A Appendix

A.1 Sharpe Ratio Derivation

Introduce the notation

$$\begin{aligned}
&\mu_1 = \mathbb{E}[g_1(X_T^x)], \quad \mu_2 = \mathbb{E}[g_2(X_T^x)],\\
&\sigma_1 = \mathrm{Var}[g_1(X_T^x)], \quad \sigma_2 = \mathrm{Var}[g_2(X_T^x)],\\
&\sigma_{12} = \mathrm{Cov}[g_1(X_T^x),\ g_2(X_T^x)].
\end{aligned}$$

From (2.3), we can write,

$$\mathbb{E}[g(X_T^x)] = \boldsymbol{\omega}^T \boldsymbol{\mu}, \quad \mathrm{Var}[g(X_T^x)] = \boldsymbol{\omega}^T \Sigma \boldsymbol{\omega}. \tag{A.1}$$

where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1 & \sigma_{12} \\ \sigma_{12} & \sigma_2 \end{bmatrix}. \tag{A.2}$$

Here, the investor seeks to maximize the Sharpe Ratio $SR(\boldsymbol{\omega})$ in (3.1). As a result, the investor is solving the following optimization problem:

$$\begin{aligned}
\max_{\boldsymbol{\omega}} \ & \frac{\boldsymbol{\omega}^T \boldsymbol{\mu} - (1 + r_{f,T})x}{\sqrt{\boldsymbol{\omega}^T \Sigma \boldsymbol{\omega}}}\\
\text{subject to} \ & \boldsymbol{\omega}^T \mathbf{1} = 1,\\
& \boldsymbol{\omega} \geq 0.
\end{aligned} \tag{A.3}$$

First, we can simplify (A.3) by considering the payoff functions:

$$\tilde{g}_1(X_T^x) = g_1(X_T^x) - (1 + r_{f,T})x, \tag{A.4}$$

$$\tilde{g}_2(X_T^x) = g_2(X_T^x) - (1 + r_{f,T})x. \tag{A.5}$$

Then, we can easily verify that

$$\begin{aligned}
&\tilde{\mu}_1 = \mathbb{E}[\tilde{g}_1(X_T^x)] = \mu_1 - (1 + r_{f,T})x, \quad \tilde{\mu}_2 = \mathbb{E}[\tilde{g}_2(X_T^x)] = \mu_2 - (1 + r_{f,T})x,\\
&\tilde{\sigma}_1^2 = \mathrm{Var}[\tilde{g}_1(X_T^x)] = \sigma_1^2, \quad \tilde{\sigma}_2^2 = \mathrm{Var}[\tilde{g}_2(X_T^x)] = \sigma_2^2, \quad \tilde{\sigma}_{12} = \mathrm{Cov}[\tilde{g}_1(X_T^x), \tilde{g}_2(X_T^x)] = \sigma_{12}.
\end{aligned}$$

So, the problem (A.3) can be expressed as follows,

$$\max_{\boldsymbol{\omega}} \frac{\boldsymbol{\omega}^T \tilde{\boldsymbol{\mu}}}{\sqrt{\boldsymbol{\omega}^T \tilde{\Sigma} \boldsymbol{\omega}}}$$
$$\text{subject to } \boldsymbol{\omega}^T \mathbf{1} = 1,$$
$$\boldsymbol{\omega} \geq 0. \tag{A.6}$$

where

$$\tilde{\boldsymbol{\mu}} := \begin{bmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \end{bmatrix}, \quad \tilde{\Sigma} = \begin{bmatrix} \tilde{\sigma}_1 & \tilde{\sigma}_{12} \\ \tilde{\sigma}_{12} & \tilde{\sigma}_2 \end{bmatrix}. \tag{A.7}$$

Now, we can rewrite (A.6) as a standard quadratic programming problem if we assume $\tilde{\mu}_1 > 0$ and $\tilde{\mu}_2 > 0$. This is a natural assumption here because the investor anticipates a higher expected return than the risk-free rate when investing in the hedge fund. If we let $f(\boldsymbol{\omega})$ denote the objective function in (A.6), then it can be verified that for any real number $\lambda > 0$, $f(\boldsymbol{\omega}) = f(\lambda\boldsymbol{\omega})$. Thus, the problem (A.6) is equivalent to the optimization problem:

$$\max_{\boldsymbol{\omega}} \frac{1}{\sqrt{\boldsymbol{\omega}^T \tilde{\Sigma} \boldsymbol{\omega}}}$$
$$\text{subject to } \boldsymbol{\omega}^T \tilde{\boldsymbol{\mu}} = 1,$$
$$\boldsymbol{\omega} \geq 0. \tag{A.8}$$

