C&O 355
Lecture 16

[N. Harvey](#)

Topics

- Review of Fourier-Motzkin Elimination
- Linear Transformations of Polyhedra
- Convex Combinations
- Convex Hulls
- Polytopes & Convex Hulls

Fourier-Motzkin Elimination

Joseph Fourier

Theodore Motzkin

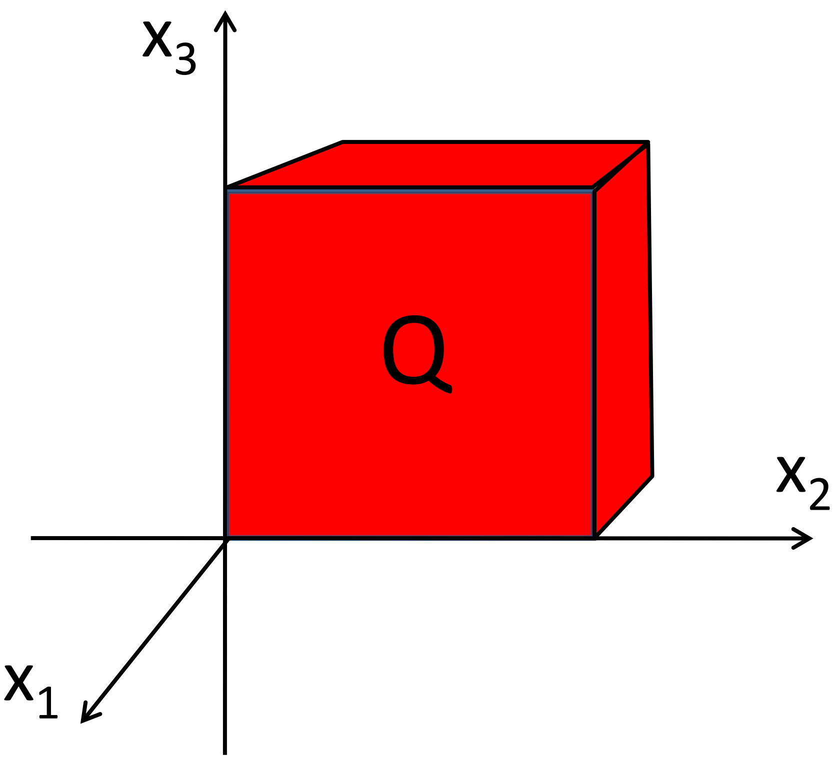
- Given a polyhedron $Q \subseteq \mathbb{R}^n$,
  we want to find the set $Q' \subseteq \mathbb{R}^{n-1}$ satisfying

$$(x_1,\ldots,x_{n-1}) \in Q' \quad \Leftrightarrow \quad \exists x_n \text{ s.t. } (x_1,\ldots,x_{n-1},x_n) \in Q$$

- Q' is called the **projection** of Q onto first $n-1$ coordinates
- Fourier-Motzkin Elimination constructs Q' by generating (finitely many) constraints from the constraints of Q.

