A proof of a conjecture for the number of ramified coverings of the sphere by the torus*

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Abstract

An explicit expression is obtained for the generating series for the number of ramified coverings of the sphere by the torus, with elementary branch points and prescribed ramification type over infinity. This proves a conjecture of Goulden, Jackson and Vainshtein for the explicit number of such coverings.

1 Introduction

Let X be a compact connected Riemann surface of genus $g \geq 0$. A *ramified covering* of $\mathbb{S}^2$ of degree n by X is a non-constant meromorphic function $f\colon \mathsf{X} \longrightarrow \mathbb{S}^2$ such that $|f^{-1}(q)| = n$ for all but a finite number of points $q \in \mathbb{S}^2$, which are called *branch points.* Two ramified coverings f_1 and f_2 of $\mathbb{S}^2$ by X are said to be *equivalent* if there is a homeomorphism $\pi\colon \mathsf{X} \longrightarrow \mathsf{X}$ such that $f_1 = f_2\pi$. A ramified covering f is said to be *simple* if $|f^{-1}(q)| = n-1$ for each branch point of f, and is *almost simple* if $|f^{-1}(q)| = n-1$ for each branch point but one, that is denoted by ∞. The preimages of ∞ are the poles of f. If $\alpha_1, \dots, \alpha_m$ are the orders of the poles of f, where $\alpha_1 \geq \dots \geq \alpha_m \geq 1$, then $\alpha = (\alpha_1, \dots, \alpha_m)$ is a partition of n and is called the *ramification type* of f.

Let $\mu_m^{(g)}(\alpha)$ be the number of almost simple ramified coverings of $\mathbb{S}^2$ by X with ramification type α. The problem of determining an (explicit) expression for $\mu_m^{(g)}(\alpha)$ is called the *Hurwitz Enumeration Problem.* The purpose of this paper is to prove the following result for the torus, giving an explicit expression for $\mu_m^{(1)}(\alpha)$ for an arbitrary partition $\alpha = (\alpha_1, \dots, \alpha_m)$. Theorem 1.1 was previously conjectured by Goulden, Jackson and Vainshtein in [4] where it was proved for all partitions α with $m \leq 6$, and for the particular partition (1^m), for any $m \geq 1$. Let $\mathcal{C}_\alpha$ be the conjugacy class of the symmetric group $\mathfrak{S}_n$ on n symbols indexed by the partition α of n.

Theorem 1.1

$$\mu_m^{(1)}(\alpha) = \frac{|\mathcal{C}_\alpha|}{24\,n!}(n+m)!\left(\prod_{i=1}^{m} \frac{\alpha_i^{\alpha_i}}{(\alpha_i-1)!}\right)\left(n^m - n^{m-1} - \sum_{i=2}^{m}(i-2)!e_i n^{m-i}\right)$$

where e_i is the i-th elementary symmetric function in $\alpha_1, \dots, \alpha_m$ and $e_1 = \alpha_1 + \cdots \alpha_m = n$.

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Previously Hurwitz [5] had shown (see also Goulden and Jackson [2] and Strehl [6]) that, for the sphere,

$$\mu_m^{(0)}(\alpha) = \frac{|\mathcal{C}_\alpha|}{n!}(n+m-2)!n^{m-3}\left(\prod_{i=1}^{m}\frac{\alpha_i^{\alpha_i}}{(\alpha_i-1)!}\right). \tag{1}$$

The approach developed by Hurwitz is outlined in the next section.

Very recently Vakil [7] has given an independent proof of Theorem 1.1. He develops, by techniques in algebraic geometry, and solves a recurrence equation that is completely different in character from the one obtained from the differential equation in this paper.

2 Hurwitz's combinatorialization of ramified coverings

Hurwitz's approach was to represent a ramified covering f of $\mathbb{S}^2$, with ramification type α, by a combinatorial datum $(\sigma_1, \dots, \sigma_r)$ consisting of transpositions in $\mathfrak{S}_n$, whose product π is in $\mathcal{C}_\alpha$, such that $\langle\sigma_1, \dots, \sigma_r\rangle$ acts transitively on the set $\{1, \dots, n\}$ of sheet labels and that $r = n+m+2(g-1)$, where $m = l(\alpha)$, the length of α. The latter condition is a consequence of the Riemann-Hurwitz formula. Under this combinatorialization he showed that

$$\mu_m^{(g)}(\alpha) = \frac{|\mathcal{C}_\alpha|}{n!}c_g(\alpha),$$

where $c_g(\alpha)$ is the number of such factorizations of an arbitrary but fixed $\pi \in \mathcal{C}_\alpha$. He studied the action of σ_r on $\sigma_1 \cdots \sigma_{r-1}$ to derive a recurrence equation for $c_g(\alpha)$. The difficulty with Hurwitz's approach is that the recurrence equations for $c_g(\alpha)$ are intractable in all but a small number of special cases.