Clearly, (A.8) can be rewritten as an equivalent minimization problem:

$$\min_{\boldsymbol{\omega}} \boldsymbol{\omega}^T \tilde{\Sigma} \boldsymbol{\omega}$$
$$\text{subject to } \boldsymbol{\omega}^T \tilde{\boldsymbol{\mu}} = 1,$$
$$\boldsymbol{\omega} \geq 0. \tag{A.9}$$

which is a standard quadratic programming problem.

Theorem A.1. *Suppose $\tilde{\mu}_1 \geq 0$ and $\tilde{\mu}_2 \geq 0$. Let $\boldsymbol{\omega^*} = (\omega_1^*, \omega_2^*)$ be the optimal solution for (A.9). If*

$$\tilde{\sigma}_{12} \leq \min\{\frac{\tilde{\mu}_1}{\tilde{\mu}_2}\tilde{\sigma}_2^2, \ \frac{\tilde{\mu}_2}{\tilde{\mu}_1}\tilde{\sigma}_1^2\}, \tag{A.10}$$

is satisfied, then

$$\omega_1^* = \frac{\tilde{\mu}_1\tilde{\sigma}_2^2 - \tilde{\mu}_2\tilde{\sigma}_{12}}{C^*} > 0, \quad \omega_2^* = \frac{\tilde{\mu}_2\tilde{\sigma}_1^2 - \tilde{\mu}_1\tilde{\sigma}_{12}}{C^*} > 0, \tag{A.11}$$

where $C^* = (\tilde{\mu}_1\tilde{\sigma}_2 - \tilde{\mu}_2\tilde{\sigma}_1)^2 + 2\tilde{\mu}_1\tilde{\mu}_2(\tilde{\sigma}_1\tilde{\sigma}_2 - \tilde{\sigma}_{12})$. *Otherwise,*

$$\boldsymbol{\omega}^* = \begin{cases} (1/\tilde{\mu}_1, 0), & \textit{if } \frac{\tilde{\sigma}_1^2}{\tilde{\mu}_1^2} \leq \frac{\tilde{\sigma}_2^2}{\tilde{\mu}_2^2}, \\ (0, 1/\tilde{\mu}_2), & \textit{o.w.} \end{cases} \tag{A.12}$$

Proof. By Best (2010, Chapter 9, Page 192), we can obtain the following optimality conditions for (A.9),

$$\begin{cases} \boldsymbol{\omega}^T\tilde{\boldsymbol{\mu}} = 1, \\ \mathbf{I}\lambda - \mathbf{2\Sigma}\boldsymbol{\omega} = \nu\boldsymbol{\mu}, \\ \lambda^T\mathbf{I}\boldsymbol{\omega} = \mathbf{0}, \\ \boldsymbol{\omega} \geq 0, \\ \lambda \geq 0. \end{cases} \tag{A.13}$$

where ν is the multiplier for the constraint $\boldsymbol{\omega}^T\tilde{\boldsymbol{\mu}} = 1$ and λ is the vector of multipliers for the constraints $\boldsymbol{\omega} \geq 0$. More explicitly, this leads to the linear system:

$$\omega_1\tilde{\mu}_1 + \omega_2\tilde{\mu}_2 = 1, \tag{A.14}$$

$$\lambda_1 - 2\omega_1\tilde{\sigma}_1^2 - 2\omega_2\tilde{\sigma}_{12} = \nu\tilde{\mu}_1, \tag{A.15}$$

$$\lambda_2 - 2\omega_1\tilde{\sigma}_{12} - 2\omega_2\tilde{\sigma}_2^2 = \nu\tilde{\mu}_2, \tag{A.16}$$

$$\lambda_1\omega_1 + \lambda_2\omega_2 = 0. \tag{A.17}$$

with constraints $\boldsymbol{\omega} \geq 0$ and $\lambda \geq 0$. The constraints imply the solution to (A.17) must satisfy one of the following three cases: (i) $\lambda_1 = 0$ and $\lambda_2 = 0$. (ii) $\lambda_1 > 0$ and $\lambda_2 = 0$. (iii) $\lambda_1 = 0$ and $\lambda_2 > 0$.