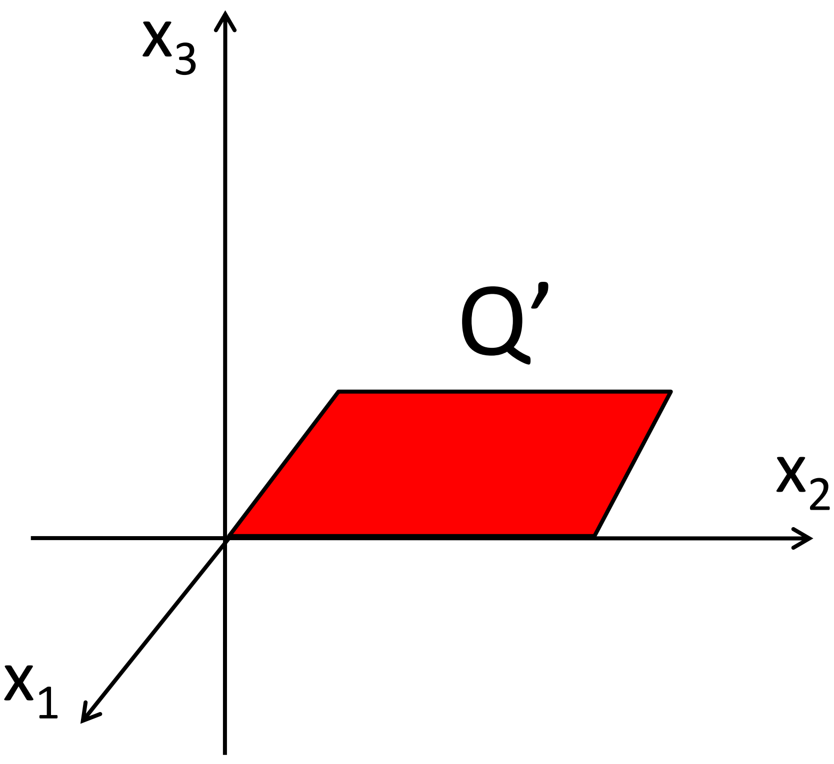
- **Corollary:** Q' is a polyhedron.

Elimination Example



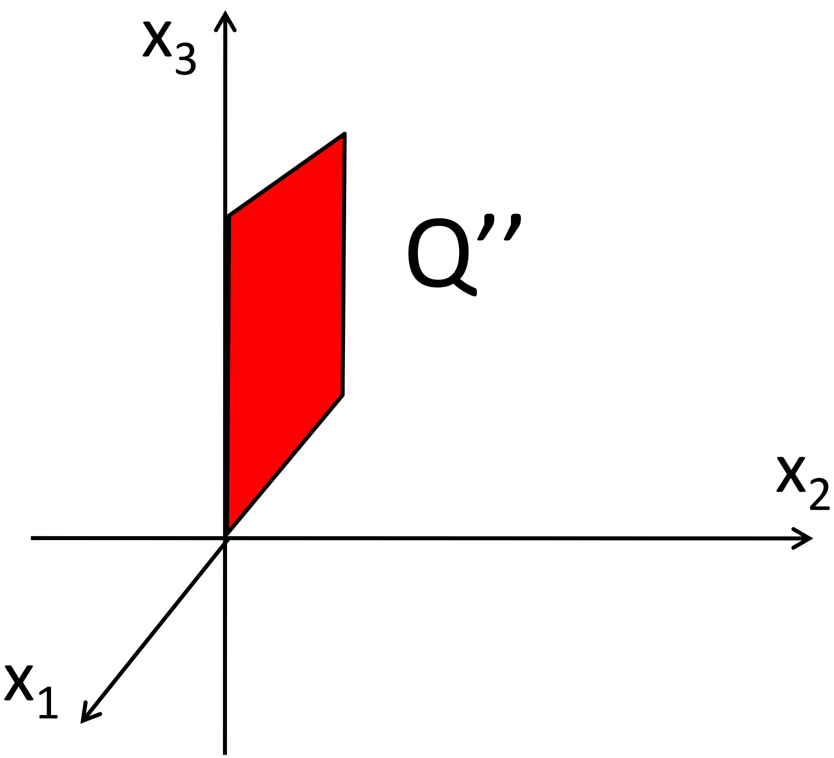
- Project Q onto coordinates $\{x_1, x_2\}$...