It appears that his approach can be made more tractable by the introduction of *cut operators* and *join operators* that have been developed for combinatorial purposes by Goulden [1], Goulden and Jackson [2] and Goulden, Jackson and Vainshtein [4]. These are partial differential operators in indeterminates $p_1, p_2, \dots$ that take account of the enumerative consequences of the action of σ_r on $\rho = \sigma_1 \cdots \sigma_{r-1}$, when summed over all such ordered transitive factorizations. There are two cases. The action of σ_r on ρ is either to join an i-cycle and a j-cycle of ρ to produce an $i+j$-cycle, or to cut an $i+j$-cycle of ρ to produce an i-cycle and a j-cycle. In the first case the operators are the join operators

$$p_{i+j}\frac{\partial^2}{\partial p_i \partial p_j} \quad \text{and} \quad p_{i+j}\left(\frac{\partial}{\partial p_i}\right)\left(\frac{\partial}{\partial p_j}\right),$$

and in the second case the operator is the cut operator

$$p_i p_j \frac{\partial}{\partial p_{i+j}}.$$

A "cut-and-join" analysis of the action of σ_r on ρ therefore leads to a nonhomogeneous partial differential equation in a countably infinite number of variables (indeterminates) for the generating series Φ for $c_g(\alpha)$. The type of Φ is determined by the combinatorial properties of the cut-and-join analysis.

The advantage of this approach to the Hurwitz Enumeration Problem is that it facilitates the transformation of the differential equation for Φ by an implicit change of variables. The series that is involved with this transformation is denoted by $s = s(x, \mathbf{p})$ throughout, where $\mathbf{p} = (p_1, p_2, \dots)$, and appears to be fundamental to the problem.

3 The differential equation

Let $\mathbf{p}_\alpha = p_{\alpha_1}\cdots p_{\alpha_m}$ where $\alpha = (\alpha_1, \ldots \alpha_m)$. Let

$$\Phi(u,x,z,\mathbf{p}) = \sum_{\substack{n,m\geq 1\\ g\geq 0}} \sum_{\substack{\alpha\vdash n\\ l(\alpha)=m}} |\mathcal{C}_\alpha| c_g(\alpha) \frac{u^{n+m+2(g-1)}}{(n+m+2(g-1))!} \frac{x^n}{n!} z^g p_\alpha, \tag{2}$$

the generating series for $c_g(\alpha)$, where $\alpha \vdash n$ signifies that α is a partition of n. It was shown in [4] that $f = \Phi(u,1,z,\mathbf{p})$ satisfies the partial differential equation

$$\frac{\partial f}{\partial u} = \frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}z\frac{\partial^2 f}{\partial p_i \partial p_j} + ijp_{i+j}\frac{\partial f}{\partial p_i}\frac{\partial f}{\partial p_j} + (i+j)p_i p_j \frac{\partial f}{\partial p_{i+j}}\right). \tag{3}$$

By replacing p_i by $x^i p_i$ for $i \geq 1$ it is readily seen that $f = \Phi(u,x,z,\mathbf{p})$ satisfies (3). But $\Phi(u,x,z,\mathbf{p}) \in \mathbb{Q}[u,z,\mathbf{p}][[x]]$, and it is also readily seen that (3) has a unique solution in this ring.

Let $F_i(x,\mathbf{p}) = [z^i]\Phi(1,x,z,\mathbf{p})$ for $i = 0,1$, where $[z^i]f$ denotes the coefficient of z^i in the formal power series f. Then F_0 is the generating series for the numbers $c_0(\alpha)$, which have been determined by Hurwitz, so F_0 is known. F_1 is the generating series for $c_1(\alpha)$. The next result gives the linear first order partial differential equation for F_1 that is induced by restricting (3) above to terms of degree at most one in z.