It is easy to see check that case (i) can be viewed as an optimization problem:

$$\min_{\boldsymbol{\omega}} \boldsymbol{\omega}^T\tilde{\Sigma}\boldsymbol{\omega}$$
$$\text{Subject to } \boldsymbol{\omega}^T\tilde{\boldsymbol{\mu}} = 1. \tag{A.18}$$

Let $\boldsymbol{\omega}^{**} = (w_1^{**}, w_2^{**})$ be the solution to (A.18). A simple calculation gives:

$$w_1^{**} = \frac{\tilde{\mu}_1\tilde{\sigma}_2^2 - \tilde{\mu}_2\tilde{\sigma}_{12}}{C^*}, \quad w_2^{**} = \frac{\tilde{\mu}_2\tilde{\sigma}_1^2 - \tilde{\mu}_1\tilde{\sigma}_{12}}{C^*}, \tag{A.19}$$

where $C^* = (\tilde{\mu}_1\tilde{\sigma}_2 - \tilde{\mu}_2\tilde{\sigma}_1)^2 + 2\tilde{\mu}_1\tilde{\mu}_2(\tilde{\sigma}_1\tilde{\sigma}_2 - \tilde{\sigma}_{12})$. This is the solution to (A.9) without the constraint $\boldsymbol{\omega} \geq 0$. Therefore, $\boldsymbol{\omega}^* = \boldsymbol{\omega}^{**}$ when $\boldsymbol{\omega}^{**}$ is a feasible solution to (A.9). Next, note that $C^* > 0$, so $\boldsymbol{\omega}^{**}$ is feasible for (A.9) if and only if $\tilde{\mu}_1\tilde{\sigma}_2^2 - \tilde{\mu}_2\tilde{\sigma}_{12} \geq 0$ and $\tilde{\mu}_2\tilde{\sigma}_1^2 - \tilde{\mu}_1\tilde{\sigma}_{12} \geq 0$. That is:

$$\tilde{\sigma}_{12} \leq \min\left\{\frac{\tilde{\mu}_1}{\tilde{\mu}_2}\tilde{\sigma}_2^2,\ \frac{\tilde{\mu}_2}{\tilde{\mu}_1}\tilde{\sigma}_1^2\right\}. \tag{A.20}$$

Next, if (A.20) is not satisfied, then the optimal solution is the solution in either case (ii) or case (iii). Clearly, case (ii) leads to $\omega_2 = 0$, and from (A.14), it is easy to calculate $\omega_1 = 1/\tilde{\mu}_1$. On the other hand, $\omega_1 = 0$ and $\omega_2 = 1/\tilde{\mu}_2$ is the solution for case (iii). Finally, we substitute the solutions in case (ii) and (iii) back into the objective function in (A.9) and compare their values to obtain the optimal solution. This yields $\boldsymbol{\omega}^* = (1/\tilde{\mu}_1, 0)$ if $\tilde{\sigma}_1^2/\tilde{\mu}_1^2 \leq \tilde{\sigma}_2^2/\tilde{\mu}_2^2$, otherwise, $\boldsymbol{\omega}^* = (0, 1/\tilde{\mu}_2)$. □

To investigate the nature of the investor's optimal strategy in terms of the original model parameters, we need to write $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_{12}$ explicitly. Following Djerroud *et al.* (2016) we assume $m_1 = m_2 = m$. Introducing the notation:

$$\begin{aligned}
C_1 &:= \mathbb{E}[(X_T^x - mx - x)_+], & C_2 &:= \mathbb{E}[(X_T^x - mx - x)_+^2],\\
P_1 &:= \mathbb{E}[(x + mx - X_T^x)_+], & P_2 &:= \mathbb{E}[(x + mx - X_T^x)_+^2],\\
P_{1,c} &:= \mathbb{E}[((1-c)x + mx - X_T^x)_+], & P_{2,c} &:= \mathbb{E}[((1-c)x + mx - X_T^x)_+^2].
\end{aligned}$$

