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RESEARCH ARTICLE

The use of intensity-dependent weight functions to "Weberize" L^2-based methods of signal and image approximation

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Abstract
We consider the problem of modifying L^2-based approximations so that they "conform" in a better way to Weber's model of perception: Given a greyscale background intensity $I > 0$, the minimum change in intensity ΔI perceived by the human visual system is $\Delta I / I^a = C$, where $a > 0$ and $C > 0$ are constants. A "Weberized distance" between two image functions u and v should tolerate greater (lesser) differences over regions in which they assume higher (lower) intensity values in a manner consistent with the above formula. In this paper, we Weberize the L^2 metric by inserting an intensity-dependent weight function into its integral. The weight function will depend on the exponent a so that Weber's model is accommodated for all $a > 0$. We also define the "best Weberized approximation" of a function and also prove the existence and uniqueness of such an approximation.

Keywords Weber model of perception · range-dependent weight functions · Weberized image metrics · best Weberized approximation

1 Introduction

In this paper, we present a method of "Weberizing" L^2-based methods of signal and image approximation by modifying the usual L^2 metric in such a way that it "conforms" as much as possible to Weber's model of perception. (We shall define the term "conform" later in the paper.) The term "Weberized" has appeared in several papers which have incorporated Weber's model into classical image processing methods, namely, total variation (TV) restoration Shen (2003) and Mumford-Shah segmentation Shen and Jung (2006). Our methods of Weberization,

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however, are quite unique in that the metrics we produce can be used in a wide variety of applications, e.g., best approximation, denoising and other image restoration problems. The novelty of our approach lies in the fact that we are performing “range-based” approximation: The nonuniformity in the metric is based on the range values of the functions. The primary motivation for our research lies in the fact that the well known and very commonly used mean squared error (MSE) and peak signal-to-noise ratio (PSNR) – examples of L^2-based distance measures – perform poorly in terms of perceptual image quality Girod (1993); Wang and Bovik (2009).

By Weber’s model of perception we mean the following: Given a greyscale background intensity $I > 0$, the minimum change in intensity ΔI perceived by the human visual system (HVS) is related to I as follows,

$$\frac{\Delta I}{I^a} = C, \tag{1}$$

where $a > 0$ and C is constant, or at least roughly constant over a significant range of intensities I. The case $a = 1$ corresponds to the standard Weber model – often known as “Weber’s Law” – which has been employed in practically all applications Wandell (1995). Even in this standard case, different values of the constant C may hold over different regions of intensity space Li et al. (2014). There are also situations in which other values of the exponent a, in particular $a = 0.5$, may apply – see, for example, Michon (1966). These complications are well beyond the scope of this paper. Here we focus on the model in Eq. (1) with the understanding that our proposed method can be adapted to conform to more complicated behaviours.

In our previous papers, we have referred to Eq. (1) as a **generalized** Weber model of perception in order to distinguish it from the special case $a = 1$. Here, for the sake of simplicity, the words “generalization” or “generalized” will be omitted: Unless otherwise indicated, “Weber’s model” will refer to Eq. (1) for $a > 0$.

The basis of our entire program to Weberize metrics is as follows. Eq. (1) implies that the HVS will be less (more) sensitive to a given change in intensity ΔI in regions of an image at which the local image intensity $I(x)$ is high (low). As such, a Weberized distance between two functions u and v should tolerate greater (lesser) differences over regions in which they assume higher (lower) intensity values. The degree of toleration as I varies will be determined by the exponent a.

In Kowalik-Urbaniak et al. (2014), the L^2 metric was Weberized for the special case $a = 1$, Weber’s standard model in Eq. (1), by the insertion of an intensity-dependent weight function into integral. Even though seemingly *ad hoc*, our method produced a distance function which could be viewed as accommodating Weber’s standard model, $a = 1$, in terms of approximations of piecewise constant functions (see Example 1 of Kowalik-Urbaniak et al. (2014)). Its use in the approximation of functions and images is discussed in much greater detail, through many examples, in Kowalik-Urbaniak (2014).

In this paper, we show that a simple modification of the weight function used in Kowalik-Urbaniak et al. (2014); Kowalik-Urbaniak (2014) produces Weberized distance functions that accommodate the general case $a > 0$. We also provide the mathematical basis for best approximation in terms of all Weberized distance functions.

Let us also mention that the approach outlined in this paper is in no way restricted to Weber's model of perception. Other forms of the intensity-based weight function, according to need or interest, may be employed. A few comments regarding the more general case are presented in the final section of this paper.

Finally, we should draw the reader's attention to another method that has been devised to Weberize L^p-based metrics, namely, the use of appropriate measures that are supported on the (positive) range space $\mathbb{R}_g = [A, B]$ of functions to reformulate the integrals which normally define the L^p distance between two functions. (When Lebesgue measure is used on $\mathbb{R}_g$, the result is the usual L^p distance integral.) Some of the main ideas of such "range-based" or "intensity-dependent" measures appeared in Li et al. (2018) and Li et al. (2019) and are discussed in much more detail in Li (2020). Some of the mathematics used in these works has been used in this paper to provide the mathematical basis of the best Weberized approximation problem.

2 Mathematical preliminaries

The basic mathematical ingredients of our formalism are listed below.