Lemma 3.1 *The series $f = F_1$ satisfies the partial differential equation*

$$T_0 f - T_1 = 0 \tag{4}$$

where

$$\begin{aligned} T_0 &= x\frac{\partial}{\partial x} + \sum_{i\geq 1} p_i \frac{\partial}{\partial p_i} - \sum_{i,j\geq 1} ijp_{i+j}\frac{\partial F_0}{\partial p_i}\frac{\partial}{\partial p_j} - \frac{1}{2}\sum_{i,j\geq 1}(i+j)p_i p_j \frac{\partial}{\partial p_{i+j}}, \\ T_1 &= \frac{1}{2}\sum_{i,j\geq 1} ijp_{i+j}\frac{\partial^2 F_0}{\partial p_i \partial p_j}. \end{aligned}$$

Proof: Clearly, from (3),

$$u\frac{\partial}{\partial u}[z]\Phi = [z]\left(x\frac{\partial}{\partial x} + \sum_{i\geq 1} p_i \frac{\partial}{\partial p_i}\right)\Phi.$$

The result follows by applying $[z]$ to (3). □

We now turn our attention to solving this partial differential equation. Let

$$G_1(x,\mathbf{p}) = \frac{1}{24}\sum_{n,m\geq 1}\sum_{\substack{\alpha\vdash n\\ l(\alpha)=m}} |\mathcal{C}_\alpha| \left(\prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i - 1)!}\right)\left(n^m - n^{m-1} - \sum_{i=2}^m (i-2)! e_i n^{m-i}\right)\frac{x^n}{n!}p_\alpha. \tag{5}$$

Since (3) has a unique solution in $\mathbb{Q}[u,z,\mathbf{p}][[x]]$, then (4) has a unique solution in $\mathbb{Q}[\mathbf{p}][[x]]$. To establish Theorem 1.1 it therefore suffices to show that $f = G_1$ satisfies (4) (note that G_1 has a constant term of 0, so the initial condition is satisfied).

4 The generating series G_1

To obtain a convenient form for G_1 the following lemma is required that expresses the elementary symmetric function $e_k(\lambda)$ as the coefficient in a formal power series. For a partition $\alpha = (\alpha_1, \ldots, \alpha_r)$ let m_i denote the number of occurrences of i in α, and we may therefore write $\alpha = (1^{m_1}, \ldots, r^{m_r})$. Let $\vartheta(\alpha) = \prod_{i=1}^{r} i^{m_i} m_i!$. Let $\mathcal{P}$ denote the set of all partitions with the null partition adjoined.

Lemma 4.1 *For any nonnegative integer k and partition λ,*

$$e_k(\lambda) = \frac{\vartheta(\lambda)}{k!}[p_\lambda](p_1 + p_2 + \cdots)^k \sum_{\alpha \in \mathcal{P}} \frac{p_\alpha}{\vartheta(\alpha)}.$$

Proof: First

$$\sum_{\alpha \in \mathcal{P}} \frac{p_\alpha}{\vartheta(\alpha)} = \sum_{m_1, m_2, \ldots \geq 0} \frac{p_1^{m_1}}{1^{m_1} m_1!} \frac{p_2^{m_2}}{2^{m_2} m_2!} \cdots$$

and

$$(p_1 + p_2 + \cdots)^k = \sum_{\substack{i_1, i_2, \ldots \geq 0 \\ i_1 + i_2 + \cdots = k}} k! \frac{p_1^{i_1} p_2^{i_2} \cdots}{i_1! i_2! \cdots}.$$

If $\vartheta(\lambda) = 1^{j_1} j_1! 2^{j_2} j_2! \cdots$ then

$$\frac{\vartheta(\lambda)}{k!}[p_\lambda](p_1 + p_2 + \cdots)^k \sum_{\alpha \in \mathcal{P}} \frac{p_\alpha}{\vartheta(\alpha)} = \sum_{\substack{m_1, i_1, m_2, i_2, \ldots \geq 0, \\ i_1 + i_2 + \cdots = k}} \binom{j_1}{i_1} 1^{i_1} \binom{j_2}{i_2} 2^{i_2} \cdots$$

where the sum is further constrained by $i_1 + m_1 = j_1, i_2 + m_2 = j_2, \ldots$. Then

$$\begin{aligned} \frac{\vartheta(\lambda)}{k!}[p_\lambda](p_1 + p_2 + \cdots)^k \sum_{\alpha \in \mathcal{P}} \frac{p_\alpha}{\vartheta(\alpha)} &= [t^k](1+t)^{j_1}(1+2t)^{j_2}(1+3t)^{j_3} \cdots \\ &= e_k((1^{j_1}, 2^{j_2}, \ldots)) \end{aligned}$$