Elimination Example



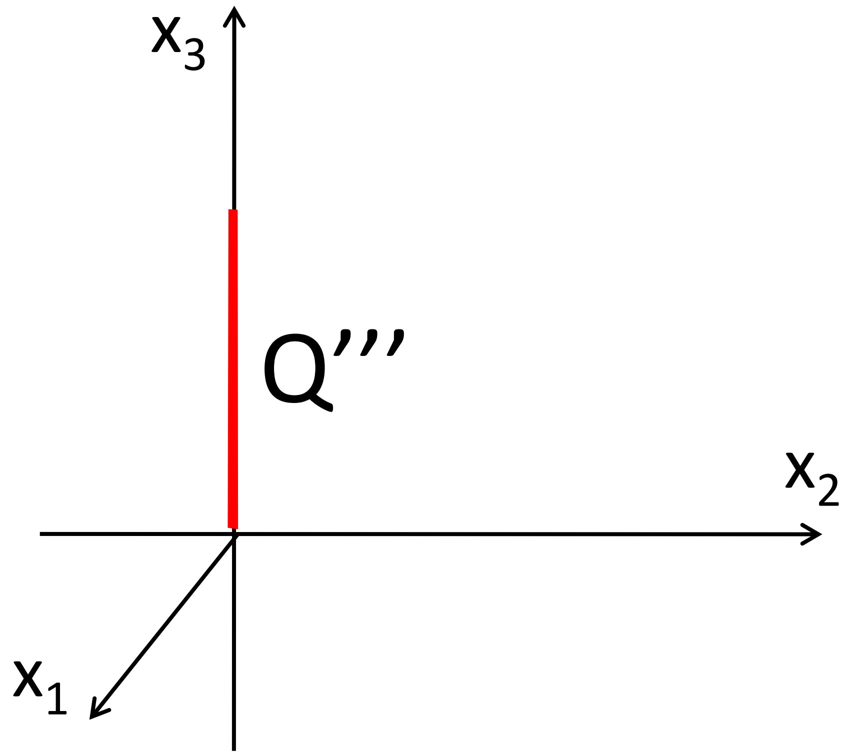
- Project Q onto coordinates $\{x_1, x_2\}$...
- **Fourier-Motzkin:** Q' is a polyhedron.
- Of course, the ordering of coordinates is irrevelant.

Elimination Example



- Of course, the ordering of coordinates is irrevelant.
- **Fourier-Motzkin:** Q'' is also a polyhedron.
- I can also apply Elimination twice…

Elimination Example



- **Fourier-Motzkin:** Q’’’ is also a polyhedron.

Projecting a Polyhedron Onto Some of its Coordinates

- **Lemma:** Given a polyhedron $Q \subseteq \mathbb{R}^n$.
  Let $S=\{s_1,\ldots,s_k\} \subseteq \{1,\ldots,n\}$ be any subset of the coordinates.
  Let $Q_S = \{ (x_{s1},\ldots,x_{sk}) : x \in Q \} \subseteq \mathbb{R}^k$.

  In other words, Q_S is projection of Q onto coordinates in S.

  Then Q_S is a polyhedron.

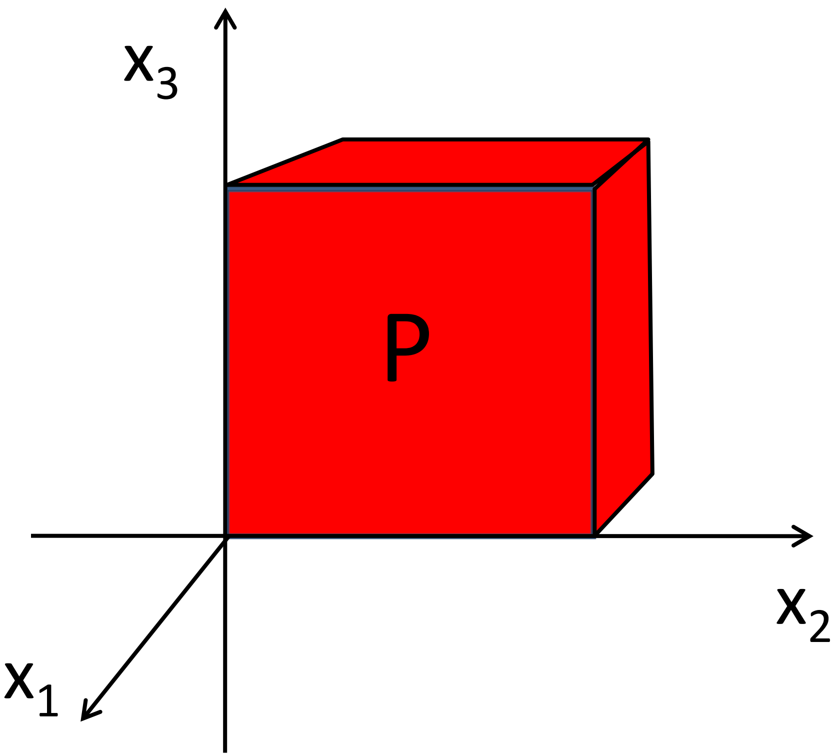
- **Proof:**
  Direct from Fourier-Motzkin Elimination.
  Just eliminate all coordinates not in S. ■

