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Recurrent Iterated Function Systems: Invariant Measures, A Collage Theorem and Moment Relations

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1. Introduction

The recurrent iterated function system (RIFS) [1-3] represents an extension of the "usual" IFS method, which has its origins in the works of Hutchinson [4] and Barnsley *et al.* [5]. (For a complete and readable treatment of normal IFS theory, see Ref. [6].) The flexibility of RIFS permits the construction of more general sets and measures which do not have to exhibit the strict self-similarity of the IFS case. Its consequences and utility in image generation have been discussed in several papers. In all of these treatments, the focus was on a probabilistic interpretation of RIFS: indeed, from the very definition, this is the most natural viewpoint. In this paper, we wish to consider a more general RIFS from the perspective of invariant measures. First, a "Markov" operator is constructed in a fashion analogous to the usual IFS case. This operator is shown to be contractive on a complete space of measures, from which follows the existence of a "fixed point" invariant measure. This provides a Collage Theorem for Measures for RIFS. Also, for the case of linear maps in $\mathbf{R}^n$, the invariance of measure permits the recursive computation of moments over the unique attractor A of the RIFS.

In order to allow those unfamiliar with the RIFS to become acquainted with it, we begin the following Section with a motivating example of a rather simple RIFS. In Section 3, we generalize this RIFS and even those introduced by Barnsley *et al.* in Ref. [1]. The appropriate space of measures and corresponding Markov operators are then defined and the theorems proved. In Section 4, for linear generalized RIFS, the invariance relations are used to derive recursion relations for moments over the attractor.

2. Simple Recurrent Iterated Function Systems

In this section we present a simple formulation of RIFS. The connection with the usual IFS will be apparent. Since many of the important features of IFS have been listed in another paper which appears in this Volume [7], we defer from repeating them here. We have attempted to preserve, as much as possible, the notation adopted in [7].

As in the usual IFS, (K,d), denotes a compact metric space with metric d. Let there exist N contraction maps on K: $w_i : K \to K$. We now associate with these maps a matrix of probabilities $\mathbf{P} = p_{ij}$ which is row stochastic, i.e. $\sum_j p_{ij}=1$, $i=1,\ldots,N$. From a *probabilistic* viewpoint, we consider a random "chaos game" sequence,

$$x_0 \in K, \quad x_{n+1} = w_{\sigma_n}(x_n), \ \ n=0,1,2,\ldots \,. \tag{2.1}$$

The fundamental difference between this process and the usual chaos game (Eq. (2.9) in Ref. [7]) is that the indices σ_n are not chosen independently, but rather with a probability that depends on the previous index σ_{n-1}:

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$$P(\sigma_{n+1}=i) = p_{\sigma_n,i}, \quad i=1,2,...,N. \tag{2.2}$$

Thus, at each step in Eq. (2.1), to compute x_{n+1}, we look at the index σ_n. The σ_nth row of **P** then gives the probabilities of choosing the next map to apply to x_n. Clearly, in the case that all rows of **P** are identical and given by the vector **p**, then the RIFS $\{K,\mathbf{w},\mathbf{P}\}$ reduces to the usual IFS $\{K,\mathbf{w},\mathbf{p}\}$. In all cases, we assume the matrix **P** to be *irreducible* [8], i.e. for any $1\leq i,j\leq N$, there exists a sequence $i_1,i_2,\ldots,i_n$ with $i_1=i$ and $i_n=j$ such that $p_{i_1i_2}p_{i_2i_3}\cdots p_{i_{n-1}i_n}>0$. (In other words, for any i,j, if we apply map w_i in the sequence, there is a nonzero probability that we will apply map w_j in the future.) A major result (we are not intending to be complete in this section) for RIFS is the following [1]:

There exists a unique stationary or invariant measure μ of the random walk in Eq. (2.1). If A is the support of μ, then there exist unique compact sets A_i, $i=1,...,N$, such that

$$A = \bigcup_{i=1}^{N} A_i, \quad A_i = \bigcup_{j:p_{ji}>0}^{N} w_i(A_j). \tag{2.3}$$

