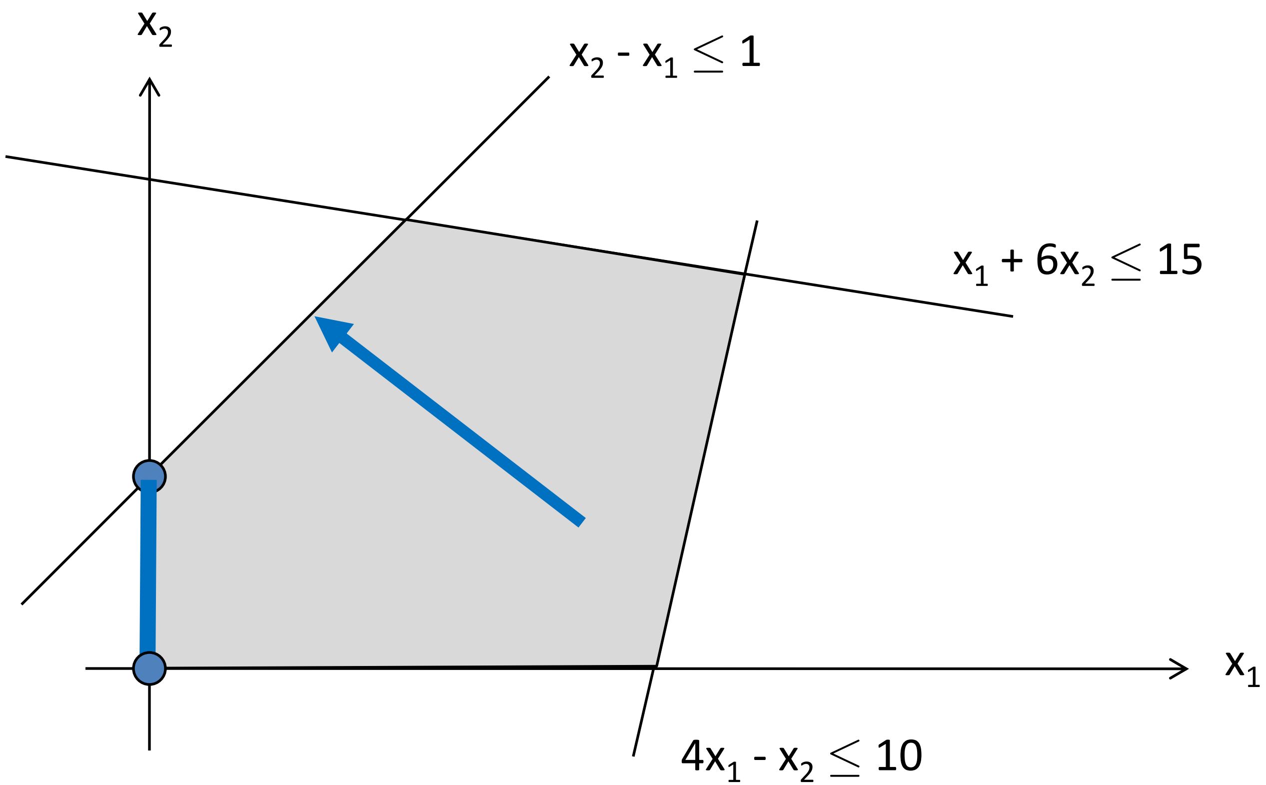
Where are optimal solutions?



Where are optimal solutions?



Where are optimal solutions?



Lemma: For any objective function, a "corner point" is an optimal solution.