Linear Transformations of Polyhedra

- **Lemma:** Let P = $\{ x : Ax \leq b \} \subseteq \mathbb{R}^n$ be a polyhedron.
  Let M be any matrix of size p×n.
  Let Q = $\{ Mx : x \in P \} \subseteq \mathbb{R}^p$. Then Q is a polyhedron.

Let M =

1	0	0
-1	1	0
0	0	1

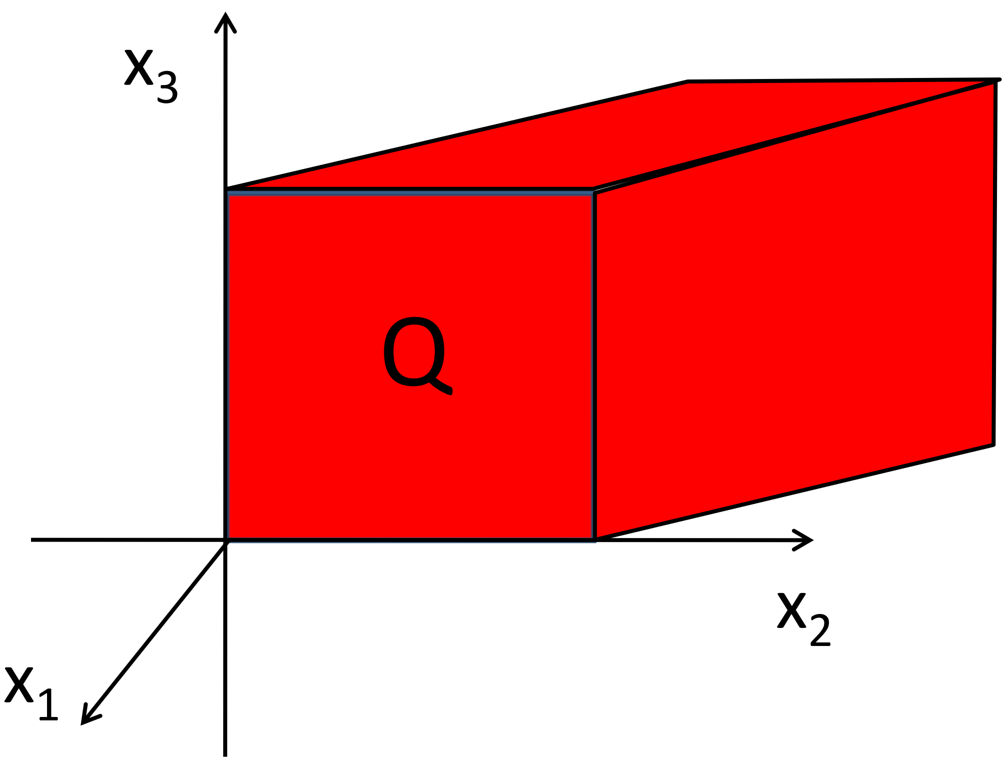


Linear Transformations of Polyhedra

- **Lemma:** Let P = $\{ x : Ax \leq b \} \subseteq \mathbb{R}^n$ be a polyhedron.
  Let M be any matrix of size p$\times$n.
  Let Q = $\{ Mx : x \in P \} \subseteq \mathbb{R}^p$. Then Q is a polyhedron.