1. *The base (or pixel) space* $X \subset \mathbb{R}^n$ on which our signals/images are supported. Here, without loss of generality since our discussion is purely theoretical, we simply consider the one-dimensional case $X = [0, 1] \subset \mathbb{R}$. The extension to higher-dimensional cases is rather straightforward. We also mention that our discussion easily extends to the discrete case encountered in practice, where X is comprised of **pixels** or **voxels** – for example, $X = \{1, 2, \cdots, n_1\} \times \{1, 2, \cdots, n_2\}$, in which case the images are $n_1 \times n_2$ arrays of numbers.
2. *The (greyscale or intensity) range space* For an $A > 0, \mathbb{R}_g = [A, B]$, where $B < \infty$. Once again, our discussion can be extended to the discrete case, e.g., N bit-per-pixel digital images for which $\mathbb{R}_g = \{0, 1, \cdots, 2^N - 1\}$.
3. *Set of (signal/image) functions* $\mathcal{F}(X) = \{u : X \to \mathbb{R}_g \mid u \text{ measurable } \}$. From our definition of the greyscale range $\mathbb{R}_g$, $u \in \mathcal{F}(X)$ is positive and bounded almost everywhere, i.e., $0 < A \leq u(x) \leq B < \infty$ for almost every $x \in X$. A consequence of this boundedness is that $\mathcal{F}(X) \subset L^p(X)$ for all $p \geq 1$, where the $L^p(X)$ function spaces are defined in the usual way. In this paper, we shall be using the L^2 metric on X,

$$d_2(u, v) = \|u - v\|_2 = \left[\int_X [u(x) - v(x)]^2 \, dx \right]^{1/2}, \quad u, v \in L^2(X). \tag{2}$$

It is important to mention that because of the restrictions on the range values, $\mathcal{F}(X)$ is not a linear space: Given $u, v \in \mathcal{F}(X)$, it does not follow that $c_1 u + c_2 v \in \mathcal{F}(X)$ for all $c_1, c_2 \in \mathbb{R}$. Moreover, the zero function is not an element of $\mathcal{F}(X)$. As will be seen below, the restriction to nonnegative range values is necessary because of the form of the weight functions used in our Weberized distance integrals.

Theorem 1 *The set $\mathcal{F}(X)$ is bounded, closed and convex.*

Proof

(a) Since $0 < A \leq u(x) \leq B < \infty$ for a.e. $x \in X$, it follows that $\|u\|_2 \leq m(X)B$, where $m(X)$ denotes the Lebesgue measure of X. Hence the set $\mathcal{F}(X)$ is bounded.
(b) To show that $\mathcal{F}(X)$ is closed, let $\{u_n\} \subset \mathcal{F}(X)$ be a convergent sequence with limit u, i.e., $d_2(u_n, u) \to 0$ as $n \to \infty$. This implies that there exists a subsequence $\{u_{n_k}\}$ which converges to u pointwise. Since $\{u_{n_k}\} \subset \mathcal{F}(X)$, it follows that for all n_k, $A \leq u_{n_k}(x) \leq B$ for a.e. $x \in X$. Taking the pointwise limit as $k \to \infty$ yields $A \leq u(x) \leq B$ for a.e. $x \in X$, which implies that $u \in \mathcal{F}(X)$. Therefore $\mathcal{F}(X)$ is closed.
(c) To show that $\mathcal{F}(X)$ is convex, let $u, v \in \mathcal{F}(X)$. Then for any $\lambda \in [0, 1]$,

$$\lambda A \leq \lambda u(x) \leq \lambda B \quad \text{and} \quad (1-\lambda)A \leq (1-\lambda)v(x) \leq (1-\lambda)B \tag{3}$$

for a.e. $x \in X$. Adding the two inequalities yields,

$$A \leq \lambda u(x) + (1-\lambda)v(x) \leq B \quad \text{for a.e.} \quad x \in X\,, \tag{4}$$

which implies the convexity of $\mathcal{F}(X)$.

□

3 Intensity-dependent weight functions which produce Weberized distance functions

In the usual L^2-based methods of approximation employed in signal and image processing, the L^2 metric in Eq. (2) is used. This metric, and indeed all other L^p- based metrics, $p \geq 1$, are not adapted to Weber's model of perception since they involve integrations over appropriate powers of intensity differences, $|u(x) - v(x)|$, with no consideration of the magnitudes of $u(x)$ or $v(x)$.

Recall that a Weberized distance between two functions u and v should tolerate greater (lesser) differences over regions in which they assume higher (lower) intensity values. One way to accomplish such a "Weberization" of the L^2 metric is to insert an intensity-dependent weight function in the integrand of Eq. (2). The general form or forms of such a weight function is an open problem worthy of exploration, the discussion of which is beyond the scope of this paper. Here we simply mention that one possibility is to consider weight functions which are dependent upon one or both of the intensities of the image functions $u(x)$ and $v(x)$. The resulting weighted L^2 metric may be written in the generic form,

$$d_{2W}(u, v) = \left[\int_X g(u(x), v(x))[u(x) - v(x)]^2\, dx\right]^{1/2}, \tag{5}$$

where $g : \mathbb{R}_g \times \mathbb{R}_g \to \mathbb{R}_+$ denotes the intensity-dependent weight function.