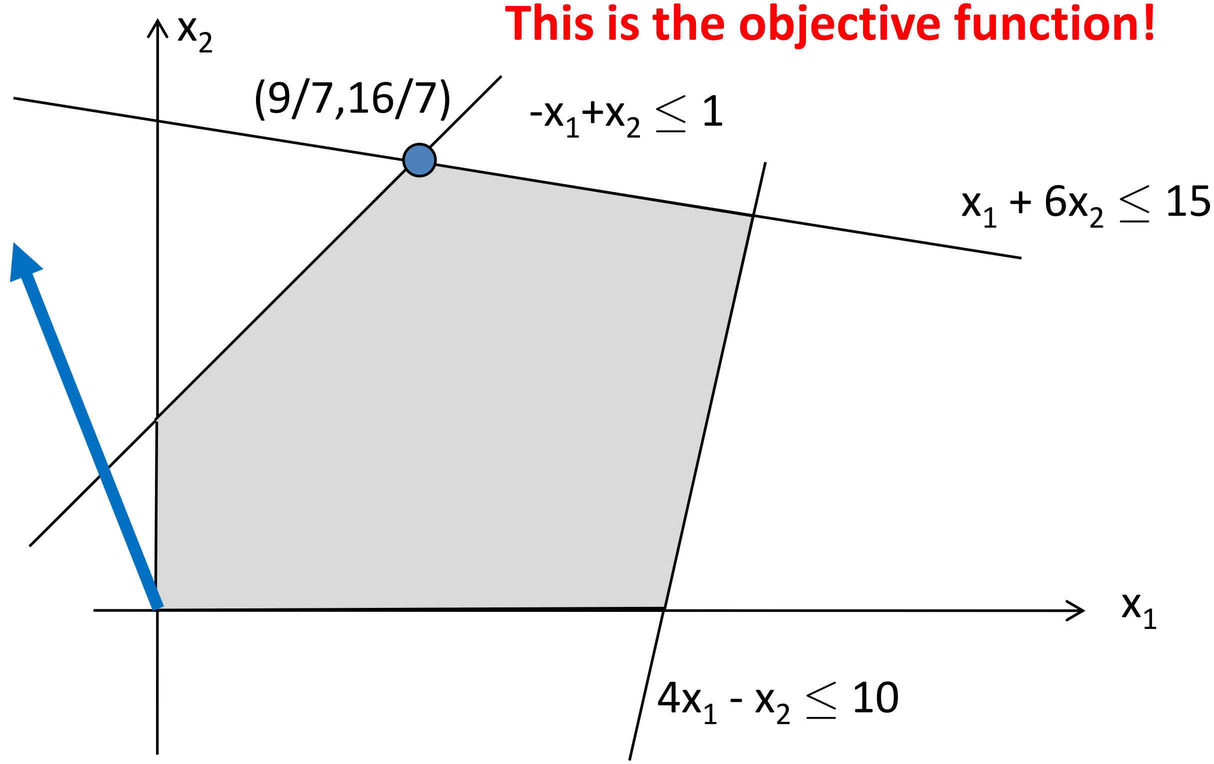
(Assuming an optimal solution exists and some corner point exists).

Proof: Later in the course!

Proving optimality

- Question: What is optimal point in direction c = (-7,14)?
- Solution: Optimal point is x=(9/7,16/7), optimal value is 23.
- How can I be sure?
  - **Every** feasible point satisfies $x_1+6x_2 \leq 15$
  - **Every** feasible point satisfies $-x_1+x_2 \leq 1 \;\Rightarrow\; -8x_1+8x_2 \leq 8$
  - **Every** feasible point satisfies their sum: $\underbrace{-7x_1+14x_2}_{\text{This is the objective function!}} \leq 23$

$$
\begin{array}{lll}
\max & -7x_1 + 14x_2 & \\
\text{s.t.} & -x_1 + x_2 & \leq 1 \\
 & x_1 + 6x_2 & \leq 15 \\
 & 4x_1 - x_2 & \leq 10 \\
 & x & \geq 0
\end{array}
$$



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  - **Every** feasible point satisfies their sum: $\underbrace{-7x_1+14x_2}_{\text{This is the objective function!}} \leq 23$

- **Certificates**
  - To convince you that optimal value is $\geq k$, I can find x such that $c^T x \geq k$.
  - To convince you that optimal value is $\leq k$, I can find a linear combination of the constraints which proves that $c^T x \leq k$.
- **Theorem:** Such certificates always exists.
- **Proof:** Later in the course!

Local-Search Algorithm
(The "Simplex Method")

- "The obvious idea of moving along edges from one vertex of a convex polygon to the next" [Dantzig, 1963]

Algorithm
Let x be any corner point
For each neighbor y of x
    If $c^T y > c^T x$ then set x=y
Halt

- In practice, very efficient
- Its analysis proves all the theorems mentioned earlier
- Every known variant of this algorithm takes exponential time
- **Open problem**: does some variant run in polynomial time?

Pitfalls and missing details

Algorithm

Let x be any corner point

For each neighbor y of x

    If $c^T y > c^T x$ then set $x = y$

Halt

1. What is a corner point?
2. What if there are no corner points?
3. What are the “neighboring” corner points?
4. What if there are no neighboring corner points?
5. How can I find a starting corner point?
6. Does the algorithm terminate?
7. Does it produce the right answer?