## C\&O 355 <br> Lecture 11

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## Topics

- The Ellipsoid Method
- Covering Half-Ellipsoids by Ellipsoids


## Our Hero


"Intelligence gathered by this and other governments leaves no doubt that the Iraq regime continues to possess and conceal some of the most lethal weapons ever devised" George W. Bush, 3/18/2003

## WMD in Iraq


"We are learning more as we interrogate or have discussions with Iraqi scientists and people within the Iraqi structure, that perhaps he destroyed some, perhaps he dispersed some. And so we will find them." George W. Bush, 4/24/2003

## Finding WMD

- UN have a satellite with a WMD detector
- The detector scans a round region of the earth
- It can compare two halves of the region, and decide which half is "more likely" to have WMD



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## Finding WMD

- It continues by rescanning the "more likely" half
- If region is so small that it obviously contains no WMD, then conclude: Iraq has no WMD
"No one was more surprised than I that we didn't find [WMDs]." U.S. General Tommy Franks, 12/2/2005



## Generalization to Higher Dimensions



Leonid Khachiyan
Even smarter than George W. Bush!

## The Ellipsoid Method

- Want to find $x \in P$
- Have ellipsoid $E(M, z) \supseteq P$
- If $z \notin P$ then it violates a constraint " $a_{i}{ }^{\top} x \leq b_{i}$ "
- So $P \subseteq\left\{x: a_{i}^{\top} x \leq a_{i}^{\top} z\right\}$
- So $P \subseteq E(M, z) \cap\left\{x: a_{i}^{\top} x \leq a_{i}^{\top} z\right\}$



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- So $P \subseteq E(M, z) \cap\left\{x: a_{i}^{\top} x \leq a_{i}^{\top} z\right\}$
- Let $E\left(M^{\prime}, z^{\prime}\right)$ be ellipsoid covering $E(M, z) \cap\left\{x: a_{i}^{\top} x \leq a_{i}^{\top} z\right\}$
- Repeat...



## The Ellipsoid Method

- Input: A polytope $P=\{A x \leq b\}$ and $K$ and $k$. (e.g., $P=W M D)$
- Output: A point $x \in P$, or announce " $P$ is empty"

Let $E(M, z)$ be an ellipsoid s.t. $P \subseteq E(M, z)$ If vol $E(M, z)<$ vol $B(0, r)$ then Halt: " $P$ is empty" If $z \in P$, Halt: " $z \in P$ "
Else

## How to find this?

Let " $a_{i}^{\top} x \leq b_{i}$ " be a constraint of $P$ violated by $z$ (i.e., $a_{i}^{\top} z>b_{i}$ ) Let $H=\left\{x: a_{i}^{\top} x \leq a_{i}^{\top} z\right\} \quad$ (so $\left.P \subseteq E(M, z) \cap H\right)$ Let $E\left(M^{\prime}, z^{\prime}\right)$ be an ellipsoid covering $E(M, z) \cap H$ Set $M \leftarrow M^{\prime}$ and $z \leftarrow z^{\prime}$ and go back to Start

- Notation: Let $\mathrm{B}(\mathrm{z}, \mathrm{k})=$ ball of radius k around point $\mathrm{z} \in \mathbb{R}^{\mathrm{n}}$
- Assumptions:
"The WMD is in Iraq": $\exists \mathrm{K}>0$ such that $\mathrm{P} \subseteq \mathrm{B}(0, \mathrm{~K})$ "WMD bigger than cow": If $\mathrm{P} \neq \emptyset$ then $\exists \mathrm{k}>0, \mathrm{z} \in \mathbb{R}^{\mathrm{n}}$ s.t. $\mathrm{B}(\mathrm{z}, \mathrm{k}) \subseteq \mathrm{P}$


## Covering Half-ellipsoids by Ellipsoids



- Let $E$ be an ellipsoid centered at $z$
- Let $H_{a}=\left\{x: a^{\top} x \geq a^{\top} z\right\}$
- Find a small ellipsoid E' that covers $E \cap H_{a}$

Use our solution for hemispheres!



- Find a linear transformation $T$ that
- Maps B to E
- Maps H to Ha
- Maps B' to E'


## Householder Reflection



- Want linear map that reflects points in hyperplane orthogonal to v
- Vectors orthogonal to v should be unchanged
- The vector v should change sign