  Geometrically obvious, but not easy to prove...

Let M = $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



Linear Transformations of Polyhedra

- **Lemma:** Let $P = \{ x : Ax \leq b \} \subseteq \mathbb{R}^n$ be a polyhedron.
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  Let $Q = \{ Mx : x \in P \} \subseteq \mathbb{R}^p$. Then Q is a polyhedron.

  Geometrically obvious, but not easy to prove…

  …unless you know Fourier-Motzkin Elimination!
- **Proof:**

  Let $P' = \{ (x,y) : Mx=y,\ Ax \leq b \} \subseteq \mathbb{R}^{n+p}$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$.

  P' is obviously a polyhedron.

  Note that Q is projection of P' onto y-coordinates.

  By previous lemma, Q is a polyhedron. ■

Convex Sets

- Let $S \subseteq \mathbb{R}^n$ be any set.
- **Recall:** S is convex if
$$\lambda x + (1-\lambda)y \in S \quad \forall x, y \in S \text{ and } \forall \lambda \in [0,1]$$
- **Proposition:** The intersection of any collection of convex sets is itself convex.
- **Proof:** Exercise.

Convex Combinations

- Let $S \subseteq \mathbb{R}^n$ be any set.
- **Definition:** A **convex combination** of points in S is any point $p = \sum_{i=1}^{k} \lambda_i s_i$ where k is finite and $s_i \in S \ \forall i, \ \lambda_i \geq 0 \ \forall i, \ \sum_{i=1}^{k} \lambda_i = 1$
- **Theorem:** (Carathéodory's Theorem) It suffices to take $k \leq n+1$.
- **Proof:** This was Assignment 2, Problem 4.

Convex Combinations

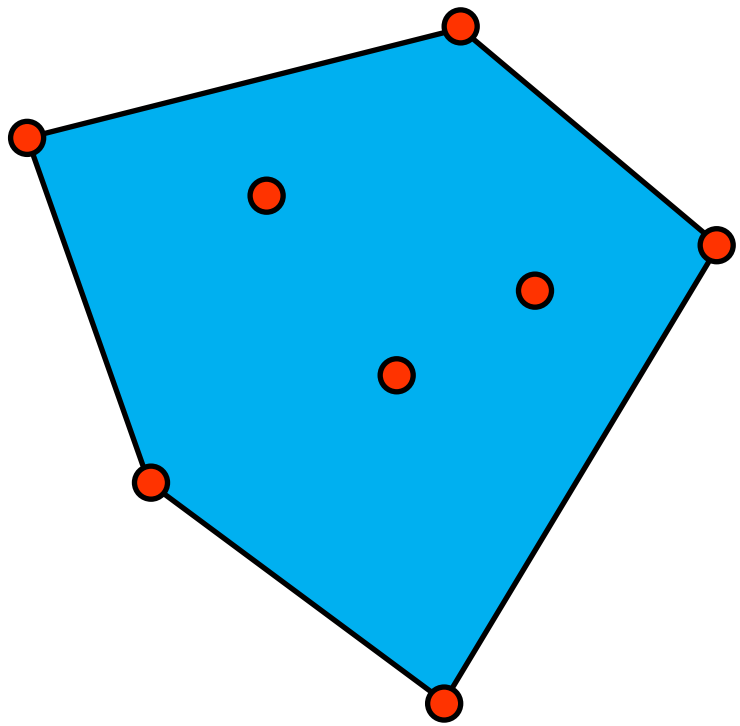
- **Theorem:** Let $S \subseteq \mathbb{R}^n$ be a **convex** set.
  Let comb(S) = { p : p is a convex comb. of points in S }.
  Then comb(S) = S.
- **Proof:** $S \subseteq$ comb(S) is trivial.