and the result follows. $\square$

Let

$$\psi_i(x, \mathbf{p}) = \sum_{r \geq 1} r^{i-1} a_r p_r x^r, \tag{6}$$

where i is an integer and $a_r = r^r/(r-1)!$, for $r \geq 1$. Let $s \equiv s(x, \mathbf{p})$ be the unique solution of the functional equation

$$s = x e^{\psi_0(s, \mathbf{p})} \tag{7}$$

in the ring $\mathbb{Q}[\mathbf{p}][[x]]$. An explicit series expansion of s can be obtained by Lagrange's Implicit Function Theorem (see [3], for example). Let ψ_i denote $\psi_i(s, \mathbf{p})$. The next result gives a closed form expression for G_1 in terms of s, and indicates the fundamental importance of the series s to the solution of the partial differential equation given in (3).

Theorem 4.2

$$G_1(x,\mathbf{p}) = \frac{1}{24}\log(1-\psi_1)^{-1} - \frac{1}{24}\psi_0.$$

Proof: For a partition $\alpha = (\alpha_1,\ldots,\alpha_m)$, let $a_\alpha = a_{\alpha_1}\cdots a_{\alpha_m}$ and

$$g(x,\mathbf{p}) = \sum_{n\geq 1}\sum_{\substack{\alpha\vdash n\\ l(\alpha)=m}} \frac{n^{m-1}}{\vartheta(\alpha)} a_\alpha p_\alpha x^n,$$

a constituent of the series G_1 given in (5), since $\vartheta(\alpha) = n!/|\mathcal{C}_\alpha|$. It is easily shown that

$$x\frac{\partial g}{\partial x} = \sum_{n\geq 1} x^n [t^n] e^{n\psi_0(t,\mathbf{p})}$$

so, by Lagrange's Implicit Function Theorem,

$$x\frac{\partial g}{\partial x} = \frac{\psi_1}{1-\psi_1}.$$

But, from (7),

$$x\frac{\partial s}{\partial x} = \frac{s}{1-\psi_1} \tag{8}$$

and from (6),

$$\frac{\partial \psi_i}{\partial s} = \frac{1}{s}\psi_{i+1}.$$

Then $x\partial(g-\psi_0)/\partial x = 0$ and, since $g(0,\mathbf{p}) = 0$, it follows that $g(x,\mathbf{p}) = \psi_0$.

Next we consider the terms of G_1 in (5) that are not included in $g(x,\mathbf{p})$. First, note that

$$\sum_{\theta\in\mathcal{P}} \frac{p_\theta}{\vartheta(\theta)} = \exp\sum_{i\geq 1}\frac{p_i}{i},$$

so, replacing p_i by ntp_ia_i for $i\geq 1$ in Lemma 4.1, we have

$$\frac{\vartheta(\alpha)}{k!}[p_\alpha t^n]\left(e^{\psi_0(t,\mathbf{p})}\right)^n \psi_1^k(t,\mathbf{p}) = n^{m-k}a_\alpha e_k(\alpha),$$

where $m = l(\alpha)$. Then

$$\begin{aligned} a_\alpha n^m - \sum_{k\geq 2}(k-2)!n^{m-k}a_\alpha e_k(\alpha) &= \vartheta(\alpha)[p_\alpha t^n]\left(1-\sum_{k\geq 2}\frac{1}{k(k-1)}\psi_1^k(t,\mathbf{p})\right)\left(e^{\psi_0(t,\mathbf{p})}\right)^n \\ &= \vartheta(\alpha)[p_\alpha x^n]\frac{1}{1-\psi_1}\left(1-\sum_{k\geq 2}\frac{1}{k(k-1)}\psi_1^k\right) \end{aligned}$$

by Lagrange's Implicit Function Theorem, for $n\geq 1$. Then

$$a_\alpha n^m - \sum_{k\geq 2}(k-2)!n^{m-k}a_\alpha e_k(\alpha) = \vartheta(\alpha)[p_\alpha x^n]\left(1+\log(1-\psi_1)^{-1}\right).$$

The result follows by combining the two expressions that have been obtained and by using the fact that $G_1(0,\mathbf{p}) = 0$. $\square$

5 The proof of Theorem 1.1

The remaining portion of the paper is concerned with the proof of Theorem 1.1.