Note how the transition matrix **P** determines which maps w_i can act on A_j. The reader will note a fundamental difference between RIFS attractors and IFS attractors: the RIFS attractors need not exhibit the self-similarity or self-tiling properties characteristic of IFS attractors, where

$$A=\bigcup_{i=1} A_i, \quad A_i=w_i(A). \tag{2.4}$$

Barnsley *et al.* [1] showed that the random walk of Eq. (2.1) is not Markov on K itself but rather on the product $K\times\{1,2,...,N\}$. Again assuming that the matrix **P** is irreducible, it admits a stationary distribution $\{m_1,m_2,\ldots,m_N\}$, where the m_i are solutions of the linear equations [8]

$$\sum_{j=1}^{N} p_{ji}m_j = m_i, \quad i=1,2,...N, \quad \text{and} \sum_{i=1}^{N} m_i = 1. \tag{2.5}$$

A convenient way to picture the RIFS is to imagine a stack of transparent planes $K_i, i=1,2,...,N$ each of which is a copy of K, cf. [1-3]. Each $A_i\subset K_i$ and we "see" A by superimposing all planes. On the other hand, we can "see" A_i by plotting points obtained immediately after applying map w_i. During the iteration sequence in Eq. (2.1), motion from K_j to K_i under the action of w_i is permitted only if $p_{ji}>0$. The invariant distribution on the indices 1 to N is $\{m_1,\ldots,m_N\}$. This may be interpreted as follows: the proportional amount of time spent by the random sequence in Eq. (2.1) on each plane K_i is precisely m_i.

We illustrate the ideas mentioned to this point with some examples.

Example 1: $N=2$, $w_1(x)=\frac{1}{2}x, w_2(x)=\frac{1}{2}x+\frac{1}{2}$. Two cases:

$$(1): \mathbf{P} = \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix}, \quad (2): \mathbf{P} = \begin{bmatrix} 2/5 & 3/5 \\ 3/5 & 2/5 \end{bmatrix}. \tag{2.6}$$

In both cases, $A=[0,1]$, $A_1=[0,\frac{1}{2}]$, $A_2=[\frac{1}{2},1]$. Histogram approximations of the invariant measures are presented in Fig. 1. Qualitatively, in case (1), the measure is seen to "spread out" toward the ends of [0,1], but in a self-similar way throughout the interval. We can understand this from a look at the transition matrix in (1): when either map is applied, there is a greater probability that the same map will be applied again, thus pulling the point closer to its respective fixed point. In case (2), when a map is applied, there is a greater probability that the other map will then be applied. The result is to focus orbits toward the center. Some moments over this invariant measure will be calculated in Section 4.

Example 2: on $\mathbf{R}^2$, $N=4$, the following four maps whose fixed points lie at the vertices of the unit square $[0,1]\times[0,1]$:

$$w_i(x,y) = (\tfrac{1}{2}x,\tfrac{1}{2}y) + b_i, \quad i=1,...,4: \; b_1=(0,0), \; b_2=(\tfrac{1}{2},0), \; b_3=(\tfrac{1}{2},\tfrac{1}{2}), \; b_4=(0,\tfrac{1}{2}).$$

along with a 4×4 matrix **P**. In the normal IFS case, i.e. $p_{ij}=p_j>0$, $j=1,...,4$, the attractor A is the unit square $[0,1]\times[0,1]$.

(1) For $p_{11}=0$ but all other $p_{ij}>0$, the attractor A is shown in Fig. 2.1. Note that changing the nonzero p_{ij} will change the invariant measure living on A. Its Hausdorff dimension is $dim(A)=ln(\frac{1}{2}(3+\sqrt{21}))/(ln\ 2)$.

(2) For $p_{11}=p_{22}=p_{33}=p_{44}=0$, and all other $p_{ij}>0$, the attractor A has the shape shown in Fig. 2.2. $dim(A)=(ln\ 3)/(ln\ 2)$.

For both cases, the reader can deduce how the zero elements in **P** produce the "holes" in $[0,1]\times[0,1]$. A symbolic dynamics viewpoint [6] helps here. The dimensions were calculated using Theorem 4.1 of [1].