We obtain:

$$\tilde{\mu}_1 = (1-\alpha_1)C_1 - P_1 - r_{f,T}x, \tag{A.21}$$

$$\tilde{\mu}_2 = (1-\alpha_2)C_1 - P_{1,c} - r_{f,T}x, \tag{A.22}$$

$$\begin{aligned}
\tilde{\sigma}_1^2 &= \mathrm{Var}[(1-\alpha_1)(X_T^x - mx - x)_+ - (x + mx - X_T^x)_+ - r_{f,T}x]\\
&= (1-\alpha_1)^2(C_2 - C_1^2) + P_2 - P_1^2 + 2(1-\alpha_1)C_1P_1,
\end{aligned} \tag{A.23}$$

$$\begin{aligned}
\tilde{\sigma}_2^2 &= \mathrm{Var}[(1-\alpha_2)(X_T^x - mx - x)_+ - ((1-c)x + mx - X_T^x)_+ - r_{f,T}x]\\
&= (1-\alpha_2)^2(C_2 - C_1^2) + P_{2,c} - P_{1,c}^2 + 2(1-\alpha_2)C_1P_{1,c}.
\end{aligned} \tag{A.24}$$

$$
\begin{aligned}
\tilde{\sigma}_{12} &= \mathbb{E}[\tilde{g}_1(X_T^x)\tilde{g}_2(X_T^x)] - \tilde{\mu}_1\tilde{\mu}_2 \\
&= (1-\alpha_1)(1-\alpha_2)C_2 - (1-\alpha_1)r_{f,T}xC_1 - (1-\alpha_2)r_{f,T}xC_1 + P_{2,c} + cxP_{1,c} \\
&\quad + r_{f,T}xP_1 + r_{f,T}xP_{1,c} + r_{f,T}^2x^2 - \tilde{\mu}_1\tilde{\mu}_2.
\end{aligned} \tag{A.25}
$$

Detailed derivations, including explicit formulas for $C_i, P_i, P_{i,c},\ i = 1,2$ in the case where X_t^x is a geometric Brownian motion (see section 4) can be found in A.3. Recalling that $\tilde{\mu}_1 > 0$ and $\tilde{\mu}_2 > 0$, we can obtain the valid ranges for α_1 and α_2 from (A.21) and (A.22):

$$
\alpha_1 < \frac{C_1 - P_1 - r_{f,T}x}{C_1} := \alpha_1^*, \quad \alpha_2 < \frac{C_1 - P_{1,c} - r_{f,T}x}{C_1} := \alpha_2^*. \tag{A.26}
$$

By (A.26) and noting that $P_1 \geq P_{1,c}$, we can easily deduce that $\alpha_1^* \leq \alpha_2^*$. This is reasonable, because the first-loss fee structure provides downside protection for the investor. In return, the investor can tolerate a higher performance fee.

A.2 Sortino Ratio Derivation

From (3.5) and noting that $r(T) = g(X_T^x)/x - 1$, we can easily obtain

$$
\begin{aligned}
\sigma_d^2 &= \mathbb{E}[\min\{r(T) - l, 0\}^2] \\
&= \mathbb{E}[\min\{\frac{g(X_T^x)}{x} - 1 - l, 0\}^2] \\
&= x^{-2}\mathbb{E}[\min\{g(X_T^x) - (1+l)x, 0\}^2].
\end{aligned} \tag{A.27}
$$

As a result, we have

$$
SOR(\boldsymbol{\omega}) = \frac{\mathbb{E}[\frac{g(X_T^x)}{x}] - l}{\sigma_d} = \frac{\mathbb{E}[g(X_T^x)] - (1+l)x}{\sqrt{\mathbb{E}[\min\{g(X_T^x) - (1+l)x, 0\}^2]}}. \tag{A.28}
$$

Recall that $g(X_T^x) = \omega_1 g_1(X_T^x) + \omega_2 g_2(X_T^x)$ and let

$$
\tilde{g}_{1,l}(X_T^x) = g_1(X_T^x) - (1+l)x \quad \text{and} \quad \tilde{g}_{2,l}(X_T^x) = g_2(X_T^x) - (1+l)x.
$$

Thus, we can further simplify equation (A.28) as follows,

$$
SOR(\boldsymbol{\omega}) = \frac{\mathbb{E}[\tilde{g}_l(X_T^x)]}{\sqrt{\mathbb{E}[\min\{\tilde{g}_l(X_T^x), 0\}^2]}}, \tag{A.29}
$$

where $\tilde{g}_l(X_T^x) = \omega_1 \tilde{g}_{1,l}(X_T^x) + \omega_2 \tilde{g}_{2,l}(X_T^x)$. Now, introduce the notation:

$$\begin{aligned}
&\mu_{1,l} := \mathbb{E}[\tilde{g}_{1,l}(X_T^x)], \quad \mu_{1,l} := \mathbb{E}[\tilde{g}_{2,l}(X_T^x)], \\
&\sigma_{1,l}(\boldsymbol{\omega}) := \mathbb{E}[\tilde{g}_{1,l}(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \leq 0\}}], \\
&\sigma_{2,l}(\boldsymbol{\omega}) := [\tilde{g}_{2,l}(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \leq 0\}}], \\
&\sigma_{12,l}(\boldsymbol{\omega}) := \mathbb{E}[\tilde{g}_{1,l}(X_T^x) \tilde{g}_{2,l}(X_T^x) \mathbf{1}_{\{\tilde{g}_l(X_T^x) \leq 0\}}].
\end{aligned} \tag{A.30}$$

and note that

$$\begin{aligned}
\mathbb{E}[\min\{\tilde{g}_l(X_T^x), 0\}^2] =& \mathbb{E}[\tilde{g}_l(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \leq 0\}}] \\
=& \mathbb{E}[(\omega_1 \tilde{g}_{1,l}(X_T^x) + \omega_2 \tilde{g}_{2,l}(X_T^x))^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \leq 0\}}] \\
=& \omega_1^2 \mathbb{E}[\tilde{g}_{1,l}(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \leq 0\}}] \\
&+ 2\omega_1\omega_2 \mathbb{E}[\tilde{g}_{1,l}(X_T^x) \tilde{g}_{2,l}(X_T^x) \mathbf{1}_{\{\tilde{g}_l(X_T^x) \leq 0\}}] \\
&+ \omega_2^2 \mathbb{E}[\tilde{g}_{2,l}(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x) \leq 0\}}].
\end{aligned} \tag{A.31}$$

We can rewrite (A.29) as

$$SOR(\boldsymbol{\omega}) = \frac{\omega_1 \mu_{1,l} + \omega_2 \mu_{1,l}}{\omega_1^2 \sigma_{1,l}(\boldsymbol{\omega}) + 2\omega_1\omega_2 \sigma_{12,l}(\boldsymbol{\omega}) + \omega_2^2 \sigma_{2,l}(\boldsymbol{\omega})}. \tag{A.32}$$

Similar to the Sharpe Ratio maximization framework in the previous section, the investor's goal is to maximize $SOR(\boldsymbol{\omega})$ at maturity T. In general, the expression (A.32) is difficult to optimize analytically.

A.3 Derivation of Explicit Formulas

Assuming $m_1 = m_2 = m$, we rewrite the investor payoffs $\tilde{g}_1(X_T^x)$ and $\tilde{g}_2(X_T^x)$ in the following more compact forms:

$$\begin{aligned}
\tilde{g}_1(X_T^x) =& X_T^x - mx - \alpha_1(X_T^x - mx - x)_+ - (1 + r_{f,T})x. \\
\tilde{g}_2(X_T^x) =& X_T^x - mx - \alpha_2(X_T^x - mx - x)_+ + (x + mx - X_T^x)_+ \\
&- ((1-c)x + mx - X_T^x)_+ - (1 + r_{f,T})x.
\end{aligned}$$

Then, using the equality $X_T^x - mx - x = (X_T^x - mx - x)_+ - (x + mx - X_T^x)_+$, we obtain that:

$$\tilde{g}_1(X_T^x) =(1-\alpha_1)(X_T^x - mx - x)_+ - (x + mx - X_T^x)_+ - r_{f,T}x, \tag{A.33}$$
$$\tilde{g}_2(X_T^x) =(1-\alpha_2)(X_T^x - mx - x)_+ - ((1-c)x + mx - X_T^x)_+ - r_{f,T}x. \tag{A.34}$$

The expressions for $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\sigma}_1^2$, and $\tilde{\sigma}_2^2$ then follow immediately. Moreover, noting that $(x+mx-X_T^x)_+((1-c)x+mx-X_T^x)_+ = ((1-c)x+mx-X_T^x)_+^2 + cx((1-c)x+mx-X_T^x)_+$ yields:

$$\begin{aligned}\tilde{\sigma}_{12} &= \mathbb{E}[\tilde{g}_1(X_T^x)\tilde{g}_2(X_T^x)] - \tilde{\mu}_1\tilde{\mu}_2 \\ &=(1-\alpha_1)(1-\alpha_2)C_2 - (1-\alpha_1)r_{f,T}xC_1 - (1-\alpha_2)r_{f,T}xC_1 + P_{2,c} + cxP_{1,c} \\ &\quad + r_{f,T}xP_1 + r_{f,T}xP_{1,c} + r_{f,T}^2x^2 - \tilde{\mu}_1\tilde{\mu}_2.\end{aligned} \tag{A.35}$$