Convex Combinations

- **Theorem:** Let $S \subseteq \mathbb{R}^n$ be a **convex** set.
  Let comb(S) = { p : p is a convex comb. of points in S }.
  Then comb(S) = S.
- **Proof:** We'll show comb(S) $\subseteq$ S.
  Consider $p = \sum_{i=1}^{k} \lambda_i s_i$. Need to show p$\in$S.
  By induction on k. Trivial if k=1 or if λ_k=1.
  So assume k>1 and λ_k<1.
  Note: $p = \sum_{i=1}^{k} \lambda_i s_i = (1-\lambda_k)\left(\underbrace{\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} s_i}_{\text{Call this point p'}}\right) + \lambda_k s_k$
  Note $\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} = 1$ so p' is a convex combination of at most k-1 points in S. By induction, p'$\in$S.
  But $p = (1-\lambda_k)p' + \lambda_k s_k$, so p$\in$S since S is convex. ■

Convex Hulls

- Let $S \subseteq \mathbb{R}^n$ be any set.
- **Definition:** The **convex hull** of S, denoted conv(S), is the intersection of all convex sets containing S.



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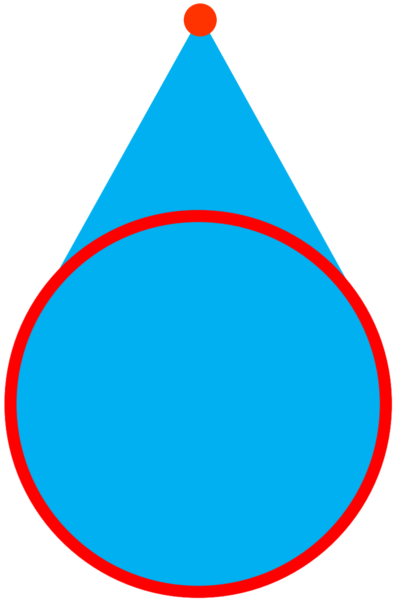

Convex Hulls

- Let $S \subseteq \mathbb{R}^n$ be any set.
- **Definition:** The **convex hull** of S, denoted conv(S), is the intersection of all convex sets containing S.
- **Claim:** conv(S) is itself convex.
- **Proof:**

  Follows from our earlier proposition
  "Intersection of convex sets is convex". ■

Convex Hulls

- Let $S \subseteq \mathbb{R}^n$ be any set.
- **Definition:** The **convex hull** of S, denoted conv(S), is the intersection of all convex sets containing S.
- Let comb(S) = { p : p is a convex comb. of points in S }.
- **Theorem:** conv(S)=comb(S).
- **Proof:** We'll show conv(S)$\subseteq$comb(S).

  Claim: comb(S) is convex.

  Proof: Consider $p = \sum_{i=1}^{k} \lambda_i s_i$ and $q = \sum_{j=1}^{\ell} \mu_j t_j$,
  where $\lambda \geq 0,\ \mu \geq 0,\ \sum_i \lambda_i = 1,\ \sum_j \mu_j = 1,\ s_i \in S,\ t_j \in S$

  So p,q$\in$comb(S). For $\alpha \in [0,1]$, consider

  $$\alpha p + (1-\alpha) q = \sum_{i=1}^{k} \alpha \lambda_i s_i + \sum_{j=1}^{\ell} (1-\alpha) \mu_j t_j$$

  Thus αp + (1-α)q $\in$ comb(S). $\square$

Convex Hulls

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  Claim: comb(S) is convex.

  Clearly comb(S) contains S.