3. Generalized RIFS and Invariant Measures

In this section, we consider the generalization of the RIFS introduced in Section 2. It is similar in form to that which first appeared in Ref. [1], Section 3.4, however, we are interested not only in the geometry but also the measures which are supported on each space. To begin, we let $(K_1,d_1),...,(K_N,d_N)$ denote compact metric spaces (they need not be copies of each other), and $\mathbf{P}=[p_{ij}]$ be an $N\times N$ row stochastic irreducible matrix for a Markov chain with state space $\{1,...,N\}$. The fact that we consider the transition probability matrix instead of the index sets $I(i)=\{j\,|p_{ji}>0\}$, $i=1,...,N$, represents a deviation from [1]. For each pair of indices (i,j), we let $w_{ij}:K_j\rightarrow K_i$ be a contractive map:

$$d_i(w_{ij}(x),w_{ij}(y)) \le s_{ij}d_j(x,y), \ \ \forall\ x,y\in K_j\ ,\ \ 0\le s_{ij}<1. \tag{3.1}$$

We also define

$$s = \max_{1\le i,j\le N}(s_{ij}) < 1. \tag{3.2}$$

(In fact, we don't need w_{ij} in the case that $p_{ji}=0$.) In Ref. [1] it was shown that there exist unique compact sets A_i, $i=1,...,N$, with $A_i\subset K_i$, such that

$$A_i = \bigcup_{j:p_{ji}>0} w_{ij}(A_j), \qquad i=1,...,N. \tag{3.3}$$

The set $A=(A_1,...,A_N)$, called the *attractor* of the RIFS $\{(K_i,d_i),(p_{ij}),(w_{ij}),1\le i,j\le N\}$, is the fixed point of an operator $\mathbf{W}$ (cf. [1], Section 3.4) that reflects the dynamics of the w_{ij}. We shall show below that by choosing a suitable combination of measures over the spaces K_i, the action of the maps w_{ij} between these different spaces defines an invariant measure which is supported over the attractor A. This will represent a generalization of the case $K_i=K$, $\forall i$ and $w_{ij}=w_i$, $\forall j$ presented in Section 3.4 of [1].

The Markov process (or "chaos game") can be thought of as "living" in the space $\overline{K}=\bigcup_{i=1}^{N}(\{i\}\times K_i)$. Starting with the element $\overline{z}_0=(i_0,x_0)\in\overline{K}$, $1\le i_0\le N$, $x_0\in K_{i_0}$, choose i_1 with the distribution given by the i_0th row of **P**. Then define $x_1=w_{i_1i_0}(x_0)$ to give $\overline{z}_1=(i_1,x_1)$, etc.. Note that $\overline{z}_n=(i_n,x_n)$ implies $x_n\in K_{i_n}$. $\{\overline{z}_n\}$ is a Markov process on $\overline{K}$, where the transition probability function is given by

$$p((s,x),\overline{B}) = \sum_{j=1}^{N}p_{sj}I_{\overline{B}}(j,w_{js}(x)), \tag{3.4}$$

which represents the probability to transfer from (s,x) to a Borel set $\overline{B}\subset\overline{K}$ in one step of the process.

Now let $\{m_1,\ldots,m_N\}$ be the stationary initial distribution of the Markov chain associated with the p_{ij}, as given by the solutions of Eq. (2.5). For an arbitrary metric space (K,d), define $M(K)$ as the set of Borel regular measures on K, and

$$\mathcal{M}_i = \{\mu\in M(K_i)\,|\mu(K_i)=m_i\} \tag{3.5}$$

and $\mathcal{M}=\mathcal{M}_1\times\mathcal{M}_2\times\cdots\times\mathcal{M}_N$. We define the distance between two measures $\overline{\mu},\overline{\nu}\in\mathcal{M}$ as

$$\overline{d}_H(\overline{\mu},\overline{\nu}) = \sum_{i=1}^{N}d_H^{(i)}(\mu_i,\nu_i), \tag{3.6}$$

where $d_H^{(i)}$ denotes the Hutchinson metric [4] between measures in $\mathcal{M}_i$. It is straightforward to show that $(\mathcal{M},\overline{d}_H)$ is a complete metric space.