$$\begin{aligned}\tilde{g}_{1,l}(X_T^x) =& X_T^x - mx - \alpha_1(X_T^x - mx - x)_+ - (1+l)x. \\ \tilde{g}_{2,l}(X_T^x) =& X_T^x - mx - \alpha_2(X_T^x - mx - x)_+ + (x + mx - X_T^x)_+ \\ & - ((1-c)x + mx - X_T^x)_+ - (1+l)x.\end{aligned}$$

Similarly, we can write $\tilde{g}_{1,l}(X_T^x)$ and $\tilde{g}_{2,l}(X_T^x)$ as follows,

$$\tilde{g}_{1,l}(X_T^x) =(1-\alpha_1)(X_T^x - mx - x)_+ - (x + mx - X_T^x)_+ - lx, \tag{A.36}$$
$$\tilde{g}_{2,l}(X_T^x) =(1-\alpha_2)(X_T^x - mx - x)_+ - ((1-c)x + mx - X_T^x)_+ - lx. \tag{A.37}$$

It is easy to check $\tilde{g}_l(X_T^x) \leq 0 \Rightarrow \omega_1\tilde{g}_{1,l}(X_T^x) + \omega_2\tilde{g}_{2,l}(X_T^x) \leq 0$, which implies that

$$\begin{aligned}&\left(\omega_1(1-\alpha_1) + \omega_2(1-\alpha_2)\right)(X_T^x - mx - x)_+ \\ &\leq \omega_1(x + mx - X_T^x)_+ + \omega_2((1-c)x + mx - X_T^x)_+ + lx.\end{aligned} \tag{A.38}$$

When $X_T^x \leq (1+m)x$, the inequality (A.38) always holds. On the other hand, when $X_T^x > (1+m)x$, we can easily obtain

$$\left(\omega_1(1-\alpha_1) + \omega_2(1-\alpha_2)\right)(X_T^x - mx - x) \leq lx$$

$$\Rightarrow X_T^x \le \frac{l}{\omega_1(1-\alpha_1)+\omega_2(1-\alpha_2)}x + (1+m)x = (1+a+m)x, \tag{A.39}$$

where $a = l/\omega_1(1-\alpha_1)+\omega_2(1-\alpha_2)$. Therefore, we have that $\tilde{g}_l(X_T^x) \le 0 \iff X_T^x \le (1+a+m)x$. It follows that

$$\begin{aligned}
\mathbb{E}[\min\{\tilde{g}_l(X_T^x), 0\}^2] =& \mathbb{E}[\tilde{g}_l(X_T^x)^2 \mathbf{1}_{\{\tilde{g}_l(X_T^x)\le 0\}}] \\
=& \mathbb{E}[(\omega_1 \tilde{g}_{1,l}(X_T^x) + \omega_2 \tilde{g}_{2,l}(X_T^x))^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\
=& \omega_1^2 \mathbb{E}[\tilde{g}_{1,l}(X_T^x)^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\
&+ 2\omega_1\omega_2 \mathbb{E}[\tilde{g}_{1,l}(X_T^x)\tilde{g}_{2,l}(X_T^x) \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\
&+ \omega_2^2 \mathbb{E}[\tilde{g}_{2,l}(X_T^x)^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}].
\end{aligned} \tag{A.40}$$

Introducing the notation:

$$\begin{aligned}
C_0^l &:= \mathbb{E}[\mathbf{1}_{\{X_T^x \le (1+a+m)x\}}], \\
C_1^l &:= \mathbb{E}[(X_T^x - mx - x)_+ \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}], \\
C_2^l &:= \mathbb{E}[(X_T^x - mx - x)_+^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}], \\
P_{1,c}^l &:= \mathbb{E}[((1-c)x + mx - X_T^x)_+ \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}], \\
P_{2,c}^l &:= \mathbb{E}[((1-c)x + mx - X_T^x)_+^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}],
\end{aligned}$$