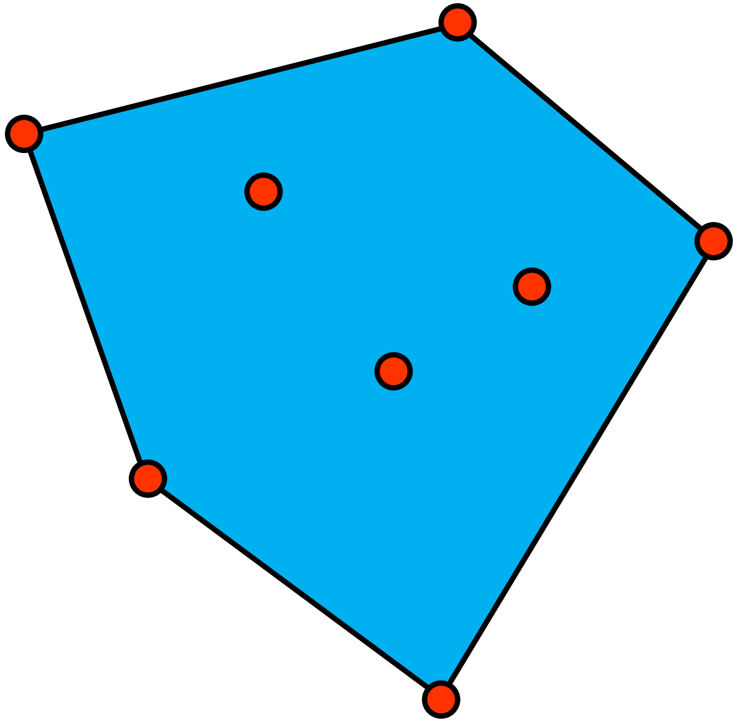
  But conv(S) is the intersection of **all** convex sets containing S, so conv(S)$\subseteq$comb(S). ■

Convex Hulls

- Let $S \subseteq \mathbb{R}^n$ be any set.
- **Definition:** The **convex hull** of S, denoted conv(S), is the intersection of all convex sets containing S.
- Let comb(S) = { p : p is a convex comb. of points in S }.
- **Theorem:** conv(S)=comb(S).
- **Proof:** Exercise: Show comb(S)$\subseteq$conv(S).

Convex Hulls of Finite Sets

- **Theorem:** Let $S=\{s_1,\ldots,s_k\}\subset\mathbb{R}^n$ be a **finite** set.
  Then conv(S) is a polyhedron.
  Geometrically obvious, but not easy to prove…



Convex Hulls of Finite Sets

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  Then conv(S) is a polyhedron.

  Geometrically obvious, but not easy to prove…

  …unless you know Fourier-Motzkin Elimination!
- **Proof:** Let M be the n×k matrix where $M_i = s_i$.

  By our previous theorem,

  $$\mathrm{conv}(S) = \{\, y \;:\; \exists x \text{ s.t. } Mx = y,\ \textstyle\sum_{i=1}^{k} x_i = 1, x \geq 0 \,\}$$

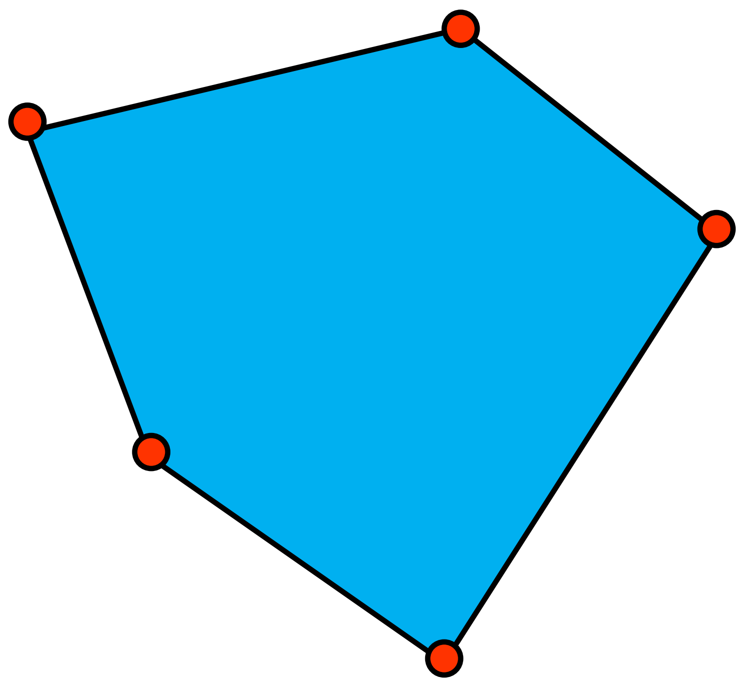
  But this is a projection of the polyhedron