We now define an appropriate "Markov" operator $T: \mathcal{M} \to \mathcal{M}$ as

$$T(\overline{\nu}) = T(\nu_1, \ldots, \nu_N) = \left(\sum_{j=1}^{N} p_{j1} \nu_j \circ w_{1j}^{-1}, \ldots, \sum_{j=1}^{N} p_{jN} \nu_j \circ w_{Nj}^{-1}\right). \tag{3.7}$$

Note that T is well defined:

$$(T\overline{\nu})_k(K_k) = \sum_{j=1}^{N} p_{jk} \nu_j(w_{kj}^{-1}(K_k)) = \sum_{j=1}^{N} p_{jk} m_j = m_k, \quad k=1,\ldots,N. \tag{3.8}$$

Its construction and the proof of its contractivity are quite analogous to the original treatment by Hutchinson [4].

Proposition: $T: \mathcal{M} \to \mathcal{M}$ is a contraction map in the metric $\overline{d}_H$ with constant s.

Proof:

$$\overline{d}_H(T\overline{\mu}, T\overline{\nu}) = \sum_{i=1}^{N} d_H^{(i)}[(T\overline{\mu})_i, (T\overline{\nu})_i] \tag{3.9}$$

$$= \sum_{i=1}^{N} \sup_{f \in Lip(K_i)} \left[\int_{K_i} f d(T\overline{\mu})_i - \int_{K_i} f d(T\overline{\nu})_i\right], \tag{3.10}$$

where $Lip(X)$ denotes the set of Lipshitz-1 functions on X. Using the fact that

$$\int_{K_i} f d(T\overline{\mu})_i = \sum_{j=1}^{N} p_{ji} \int_{K_j} f \circ w_{ij} d\mu_j, \tag{3.11}$$

we have ,

$$\overline{d}_H(T\overline{\mu}, T\overline{\nu}) = \sum_{i=1}^{N} \sup_{f \in Lip(K_i)} \sum_{j=1}^{N} p_{ji} \left[\int_{K_j} f \circ w_{ij} d\mu_j - \int_{K_j} f \circ w_{ij} d\nu_j\right] \tag{3.12}$$

$$\leq \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ji} \sup_{f \in Lip(K_i)} \left[\int_{K_j} f \circ w_{ij} d\mu_j - \int_{K_j} f \circ w_{ij} d\nu_j\right] \tag{3.13}$$

$$= \sum_{j=1}^{N}\sum_{i=1}^{N} p_{ji} \sup_{f \in Lip(K_i)} \left[\int_{K_j} f \circ w_{ij} d\mu_j - \int_{K_j} f \circ w_{ij} d\nu_j\right] \tag{3.14}$$

If $\overline{f} = s^{-1}(f \circ w_{ij})$, with $f \in Lip(K_i)$, then it follows from Eq. (3.1), that $\overline{f} \in Lip(K_j)$. Now define

$$F(K_j) = \{\overline{f} \in Lip(K_j) \mid \overline{f} = s^{-1}(f \circ w_{ij}), \text{ where } f \in Lip(K_i), \text{ for some } i \in \{1,\ldots,N\}\}. \tag{3.15}$$

It then follows that $F(K_j) \subset Lip(K_j)$. The right side of Eq. (3.14) becomes

$$\sum_{j=1}^{N}\sum_{i=1}^{N} p_{ji} \sup_{\overline{f} \in F(K_j)} s \left[\int_{K_j} \overline{f} d\mu_j - \int_{K_j} \overline{f} d\nu_j\right] \leq \sum_{j=1}^{N} s d_H^{(j)}(\mu_j, \nu_j) = s\overline{d}_H(\overline{\mu}, \overline{\nu}), \tag{3.16}$$

where we have used the fact that $\sum_i p_{ji} = 1$.

Now let $\overline{\mu}$ denote the fixed point of the Markov operator T in $\mathcal{M}$. We call $\overline{\mu}$ the invariant measure of the RIFS defined at the beginning of this section. The property $T\overline{\mu} = \overline{\mu}$ thus implies

$$\mu_i = \sum_{j=1}^{N} p_{ji} \mu_j \circ w_{ij}^{-1}, \quad i=1,\ldots,N. \tag{3.17}$$

Proposition: Let $B = (B_1, \ldots, B_N)$, where $B_i = supp(\mu_i) \subset K_i$, $i=1,\ldots,N$. Then

(1) $B_i = \bigcup_{j: p_{ji} > 0} w_{ij}(B_j)$, $i=1,\ldots,N$, and

(2) $B = A$, the attractor of the RIFS defined in Eq. (3.3).