by (A.36) and (A.37), we can obtain

$$\begin{aligned}
\mu_{1,l} =& (1-\alpha_1)C_1 - P_1 - lx, \quad \mu_{1,l} = (1-\alpha_2)C_1 - P_{1,c} - lx, \\
\sigma_{1,l}(\boldsymbol{\omega}) =& \mathbb{E}[\tilde{g}_{1,l}(X_T^x)^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\
=& \mathbb{E}[((1-\alpha_1)(X_T^x - mx - x)_+ - (x + mx - X_T^x)_+ - lx)^2 \, \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}]. \\
=& (1-\alpha_1)^2 C_2^l + P_{2,0}^l + l^2x^2C_0^l - 2(1-\alpha_1)lxC_1^l + 2lxP_{1,0}^l. \\
\sigma_{2,l}(\boldsymbol{\omega}) =& \mathbb{E}[\tilde{g}_{2,l}(X_T^x)^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\
=& \mathbb{E}[((1-\alpha_2)(X_T^x - mx - x)_+ - ((1-c)x + mx - X_T^x)_+ - lx)^2 \, \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\
=& (1-\alpha_2)^2 C_2^l + P_{2,c}^l + l^2x^2C_0^l - 2(1-\alpha_2)lxC_1^l + 2lxP_{1,c}^l \\
\sigma_{12,l}(\boldsymbol{\omega}) =& \mathbb{E}[\tilde{g}_{1,l}(X_T^x)\tilde{g}_{2,l}(X_T^x) \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\
=& (1-\alpha_1)(1-\alpha_2)C_2^l - (1-\alpha_1)lxC_1^l - (1-\alpha_2)lxC_1^l + P_{2,c}^l + cxP_{1,c}^l \\
&+ lxP_{1,0}^l + lxP_{1,c}^l + l^2x^2C_0^l.
\end{aligned}$$

In order to derive explicit formulas for $C_i, P_i, P_{i,c}, P^l_{i,c}$, $i = 1, 2$ and C^l_j, $j = 0, 1, 2$, we need the following result.

Proposition A.1. *Let* $X^x_t = x\exp\{(\mu - \frac{1}{2}\sigma^2)t - \sigma W_t\}$. *Then, for* $0 \le a \le b$,

$$\mathbb{E}[(X^x_t)^2 \mathbf{1}_{\{a<X^x_t<b\}}] = x^2 e^{2\mu t+\sigma^2 t}(\Phi(\tilde{d}_1(x, a, t)) - \Phi(\tilde{d}_1(x, b, t))). \tag{A.41}$$

Proof. Let $Z = t^{-1/2}W_t \sim N(0, 1)$,

$$d_1(x, y, t) = \frac{\log(x/y) + (\mu + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \quad d_2(x, y, t) = d_1(x, y, t) - \sigma\sqrt{t}$$

Then

$$\begin{aligned}
&\log(a/x) < (\mu - \sigma^2/2) - \sigma\sqrt{t}Z < \log(a/x)\\
\iff& \frac{\log(X_t/b) + (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} < Z < \frac{\log(x/a) + (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\\
\iff& d_2(x, b, t) < Z < d_2(x, a, t).
\end{aligned} \tag{A.42}$$

$$\begin{aligned}
\mathbb{E}[(X^x_t)^2 \mathbf{1}_{\{a<X^x_t<b\}}] &= \int_{d_2(x,b,t)}^{d_2(x,a,t)} x^2 \exp\{2(\mu - \frac{1}{2}\sigma^2)t - 2\sigma\sqrt{t}Z\}\exp\{-\frac{1}{2}z^2\}dz\\
&= x^2 e^{2\mu t}\int_{d_2(x,b,t)}^{d_2(x,a,t)} \exp\{-\frac{1}{2}(z^2 + 4\sigma\sqrt{t}z + 2\sigma^2 t)\}dz\\
&= x^2 e^{2\mu t}\int_{d_2(x,b,t)}^{d_2(x,a,t)} \exp\{-\frac{1}{2}(z + 2\sigma\sqrt{t})^2\}\exp\{\sigma^2 t\}dz\\
&= x^2 e^{2\mu t+\sigma^2 t}\int_{d_2(x,b,t)+2\sigma\sqrt{t}}^{d_2(x,a,t)+2\sigma\sqrt{t}} \exp\{-\frac{1}{2}y^2\}dy\\
&= x^2 e^{2\mu t+\sigma^2 t}(\Phi(\tilde{d}_1(x, a, t)) - \Phi(\tilde{d}_1(x, b, t))).
\end{aligned}$$

where

$$\tilde{d}_1(x, y, t) = \frac{\log(x/y) + (\mu + \frac{3}{2}\sigma^2)t}{\sigma\sqrt{t}} \tag{A.43}$$