This, of course, leads to the question of properties that should be satisfied by the weight function g as well as possible functional forms that it could assume. As discussed in Kowalik-Urbaniak et al. (2014), for d_{2W} to satisfy the properties of a metric, $g(u,v)$ should be symmetric in its arguments, i.e., $g(u,v) = g(v,u)$. Furthermore, for the d_{2W} to be Weberized, it is desirable that $g(u,v)$ be decreasing in each of its arguments. These requirements are satisfied by the family of weight functions, $g(u,v) = |uv|^{-q}$, where $q > 0$, resulting in weighted L^2 metrics of the form,

$$d_{2W,q}(u,v) = \left[\int_X \frac{1}{u(x)^q v(x)^q} [u(x) - v(x)]^2 \, dx \right]^{1/2}, \quad q > 0 \quad u,v \in \mathcal{F}(X). \tag{6}$$

The appearance of both functions in the denominator, however, complicates matters when we consider the approximation problem $u \simeq v$ where v is a linear combination of basis functions – see Sect. 4. As discussed in Kowalik-Urbaniak et al. (2014), a simplification is achieved if we consider g to be a function of only one intensity function. In that paper, for the special case $a = 1$, we considered two unsymmetric weight functions, $g_1(u(x), v(x)) = u(x)^{-2}$ and $g_2(u(x), v(x)) = v(x)^{-2}$ to produce two integral distance functions, denoted as $\Delta(u,v)$ and $\Delta(v,u)$, either of which could be bounded above and below by the other. The distance function $\Delta(u,v)$, used for the approximation problem $u \simeq v$, was shown to conform to Weber's model for $a = 1$. Referring the reader to Kowalik-Urbaniak et al. (2014) for details, we now proceed to the first major contribution of this paper, the construction of distance functions which conform to Weber's model for any $a > 0$, based upon a rather straightforward extension of the result in Kowalik-Urbaniak et al. (2014).

First of all, for an $a > 0$, consider the nonsymmetric weight function $g_1(u(x), v(x)) = u(x)^{-2a}$ so that the weighted L^2 distance in Eq. (5) becomes

$$\Delta_a(u,v) = \left[\int_X \frac{1}{u(x)^{2a}} [u(x) - v(x)]^2 \, dx \right]^{1/2}. \tag{7}$$

Now consider the nonsymmetric weight function $g_2(u(x), v(x)) = v(x)^{-2a}$ so that the weighted L^2 distance in Eq. (5) becomes

$$\Delta_a(v,u) = \left[\int_X \frac{1}{v(x)^{2a}} [u(x) - v(x)]^2 \, dx \right]^{1/2}. \tag{8}$$

Note that in general, $\Delta_a(u,v) \neq \Delta_a(v,u)$, which implies that Δ_a is not a metric in the strict mathematical sense of the term. This is once again the price paid for employing weight functions $g(u,v)$ which are not symmetric in the functions u and v. We could, of course, employ both $\Delta(u,v)$ and $\Delta(v,u)$ to construct a *bona fide* metric but this will not be necessary because of the following result, which may be easily derived from the fact that u and v are in $\mathcal{F}(X)$.

Theorem 2 *Let $u,v \in \mathcal{F}(X)$, once again recalling the assumption that the greyscale range $\mathbb{R}_g = [A,B]$ is bounded away from zero, i.e., $A > 0$. Then for $\Delta_a(u,v)$ and $\Delta_a(v,u)$ defined in Eqs.* (7) *and* (8) *respectively,*

$$\frac{1}{B^a} d_2(u, v) \leq \left\{ \begin{array}{c} \Delta_a(u, v) \\ \Delta_a(v, u) \end{array} \right\} \leq \frac{1}{A^a} d_2(u, v), \tag{9}$$

where d_2 *denotes the* L^2 *metric – see Eq.* (2) *– from which it follows that*

$$\left(\frac{A}{B}\right)^a \Delta_a(u, v) \leq \Delta_a(v, u) \leq \left(\frac{B}{A}\right)^a \Delta_a(u, v). \tag{10}$$

Comments on Theorem 2:

1. From Eq. (10) it is sufficient to consider only one of the two distance functions, $\Delta_a(u, v)$ or $\Delta_a(v, u)$, in any theoretical treatment or application. As in Kowalik-Urbaniak et al. (2014), we shall let u denote a reference function and v and approximation to it, in which case the error of the approximation $u \simeq v$ is given by $\Delta_a(u, v)$ in Eq. (7).
2. The distance functions, $\Delta_a(u, v)$ and $\Delta_a(v, u)$, and Theorem 1 above, are generalizations of the special case $a = 1$ examined in Kowalik-Urbaniak et al. (2014).
3. In the special case $a = 1$, the leftmost inequality in Eq. (10) is an improvement over the one which originally appeared in Kowalik-Urbaniak et al. (2014).

We now make the following important observation, a generalization of Example 1 in Kowalik-Urbaniak et al. (2014).

Example 1 Consider the "flat" reference image $u(x) = I$, where $I \in \mathbb{R}_g$. For an $a > 0$, let $v(x) = I + \Delta I$ be the constant approximation to $u(x)$, where $\Delta I = CI^a > 0$ is the minimum perceived change in intensity corresponding to I, according to Weber's model in Eq. (1). The L^2 distance between u and v is

$$d_2(u, v) = K \cdot \Delta I = KCI^a, \quad \text{where } K = \left[\int_X dx\right]^{1/2}. \tag{11}$$

A simple computation shows that the weighted L^2 distance in Eq. (7) is

$$\Delta_a(u, v) = K\frac{\Delta I}{I^a} = KC. \tag{12}$$

Note that the L^2 distance in Eq. (11) increases with the intensity level I which is expected since ΔI increases with I. However, the weighted L^2 distance in Eq. (12) remains constant. As such, we claim that $\Delta_a(u, v)$ **accommodates, or "conforms to", Weber's model of perception for** $a > 0$: Perturbations ΔI of image intensities I according to Eq. (1) yield the same distance measure, independent of I.