Proof: (1): in two steps. First, we prove the following inclusion by contradiction:

$$B_i \subset \bigcup_{j:p_{ji}>0} w_{ij}(B_j).$$

Let $x \in B_i$ so that for any neigbourhood $V(x)$, $\mu_i(V(x))>0$, and let us suppose that for all j such that $p_{ji}>0$, $x \notin w_{ij}(B_j)$. Since B_j is closed, therefore compact, $w_{ij}(B_j)$ is closed in K_i. It then follows that for each j there exists a neighbourhood of x, $U_j(x)$, such that $U_j(x) \cap w_{ij}(B_j)=\emptyset$ and $\mu_j(w_{ij}^{-1}(U_j(x)))=0$. If $V(x)=\bigcap_{j:p_{ji}>0} U_j(x)$, then $V(x)$ is a neighbourhood of x such that $V(x) \subset U_j(x), \forall j$. Thus, we have

$$\mu_i(V(x)) = \sum_{j=1}^{N} p_{ji}\mu_j(w_{ij}^{-1}(V(x)) = 0.$$

which contradicts the original hypothesis.

We now prove the other inclusion:

$$\bigcup_{j:p_{ji}>0} w_{ij}(B_j) \subset B_i.$$

Let j_0 be such that $p_{j_0 i}>0$. If $x \in w_{ij_0}(B_{j_0})$, and $U(x)$ is any neighbourhood of x, then for $a_{j_0} \in w_{ij_0}^{-1}(x)$, $w_{ij_0}^{-1}(U(x))$ is a neighbourhood of a_{j_0}. From the definition of the support of a measure, it follows that $\mu_{j_0}(w_{ij_0}^{-1}(U(x)))>0$. Then

$$\mu_i(U(x)) = \sum_{j=1}^{N} p_{ji}\mu_j(w_{ij}^{-1}(U(x))) \geq p_{j_0 i}\mu_{j_0}(w_{ij_0}^{-1}(U(x))) > 0.$$

Therefore, $x \in supp(\mu_i)=B_i$.

(2): Since A is the unique set satisfying the relations in Eq. (3.3), i.e. $\mathbf{W}(A)=A$, it is enough to show that $\mathbf{W}(B)=B$. But this is the result of (1), so the proof is complete.

From these results, a collage theorem for invariant measures on recurrent IFS now follows, in complete analogy to that for normal IFS[6]:

RIFS Collage Theorem: Let $\overline{\nu} \in \mathcal{M}$ be a measure over the metric spaces (K_i,d_i), $i=1,\dots,N$ as defined earlier, and suppose that there exists a RIFS $\{(K_i,d_i),\mathbf{P},\mathbf{w}, i=1,\dots,N\}$ with contractivity factor s, so that

$$\overline{d}_H(\overline{\nu},T(\overline{\nu})) < \epsilon. \tag{3.18}$$

Then

$$\overline{d}_H(\overline{\mu},\overline{\nu}) < \frac{\epsilon}{(1-s)}, \tag{3.19}$$

where $\overline{\mu}$ is the invariant measure of the RIFS.

4. Moments of Invariant Measures of RIFS

We now consider the special case where the compact metric spaces (K_i,d_i) are subsets $K_i \subset \mathbf{R}^n$, with usual Euclidean metric. For a given RIFS with attractor A and invariant measure $\overline{\mu}$, we define the power moments of $\overline{\mu}$ by the integrals

$$g_{i_1 i_2 \cdots i_n} = \int_A x_1^{i_1} \cdots x_n^{i_n} d\overline{\mu}, \qquad g_{00\dots0} = \int_A d\overline{\mu} = 1. \tag{4.1}$$

Just as for the usual IFS, when the maps w_{ij} are linear, then the invariance property for integrals, Eq. (3.17) permits a recursive computation of the moments. We illustrate this property for the one dimensional case, i.e. $K_i \subset \mathbf{R}$. The maps w_{ij} will assume the following general form

$$w_{ij}(x) = s_{ij}x + a_{ij}, \quad |s_{ij}| < 1, \quad i=1,\dots,N. \tag{4.2}$$