  $$\{\, (x,y) \;:\; Mx = y,\ \textstyle\sum_{i=1}^{k} x_i = 1, x \geq 0 \,\}$$

  By our lemma on projections of polyhedra,
  conv(S) is also a polyhedron. ■

Polytopes & Convex Hulls

- Let $P \subset \mathbb{R}^n$ be a **polytope.** (i.e., a **bounded** polyhedron)
- Is P the convex hull of anything?
- Since P is convex, $P = \text{conv}(P)$. Too obvious…
- Maybe $P = \text{conv}($ extreme points of P $)$?



Polytopes & Convex Hulls

- **Theorem:** Let $P \subset \mathbb{R}^n$ be a non-empty polytope.

  Then P = conv(extreme points of P).

- **Proof:** First we prove conv(extreme points of P) $\subseteq$ P.

  We have:

  $$\text{conv(extreme points of } P \text{)} \underbrace{\subseteq}_{\text{Obvious}} \text{conv(} P \text{)} \underbrace{\subseteq}_{\text{Our earlier theorem proved comb(P)} \subseteq \text{P}} P.$$

Polytopes & Convex Hulls

- **Theorem:** Let $P \subset \mathbb{R}^n$ be a non-empty polytope.
  Then P = conv(extreme points of P).
- **Proof:** Now we prove $P \subseteq$ conv(extreme points of P).
  Let the extreme points be $\{v_1, \ldots, v_k\}$. (Finitely many!)
  Suppose $\exists b \in P \setminus$ conv(extreme points of P).
  Then the following system has no solution:
  $$\sum_{i=1}^{k} v_i \lambda_i = b$$
  $$\sum_{i=1}^{k} \lambda_i = 1$$
  $$\lambda_i \geq 0 \ \forall i$$
  By Farkas' lemma, $\exists u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ s.t.
  $$u^{\mathsf{T}} v_i + \alpha \geq 0 \ \forall i \qquad \text{and} \qquad u^{\mathsf{T}} b + \alpha < 0$$
  So $-u^{\mathsf{T}}b > -u^{\mathsf{T}}v_i$ for every extreme point v_i of P.

Polytopes & Convex Hulls

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- **Proof:** Now we prove $P \subseteq$ conv(extreme points of P).
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  ...

  By Farkas' lemma, $\exists u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ s.t.

$$u^\mathsf{T} v_i + \alpha \geq 0 \;\forall i \qquad \text{and} \qquad u^\mathsf{T} b + \alpha < 0$$

  So $-u^\mathsf{T}b > -u^\mathsf{T}v_i$ for every extreme point v_i of P.
  Consider the LP $\max \{ -u^\mathsf{T}x : x \in P \}$.
  It is not unbounded, since P is bounded.
  Its optimal value is **not attained at an extreme point**.
  This is a contradiction. ■

Generalizations

- **Theorem:**
  Every polytope in $\mathbb{R}^n$ is the convex hull of its extreme points.

- **Theorem:** [Minkowski 1911]
  Every compact, convex set in $\mathbb{R}^n$ is the convex hull of its extreme points.

- **Theorem:** [Krein & Milman 1940]
  Every compact convex subset of a locally convex Hausdorff linear space is the closed convex hull of its extreme points.

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