4 Best approximation in terms of Weberized distance functions $\Delta_a(u, v)$

In what follows, we let $\{\phi_k\}_{k=1}^{\infty}$ denote a set of real-valued functions that form a complete basis of $L^2(X)$. Now let $u \in \mathcal{F}(X) \subset L^2(X)$ denote the reference signal/image function to be approximated. We are interested in best approximations to u having the form,

$$u \simeq v_n = \sum_{k=1}^{n} c_k \phi_k , \tag{13}$$

for $n \geq 1$. As is well known, in the special case that the $\{\phi_k\}$ functions comprise an orthonormal basis, the best L^2 approximation to u in the subspace $V_n = \text{ span } \{\phi_1, \cdots, \phi_n\}$ is the minimizer of the L^2 distance $\|u - v\|_2$ over all $v \in V_n$. It is uniquely defined by the **Fourier coefficients** of u in the $\{\phi_k\}$ basis, i.e.,

$$c_k = \langle u, \phi_k \rangle = \int_X u(x)\phi_k(x)\, dx, \quad 1 \leq k \leq n . \tag{14}$$

Here, however, we wish to find the **"best Weberized" approximation** to u, i.e., for a given $n \geq 1$ and $a > 0$, find the expansion in Eq. (13) which minimizes the weighted L^2 distance $\Delta_a(u, v_n)$.

Technically, it should be guaranteed that the approximation function $v_n(x)$ in Eq. (13) lies in the space $\mathcal{F}(X)$, i.e., $A \leq v_n(x) \leq B$ for a.e. $x \in X$. To address this complication we define, for each $n \geq 1$, the following feasible parameter set $C_n \in \mathbb{R}^n$,

$$C_n = \left\{ \mathbf{c} = (c_1, \cdots, c_n) \in \mathbb{R}^n \;\middle|\; v_n(x) = \sum_{k=1}^{n} c_k \phi_k(x) \in \mathcal{F}(X) \right\} . \tag{15}$$

By definition, $C_n \subset C_{n+1}$ for $n \geq 1$. The subsets C_n depend on the choice of basis set $\{\phi_k\}_{k=1}^{\infty}$. In what follows, we assume that the ϕ_k functions satisfy some rather generic conditions:

1. $\phi_1(x) = K > 0$, a constant, for all $x \in X$.
2. There exists a constant $M > 0$, such that $|\phi_k(x)| \leq M$ for all $k \geq 2$. (In the case of an unnormalized sine/cosine basis, $M = 1$.)

Theorem 3 *For all $n \geq 1$, the subsets $C_n \subset \mathbb{R}^n$ are compact and convex.*

Proof For any $n \geq 1$, we have the condition that $A \leq v_n(x) \leq B$ for all $x \in X$, or

$$A \leq \sum_{k=1}^{n} c_k \phi_k(x) \leq B . \tag{16}$$

We start with the case $n = 1$. From the assumption that $\phi_1(x) = K$, it follows that

$$A \le c_1 K \le B \quad \Longrightarrow \quad \frac{A}{K} \le c_1 \le \frac{B}{K}. \tag{17}$$

Therefore $C_1 = [A/K, B/K]$ which is compact.

We now consider the case $n = 2$, i.e.,

$$A \le c_1\phi_1(x) + c_2\phi_2(x) \le B. \tag{18}$$

If we set $c_2 = 0$, the above inequality becomes (17), implying that C_2 lies between the hyperplanes $c_1 = A/K$ and $c_1 = B/K$. We now examine nonzero values of c_2. First of all, the minimum value of $c_1\phi_1(x)$ is A, in which case the inequality in (18) becomes

$$0 \le c_2\phi_2(x) \le B - A. \tag{19}$$

Now consider the case when $c_2 > 0$. Since the maximum value of $\phi_2(x)$ is M, it follows from (19) that

$$0 \le c_2 \le \frac{B - A}{M}. \tag{20}$$

Now consider the case when $c_2 < 0$. Since the minimum value of $\phi_2(x)$ is $-M$, it follows from (19) that

$$0 \ge c_2 \ge \frac{A - B}{M}. \tag{21}$$

In summary, we have that

$$\frac{A - B}{M} \le c_2 \le \frac{B - A}{M}. \tag{22}$$

These inequalities are also obtained when one considers the case where $c_1\phi_1(x)$ achieves its maximum value of B in (18).

We may now proceed in a recursive manner. Consider the inequalities satisfied by v_{n+1}, written as follows,

$$A \le [c_1\phi_1(x) + \cdots c_n\phi_n(x)] + c_{n+1}\phi_{n+1}(x) \le B. \tag{23}$$

The minimum and maximum values of the expression in brackets, namely $v_n(x)$, are A and B, respectively. As in the case $n = 2$, this leads to the following result,

$$\frac{A - B}{M} \le c_{n+1} \le \frac{B - A}{M}. \tag{24}$$

This shows that the above inequality holds for all c_n, $n \ge 2$. Therefore, the sets C_n are bounded for $n \ge 1$.