(Note that the maps w_{ij}, $j=1,\ldots,N$ are not necessarily identical, as in the usual IFS case.) We first define the power moments over A as

$$g_n = \int_A x^n d\bar{\mu} = \sum_{j=1}^{N} \int_{A_j} x^n d\mu_j = \sum_{j=1}^{N} g_n^{(j)}. \tag{4.3}$$

Using the Markov operator of Eq. (3.7), we have

$$g_n^{(i)} = \int_{A_i} x^n d\mu_i = \sum_{j=1}^{N} p_{ji} \int_{A_j} (s_{ij}x+a_{ij})^n d\mu_j. \tag{4.4}$$

Expanding the polynomials, and integrating, we obtain the relations

$$g_n^{(i)} = \sum_{j=1}^{N} p_{ji} \sum_{k=0}^{n} \binom{n}{k} s_{ij}^k a_{ij}^{n-k} g_k^{(j)} \tag{4.5}$$

which, for $n\geq 1$, may be rewritten as the following systems of linear equations in the moments $g_n^{(i)}$, $i=1,\ldots,N$:

$$\sum_{j=1}^{N} (p_{ji} s_{ij}^n - \delta_{ij}) g_n^{(j)} = -\sum_{k=0}^{n-1} \binom{n}{k} \left[\sum_{j=1}^{N} s_{ij}^k a_{ij}^{n-k} p_{ji} g_k^{(j)} \right], \quad i=1,\ldots,N, \quad n\geq 1. \tag{4.6}$$

When $n=0$, Eq. (4.5) yields precisely the linear equations of (2.5). We thus set $g_0^{(i)}=m_i$, so that $g_0=1$. This is in agreement with the definition of the measure space $\mathcal{M}$ constructed from the measures in Eq. (3.5). With this normalization, the higher moments $g_n^{(i)}$, $i=1,2,\ldots,N$ are calculated for $n=1,2,\ldots$ recursively. The special case

$$w_{ij}(x)=w_i(x) = s_i x + a_i, \; j=1,\ldots,N, \tag{4.7}$$

corresponds to the simple RIFS of Section 2.

It follows that derivatives of moments with respect to the RIFS parameters could also be expressed in closed form as solutions of simultaneous linear equations. This method could be extended, in principle, to RIFS on $\mathbf{R}^2$. The relations would be rather complicated as are their counterparts for the usual IFS in two dimensions [9].

4.1 Some Simple Moment Calculations

From Eqs. (4.6) and (4.7), the first five moments over the attractor [0,1] for the simple RIFS of Example 1 in Section 1 have been computed. The results are

$$\begin{aligned} &(1): \quad g_1=\frac{1}{2},\; g_2=\frac{19}{54},\; g_3=\frac{5}{18},\; g_4=\frac{2449}{10530},\; g_5=\frac{425}{2106}. \\ &(2): \quad g_1=\frac{1}{2},\; g_2=\frac{7}{22},\; g_3=\frac{5}{22},\; g_4=\frac{783}{4510},\; g_5=\frac{125}{902}. \end{aligned} \tag{4.8}$$

In fact, let us extend our analysis of this simple two-map RIFS. Suppose that the transition probability matrix $\mathbf{P}$ for this system is given by the general form

$$\mathbf{P}(\epsilon) = \begin{bmatrix} \frac{1}{2}+\epsilon & \frac{1}{2}-\epsilon \\ \frac{1}{2}-\epsilon & \frac{1}{2}+\epsilon \end{bmatrix}. \tag{4.9}$$

The two cases given in Eq. (4.8) correspond to (1) $\epsilon=1/10$ and (2) $\epsilon=-1/10$, respectively. Note that we have chosen to expand $\mathbf{P}$ about the "unperturbed" matrix

$$\mathbf{P}^{(0)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{4.10}$$

In this case, for $\epsilon=0$, the RIFS reduces to the IFS $\{w_1,w_2,p_1=p_2=\frac{1}{2}\}$ with $A=[0,1]$ and $\mu=$ uniform Lebesgue measure. Hence, the moments are

$$g_n = \frac{1}{n+1}, \quad n=0,1,2,\ldots \tag{4.11}$$

Two other special cases are easy to determine:

1. $\epsilon = ½$, $\mathbf{P}=\mathbf{I}$ (not irreducible). Invariant measure $\mu = ½\delta_0 + ½\delta_1$ (δ_x denotes unit mass of measure at x). Hence $g_n = ½$, $n \geq 0$.
2. $\epsilon = -½$ (two-cycle at (1/3,2/3)). $\mu = ½\delta_{1/3} + ½\delta_{2/3}$, with moments

$$g_n = \frac{1}{2}\left[\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n\right].$$

Using the algebraic computation language Maple [10], the first five moments for $\mathbf{P}(\epsilon)$ in Eq. (4.9) have been computed:

$$g_1 = \frac{1}{2}, \; g_2 = \frac{1}{6}\frac{2-\epsilon}{1-\epsilon}, \; g_3 = \frac{1}{4}\frac{1}{1-\epsilon}, \; g_4 = \frac{1}{30}\frac{(\epsilon-8)(\epsilon-3)}{(\epsilon-4)(\epsilon-1)}, \; g_5 = \frac{1}{12}\frac{5\epsilon+8}{(\epsilon-4)(\epsilon-1)}. \tag{4.12}$$

5. Conclusions

A Markov operator for measures on attractors of generalized recurrent function systems, analogous to that for simple IFS, has been constructed. It is contractive and hence possesses a unique fixed point, the invariant measure for the RIFS. This provides a Collage Theorem for Measures on RIFS. As well, the moments for RIFS on $\mathbf{R}^n$ can be computed recursively. These results may prove useful in extending techniques for the inverse problem of fractal/measure construction using IFS to the RIFS. For a discussion of some inverse methods for IFS, the reader is referred to [7].

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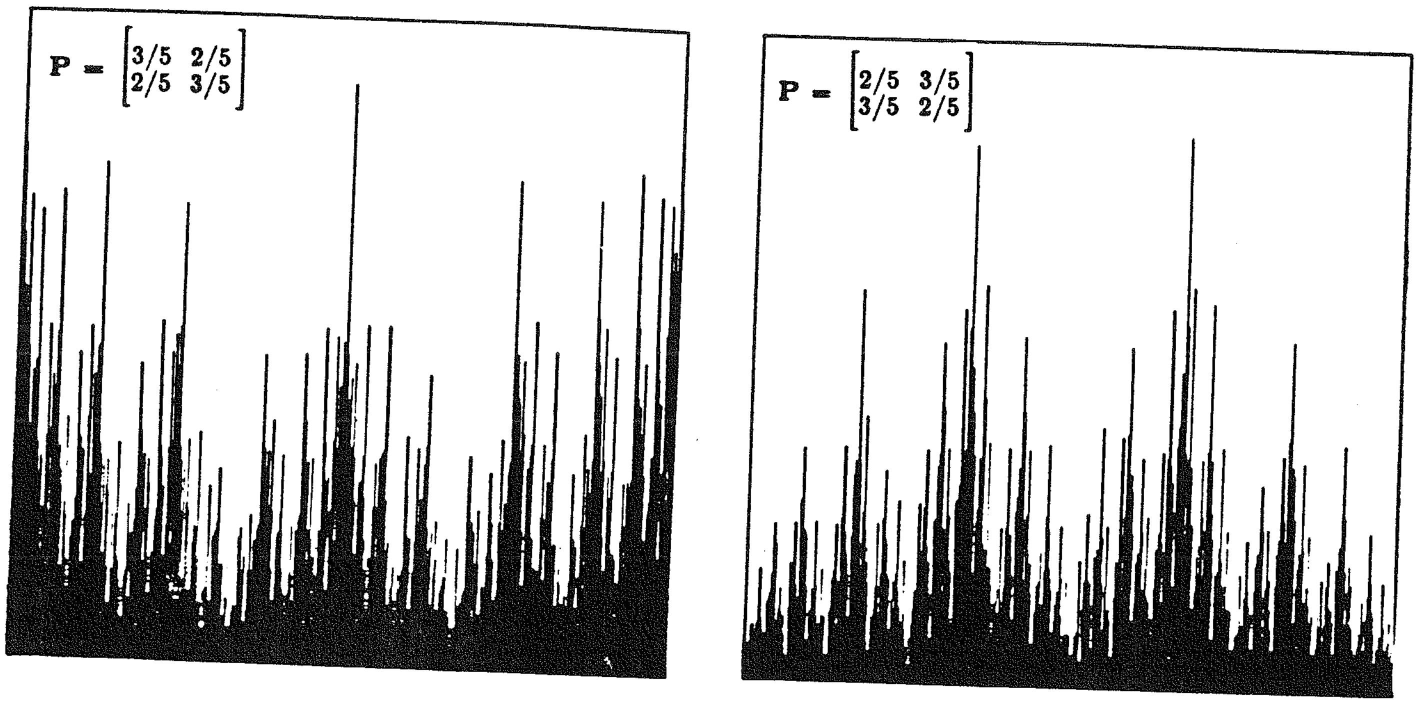