□

To simplify notation, we define

$$\tilde{d}_1(x, bx, T) := \tilde{d}_1(b), \qquad d_1(x, bx, T) := d_1(b),$$
$$d_2(x, bx, T) := d_2(b).$$

Then, we can explicitly write C_0^l, C_1, C_1^l, C_2, C_2^l, P_1, $P_{1,0}^l$, P_2, $P_{2,0}^l$, $P_{1,c}$, $P_{1,c}^l$, $P_{2,c}$ and $P_{2,c}^l$ as follows,

$$\begin{aligned}
C_0^l &= \mathbb{E}[\mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] = \Phi(-d_2((1+m+a))) \\
C_1 &= \mathbb{E}[(X_T^x - x - mx)_+] = xe^{\mu T}\Phi(d_1(1)) - (1+m)x\Phi(d_2(1)), \\
C_1^l &= \mathbb{E}[(X_T^x - mx - x)_+ \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\
&= xe^{\mu T}(\Phi(d_1((1+m)) - \Phi(d_1((1+m+a))) + (1+m)x(\Phi(d_2((1+m)) - \Phi(d_2((1+m+a))) \\
C_2 &= \mathbb{E}[(X_T^x - x - mx)_+^2] \\
&= x^2 e^{2\mu T + \sigma^2 T}\Phi(\tilde{d}_1(1)) - 2(1+m)x^2 e^{\mu T}\Phi(d_1(1)) + (1+m)^2 x^2 \Phi(d_2(1)), \\
C_2^l &= \mathbb{E}[(X_T^x - mx - x)_+^2 \mathbf{1}_{\{X_T^x \le (1+a+m)x\}}] \\
&= x^2 e^{2\mu T + \sigma^2 T}(\Phi(\tilde{d}_1((1+m)) - \Phi(\tilde{d}_1((1+m+a))) \\
&\quad - 2(1+m)x^2 e^{\mu T}(\Phi(d_1((1+m)) - \Phi(d_1((1+m+a)))) \\
&\quad + (1+m)^2 x^2(\Phi(d_2((1+m)) - \Phi(d_2((1+m+a))) \\
P_1 &= P_{1,0} = (1+m)x\Phi(-d_2(1)) - xe^{\mu T}\Phi(-d_1(1)), \\
P_{1,0}^l &= P_{1,0} = (1+m)x\Phi(-d_2(1)) - xe^{\mu T}\Phi(-d_1(1)) \\
P_2 &= P_{2,0} = (1+m)^2 x^2 \Phi(-d_2(1)) - 2(1+m)x^2 e^{\mu T}\Phi(-d_1(1)) + x^2 e^{2\mu T + \sigma^2 T}\Phi(-\tilde{d}_1(1)), \\
P_{2,0}^l &= P_{2,0} = (1+m)^2 x^2 \Phi(-d_2(1)) - 2(1+m)x^2 e^{\mu T}\Phi(-d_1(1)) + x^2 e^{2\mu T + \sigma^2 T}\Phi(-\tilde{d}_1(1)), \\
P_{1,c} &= \mathbb{E}[((1-c)x + mx - X_T^x)_+] \\
&= (1-c+m)x\Phi(-d_2(1-c)) - xe^{\mu T}\Phi(-d_1(1-c)), \\
P_{1,c}^l &= P_{1,c} \\
&= (1-c+m)x\Phi(-d_2(1-c)) - xe^{\mu T}\Phi(-d_1(1-c)), \\
P_{2,c} &= \mathbb{E}[((1-c)x + mx - X_T^x)_+^2] \\
&= (1-c+m)^2 x^2 \Phi(-d_2(1-c)) - 2(1-c+m)x^2 e^{\mu T}\Phi(-d_1(1-c)) + x^2 e^{2\mu T + \sigma^2 T}\Phi(-\tilde{d}_1(1-c)). \\
P_{2,c}^l &= P_{2,c} \\
&= (1-c+m)^2 x^2 \Phi(-d_2(1-c)) - 2(1-c+m)x^2 e^{\mu T}\Phi(-d_1(1-c)) + x^2 e^{2\mu T + \sigma^2 T}\Phi(-\tilde{d}_1(1-c)).
\end{aligned}$$