To show that C_n is closed, let $\{\mathbf{c}_j\}_{j=1}^{\infty} \subset C_n$ be a convergent sequence with limit $\mathbf{d}$, i.e., for $1 \le k \le n$, $\lim_{j\to\infty} c_{jk} = d_k$. Now define

$$f_j(x) = \sum_{k=1}^{n} c_{jk}\phi_k(x)\,, \quad \forall j \ge 1\,, \tag{25}$$

and

$$f(x) = \sum_{k=1}^{n} d_k\phi_k(x)\,. \tag{26}$$

Since $\{\mathbf{c}_j\}_{j=1}^{\infty} \subset C_n$, we have that

$$A \le f_j(x) \le B\,, \quad \forall\, j \ge 1\,. \tag{27}$$

Since $\mathbf{c}_j \to \mathbf{d}$ as $j \to \infty$,

$$\lim_{j\to\infty}[f_j(x) - f(x)] = \lim_{j\to\infty}\sum_{k=1}^{n}(c_{jk} - d_k)\phi_k(x) = 0 \quad \text{a.e.} \;\; x \in X\,. \tag{28}$$

Therefore, taking the limit $j \to \infty$ in Eq. (27), it follows that

$$A \le f(x) \le B \quad \text{a.e.} \;\; x \in X\,. \tag{29}$$

This implies that $f \in C_n$, thus proving that C_n is closed. Since $C_n \subset \mathbb{R}^n$ is both closed and bounded, it is compact (Bolzano-Weierstrass).

Finally, to show that C_n is convex, let $\mathbf{c}$ and $\mathbf{d}$ be any two elements of C_n, implying that

$$A \le \sum_{k=1}^{n} c_k\phi_k(x) \le B \quad \text{and} \quad A \le \sum_{k=1}^{n} d_k\phi_k(x) \le B \quad \text{a.e.} \;\; x \in X\,. \tag{30}$$

Now for any $\lambda \in [0, 1]$, multiply the first set of inequalities by λ and the second by $(1 - \lambda)$ and then add them to obtain the following inequality,

$$A \le \sum_{k=1}^{n}[\lambda c_k + (1 - \lambda)d_k]\phi_k(x) \le B \quad \text{a.e.} \;\; x \in X\,. \tag{31}$$

This implies that $\mathbf{b} = \lambda\mathbf{c} + (1 - \lambda)\mathbf{d} \in C_n$ for any $\lambda \in [0, 1]$ which, in turn, implies that C_n is convex. □

We now use the subsets $C_n \subset \mathbb{R}^n$ to define the following subsets $S_n \subset \mathcal{F}(X)$ for $n \ge 1$,

$$S_n = \left\{ v : X \to \mathbb{R}_g \;\middle|\; v(x) = \sum_{k=1}^{n} c_k\phi_k(x) \quad \text{for} \quad \mathbf{c} \in C_n \right\}. \tag{32}$$

Definition 1 For each $n \ge 1$, let $\psi_n : C_n \to S_n$ be defined as follows. For a given $\mathbf{a} = (a_1, \cdots, a_n) \in C_n$, define

$$v_n(x) = \sum_{k=1}^{n} a_n \phi_n(x) \quad \text{for all} \quad x \in X\,. \tag{33}$$

By construction, $v_n \in S_n$. We denote v_n as $\psi_n(\mathbf{a})$.

Theorem 4 *For each $n \geq 1$, the mapping $\psi_n : C_n \to S_n$ is a homeomorphism.*

Proof Very briefly, the linear independence of the basis functions, $\{\phi_1, \cdots, \phi_n\}$, implies that ψ_n is a bijection. From the equivalence of the D_a and d_2 metrics in S_n, we may use the d_2 metric to easily show that for $\mathbf{a}, \mathbf{b} \in C_n$,

$$d_2(\psi_n(\mathbf{a}), \psi_n(\mathbf{b})) = [(\mathbf{a} - \mathbf{b})^T \mathbf{S}(\mathbf{a} - \mathbf{b})]^{1/2}\,, \tag{34}$$

where the elements of the symmetric (and nonsingular) overlap matrix $\mathbf{S}$ are $s_{ij} = \langle \phi_i, \phi_j \rangle$. Continuity of ψ_n and ψ_n^{-1} follows. □

Corollary *For each $n \geq 1$, the subset $S_n \subset \mathcal{F}(X)$ is compact and convex.*

The S_n will play the role of approximation spaces in our best Weberized approximation problem. Let us recall that for any $n \geq 1$, the subset $S_n \subset \mathcal{F}(X)$ is **not** a linear space because of the restrictions involved in the definition of the set $\mathcal{F}(X)$. The S_n are clearly subsets – but not subspaces – of the approximation spaces $V_n = \text{ span } \{\phi_1, \cdots, \phi_n\}$ that are normally used in best L^2 approximation.

Our "best Weberized" approximation problem will now be defined as follows.

Definition 2 For a given $a > 0$ and a given $n \geq 1$, we define the **best Weberized approximation** $v_n \in S_n$ to $u \in \mathcal{F}(X)$ as

$$v_n = \arg\min_{v \in S_n} \Delta_a(u, v)\,, \tag{35}$$

where the weighted L^2 distance function $\Delta_a(u, v)$ corresponding to Weber's model for $a > 0$ is defined in Eq. (7). The existence and uniqueness of this best approximation will be established below.