Figure 1: Histogram approximations of invariant measures on [0,1] for RIFS of Example 1, with given transition probability matrices .

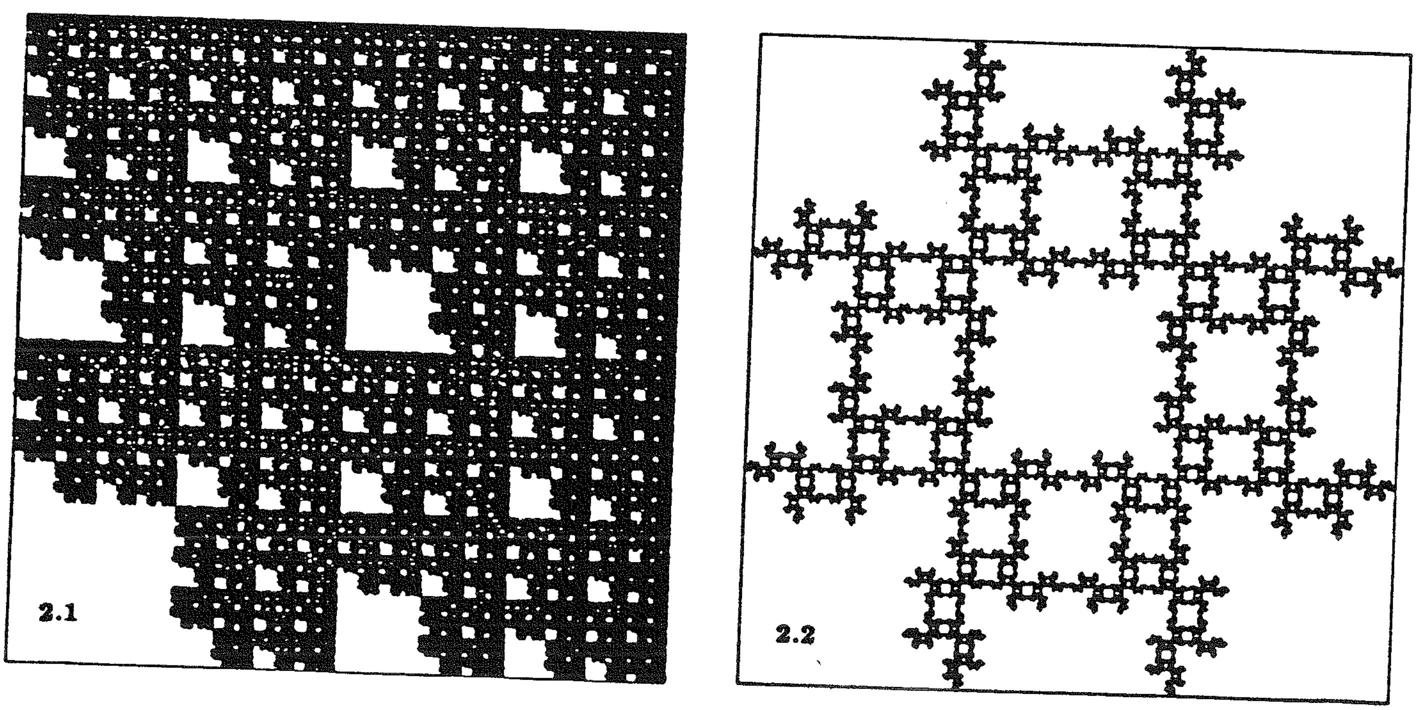

Figure 2: Attractors for RIFS of Example 2 plotted in $[0,1]\times[0,1]$.