The existence of a solution v_n to Eq. (35) is guaranteed by the following result.

Theorem 5 *For a fixed $u \in \mathcal{F}(X)$ and $a > 0$, consider the function $h : \mathcal{F}(X) \to \mathbb{R}$ defined as $h(v) = \Delta_a(u, v)$ for $v \in \mathcal{F}(X)$. Then:*

1. *$h(v)$ is a continuous function of v.*
2. *$h(v)$ is a convex function of v.*

Proof From Eq. (7), $h(v) = \|u^{-a}(u - v)\|_2$, where $\|\cdot\|_2$ denotes the standard L^2 norm – see Eq. (2). The continuity and convexity of $h(v)$ trivially follow from, respectively, the continuity and convexity of the L^2 norm.

For each $n \geq 1$, it follows from the continuity of $h : \mathcal{F}(X) \to \mathbb{R}$ and the compactness of S_n that a solution to Eq. (35) exists.

We now establish the uniqueness of the solution v_n to Eq. (35). We use the facts that (i) $\mathcal{F}(X) \subset L^2(X)$ and (ii) $L^2(X)$ is a **strictly normed space** Lebedev et al. (2003), i.e., if

$$\|x + y\|_2 = \|x\|_2 + \|y\|_2 \,, \quad x \neq 0 \,, \tag{36}$$

then $y = \lambda x$ and $\lambda \geq 0$, where $\|\cdot\|_2$ denotes the L^2 norm.

Theorem 6 *Let $u \in \mathcal{F}(X)$. Then for a given $n \geq 1$ and $a > 0$, the solution to Eq. (35) is unique.*

Proof If $u \in S_n$, then there is only one minimizer, $v_n = u$. Now suppose that $u \notin S_n$ and that there are two minimizers of $\Delta_a(u, v)$, namely $v_{n,1}, v_{n,2} \in S_n$ with $v_{n,1} \neq v_{n,2}$. Thus,

$$\Delta_a(u, v_{n,1}) = \Delta_a(u, v_{n,2}) = \min_{v \in S_n} \Delta_a(u, v) = d > 0 \,. \tag{37}$$

Recalling the definition of $h(v)$ in Theorem 6, we have that

$$h(v_{n,1}) = h(v_{n,2}) = d \,, \tag{38}$$

which may be expressed in terms of the L^2 norm as follows,

$$\|pu - pv_{n,1}\|_2 = \|pu - pv_{n,2}\|_2 = d > 0 \,, \tag{39}$$

where $p(x) = u(x)^{-a}$. Since S_n is convex, $w_n = \frac{1}{2}(v_{n,1} + v_{n,2}) \in S_n$ so that

$$\Delta_a(u, w_n) = h(w_n) = \|pu - pw_n\|_2 \geq d \,. \tag{40}$$

But

$$\begin{aligned} \|pu - pw_n\|_2 &= \left\| pu - \frac{1}{2}(pv_{n,1} + pv_{n,2}) \right\|_2 \\ &\leq \left\| \frac{1}{2}(pu - pv_{n,1}) \right\|_2 + \left\| \frac{1}{2}(pu - pv_{n,2}) \right\|_2 \\ &\leq \frac{1}{2} \|pu - pv_{n,1}\|_2 + \frac{1}{2} \|pu - pv_{n,2}\|_2 \\ &= d \,. \end{aligned} \tag{41}$$

From (40) and (41) it follows that $\|pu - pw_n\|_2 = d$ which may be expressed as follows,

$$\left\| pu - \frac{1}{2}(pv_{n,1} + pv_{n,2}) \right\|_2 = \left\| \frac{1}{2}(pu - pv_{n,1}) \right\|_2 + \left\| \frac{1}{2}(pu - pv_{n,2}) \right\|_2 . \tag{42}$$

Now using the fact that $L^2(X)$ is a strictly normed space and making the following identifications in Eq. (36),

$$x = \frac{1}{2}(pu - pv_{n,1}), \quad y = \frac{1}{2}(pu - pv_{n,2}), \tag{43}$$

it follows that $y = \lambda x$ for some $\lambda \geq 0$, i.e.,

$$pu - v_{n,1} = \lambda(pu - pv_{n,2}). \tag{44}$$

Taking norms, we have

$$\|pu - v_{n,1}\|_2 = \lambda\|pu - pv_{n,2}\|_2. \tag{45}$$

From Eq. (39), $\lambda = 1$ which, from Eq. (44), implies that $v_{n,1} = v_{n,2}$. This contradicts the original assumption that the two minimizers are unequal. Therefore there can be at most one minimizer of $\Delta_a(u, v)$ in S_n. □

For the practical problem of finding best approximations, it is more convenient to work with the squared distance function,

$$[\Delta_a(u, v_n)]^2 = \int_X g(x)\left[u(x) - \sum_{k=1}^{n} c_k\phi_k(x)\right]^2 dx =: f(\mathbf{c}), \tag{46}$$

where $\mathbf{c} = (c_1, c_2, \cdots, c_n)$. Here, the weight function is $g(x) = u(x)^{-2a}$ but the algebraic expressions presented below apply to any weight function $g(x)$.

The squared distance function $f(\mathbf{c})$ is a quadratic form in the expansion coefficients c_k, $1 \leq k \leq n$, i.e.,

$$f(\mathbf{c}) = \mathbf{c}^T\mathbf{A}\mathbf{c} + 2\mathbf{b}^T\mathbf{c} + d, \tag{47}$$

where the elements of the $n \times n$ matrix $\mathbf{A}$ and the n-vector $\mathbf{b}$ are given by

$$a_{ij} = \int_X g(x)\phi_i(x)\phi_j(x)\,dx, \quad b_j = \int_X g(x)u(x)\phi_j(x)\,dx, \quad 1 \leq i, j \leq n \tag{48}$$

and

$$d = \int_X g(x)u(x)^2\,dx. \tag{49}$$

The optimization problem in Eq. (35) may then be replaced by the following problem,

$$\mathbf{a} = (a_1, \cdots, a_n) = \arg\min_{\mathbf{c}\in C_n} f(\mathbf{c}), \tag{50}$$

the solution of which yields the best Weberized approximation $\psi_n(\mathbf{a}) = v_n \in S_n$,

$$v_n(x) = \sum_{k=1}^{n} a_k\phi_k(x). \tag{51}$$

Since $f(\mathbf{c})$ is a continuous function of its arguments $c_k, 1 \leq k \leq n$, it achieves a minimum value on the compact set S_n, implying the existence of a solution to (50). The uniqueness of this solution is guaranteed by Theorems 4 and 6.

If, for an $n \geq 1$, the minimum of $f(\mathbf{c})$ is achieved at an interior point of the feasible set S_n, the best approximation v_n can be found by solving the following linear system of equations,

$$\mathbf{A}\mathbf{c} = \mathbf{b}\,, \tag{52}$$

which results from the stationarity conditions,

$$\frac{\partial f}{\partial c_k} = 0\,, \quad 1 \leq k \leq n\,. \tag{53}$$

(Note that in the special case $g(x) = 1$, the matrix $\mathbf{A} = \mathbf{I}$, the $n \times n$ identity matrix, and the solution reduces to the Fourier coefficients in Eq. (14).)

Example 2 Consider the following step function on $X = [0, 1]$,

$$u(x) = \begin{cases} 2, & 0 \leq x \leq 1/2, \\ 4, & 1/2 < x \leq 1. \end{cases} \tag{54}$$

We use the following set of functions,

$$\phi_1(x) = 1,\ \phi_k(x) = \sqrt{2}\cos((k-1)\pi x), \quad k \geq 2\,, \tag{55}$$

which form an orthonormal basis in the space of functions $L^2[0, 1]$. In Fig. 1 are presented plots of the best Weberized approximations v_n to u using $n = 5$, $n = 10$ and $n = 20$ basis functions for the cases $a = 0.25, 0.5, \cdots, 2.0$. The best L^2 approximations, u_n, which actually correspond to the case $a = 0$, are also shown for comparison. As expected, the best Weberized approximations v_n yield better approximations of $u(x)$ than u_n over [0, 0.5] and poorer approximations over [0.5, 1]. Also as expected, the degree of "betterness" over [0, 0.5] and "worseness" over [0.5, 1] of the Weberized approximations increases with the Weber exponent a since the weight function $g(u) = u^{-2a}$ decreases more rapidly with increasing a.

Example 3 Consider a general $u \in \mathcal{F}(X)$ on $X \subset \mathbb{R}$. For a given $a \geq 0$, the best constant approximation, $u \simeq c_a$, to u on X is obtained by minimizing the following squared distance function, cf. Eq. (46),

$$[\Delta_a(u, c)]^2 = \int_X \frac{1}{u(x)^{2a}}[u(x) - c]^2 dx = f(c)\,. \tag{56}$$

The global minimum point of this function is easily found to be as follows,

Fig. 1 Best Weberized approximations v_n to step function in Eq. (54) for $a = 0.25, 0.50 \cdots, 2.0$ along ▶ with best L^2 approximation ($a = 0$) for comparison using cosine basis functions in Eq. (55). *Top*: $n = 5$ basis functions used. *Middle*: $n = 10$. *Bottom*: $n = 20$

$$c_a = \left[\int_X u(x)^{1-2a} dx\right]\left[\int_X u(x)^{-2a} dx\right]^{-1}. \quad (57)$$

In the case $a = 0$, i.e., a lack of the Weber effect, c_0 is the mean value of u on X, as expected.

5 Concluding remarks

In this paper, we have described a method to provide "best Weberized" approximations to signals and images using intensity-dependent weight functions. These weight functions decrease with intensity value so that (i) regions where the target function has lower (higher) values will be weighted more (less) in the weighted L^2 distance integral, (ii) the behaviour of the weighting function is in accordance with Weber's model of perception $\Delta I = CI^a$ for $a > 0$. Up to a multiplicative constant, we have employed the functions $g(u) = u^{-2a}$ for $u \in [A, B]$.

This approach may seem rather *ad hoc*, but it does yield approximations that behave in a "Weberized" manner. Moreover, the computation of these approximations via the linear system in Eq. (52) is much more convenient than what is required for the measure-based method discussed in Li (2020); Li et al. (2018, 2019). Whether there is any relationship between the approximations yielded by this method and those of the measure-based method for a given value of a is still an open question.

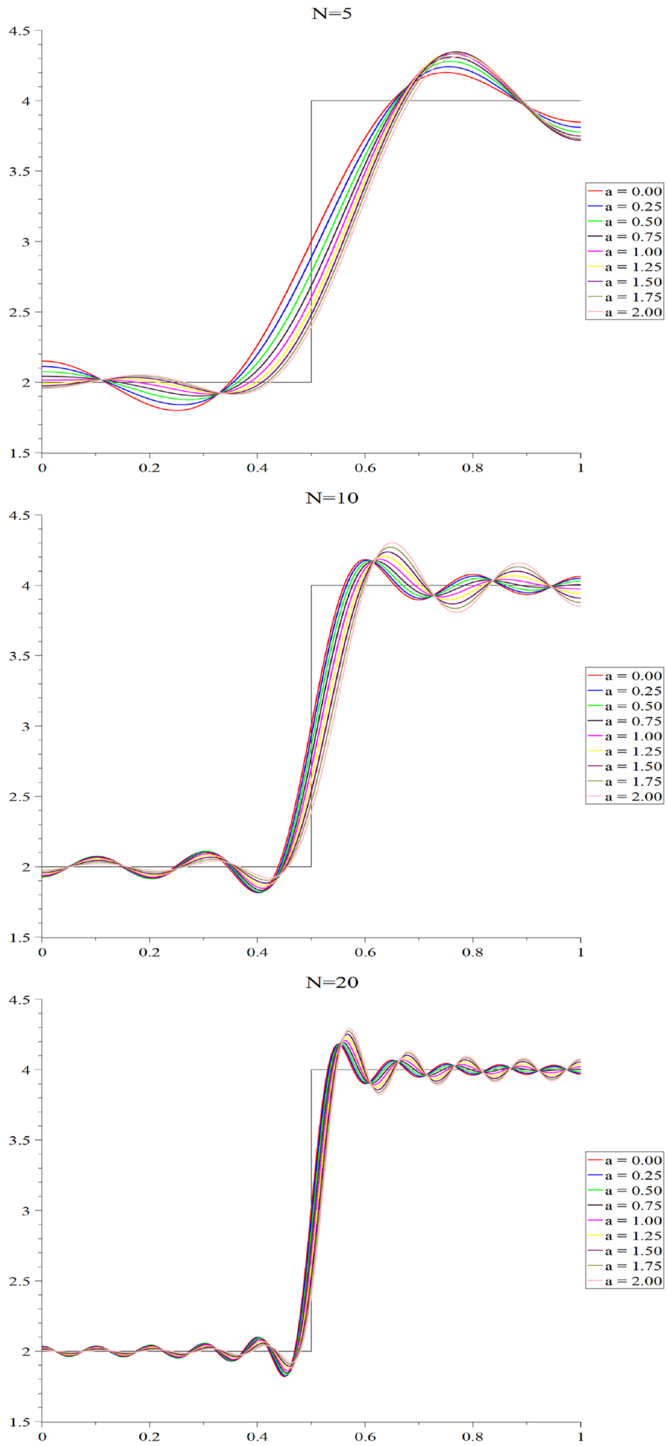
Our intensity-dependent approach may be adapted to perceptual models with differing Weber exponents over different intensity ranges – it is simply a matter of defining the weight function $g(u)$ in a piecewise manner.

Here we also mention that our approach is not restricted to Weber's model of perception. The weight function can be tailored to any desired type of behaviour over the range space $\mathbb{R}_g = [A, B]$. For example, one may wish, for some reason, to assign more weight to the middle of the greyscale interval $[A, B]$ than at the endpoints. One possible way to accomplish such a weighting is by using a logistic-type quadratic function of the target function u over $[A, B]$, i.e.,

$$g(u(x)) = C(u(x) - A)(B - u(x)) + D, \quad (58)$$

for appropriate values of the positive constants C and D.

In the more general case, let us assume that the weight function is sufficiently "nice", i.e., at least continuous on $[A, B]$, so that it achieves minimum and maximum values on $[A, B]$, to be denoted as $g_{\min}$ and $g_{\max}$, respectively, Then inequalities in Theorem 2 are modified as follows,


N=5
4.5
4
3.5
3
2.5
2
1.5
0
0.2
0.4
0.6
0.8
1
a = 0.00
a = 0.25
a = 0.50
a = 0.75
a = 1.00
a = 1.25
a = 1.50
a = 1.75
a = 2.00
N=10
N=20

$$g_{\min} d_2(u, v) \le \left\{ \begin{array}{c} \Delta_a(u, v) \\ \Delta_a(v, u) \end{array} \right\} \le g_{\max} d_2(u, v) \tag{59}$$

and

$$\frac{g_{\min}}{g_{\max}} \Delta_a(u, v) \le \Delta_a(v, u) \le \frac{g_{\max}}{g_{\min}} \Delta_a(u, v)\,. \tag{60}$$

Once again, we see that it is sufficient to consider only one distance function, $\Delta(u, v)$, given by

$$\Delta(u, v) = \left[\int_X g(u(x))[u(x) - v(x)]^2 \, dx \right]^{1/2}, \tag{61}$$

which may be viewed as the error of the best Weberized approximation $u \simeq v$. Eqs. (46) to (52) still apply in this case, assuming that our modified function $g(u(x))$ is employed.

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