Proof: Our strategy is to show that the expression for G_1 given in Theorem 4.2 satisfies the partial differential equation given in (4). We begin by considering the derivatives that are required in the determination of $T_0G_1 - T_1$. From the functional equation (7)

$$\frac{\partial s}{\partial p_k} = \frac{1}{k}\frac{a_k s^{k+1}}{1-\psi_1}. \tag{9}$$

Then, for $k \geq 1$,

$$\frac{\partial \psi_j}{\partial p_k} = k^{j-1} a_k s^k + \frac{a_k}{k}\frac{\psi_{j+1} s^k}{1-\psi_1}. \tag{10}$$

The only derivatives of F_0 that are needed are

$$\frac{\partial F_0}{\partial p_k} = \frac{a_k}{k^3} s^k - \frac{a_k}{k^2}\sum_{r\geq 1} a_r p_r \frac{s^{k+r}}{k+r}, \tag{11}$$

from Proposition 3.1 [2] and, from (9) and (11),

$$\frac{\partial^2 F_0}{\partial p_i \partial p_j} = \frac{a_i a_j}{ij}\frac{s^{i+j}}{i+j}, \tag{12}$$

for $i, j \geq 1$. For completeness we note that, from Proposition 3.1 [2],

$$\left(x\frac{\partial}{\partial x}\right)^2 F_0 = \psi_0.$$

The derivatives of G_1 that are needed are, from (9),

$$x\frac{\partial G_1}{\partial x} = \frac{1}{24}\left(\frac{\psi_2}{(1-\psi_1)^2} - \frac{\psi_1}{1-\psi_1}\right) \tag{13}$$

and, from (10), for $k \geq 1$,

$$\frac{\partial G_1}{\partial p_k} = \frac{1}{24} a_k \frac{s^k}{1-\psi_1} + \frac{1}{24}\frac{a_k}{k} s^k \left(\frac{\psi_2}{(1-\psi_1)^2} - \frac{1}{1-\psi_1}\right). \tag{14}$$

Then from Lemma 3.1 and expressions (11), (12), (13) and (14) it follows that

$$\begin{aligned} 24(1-\psi_1)^2(T_0G_1 - T_1) &= \psi_2(1+\psi_0) - \psi_0(1-\psi_1) - 12(1-\psi_1)^2 A - (1-\psi_1)B \\ &\quad +(1-\psi_1)C - (\psi_1+\psi_2-1)D + (\psi_1+\psi_2-1)E, \end{aligned} \tag{15}$$

where

$$\begin{aligned}
A &= \sum_{i,j\geq 1} \frac{a_i a_j}{i+j} p_{i+j} s^{i+j}, \\
B &= \sum_{i,j\geq 1} \frac{i a_i a_j}{j^2} p_{i+j} s^{i+j}, \\
C &= \sum_{i,j,m\geq 1} \frac{i a_i a_j a_m}{j(j+m)} p_{i+j} p_m s^{i+j+m} - \frac{1}{2} \sum_{i,j\geq 1} (i+j) a_{i+j} p_i p_j s^{i+j}, \\
D &= \sum_{i,j\geq 1} \frac{a_i a_j}{j^2} p_{i+j} s^{i+j}, \\
E &= \sum_{i,j,m\geq 1} \frac{a_i a_j a_m}{j(j+m)} p_{i+j} p_m s^{i+j+m} - \frac{1}{2} \sum_{i,j\geq 1} a_{i+j} p_i p_j s^{i+j}.
\end{aligned}$$

When the expression (15) is transformed by replacing $p_i s^i$ by y_i, for $i \geq 1$, it is immediately seen to be a polynomial in $y_1, y_2, \ldots$ of degree 3 with rational coefficients. If U_i denotes the degree i part of the expression, for $i = 1, \ldots, 3$, the transformed expression can be written in the form

$$24(1-\psi_1)^2(T_0 G_1 - T_1) = U_1 + U_2 + U_3$$

where

$$\begin{aligned}
U_1 &= \psi_2 - \psi_0 - 12A - B + D, \\
U_2 &= \psi_0(\psi_1 + \psi_2) + 24\psi_1 A + \psi_1 B + C - (\psi_1 + \psi_2) D - E, \\
U_3 &= -12\psi_1^2 A - \psi_1 C + (\psi_1 + \psi_2) E.
\end{aligned}$$

Then $U_i \in \mathcal{H}_i[y_1, y_2, \ldots]$, the set of homogeneous polynomials of degree i in $y_1, y_2, \ldots$. Let

$$\varpi_{1,\ldots,i} \colon \mathcal{H}_i[y_1, y_2, \ldots] \longmapsto \mathbb{Q}[x_1, x_2, \ldots]$$

be the symmetrization operation defined by

$$\varpi_{1,\ldots,i}(y_{\alpha_1} \cdots y_{\alpha_i}) = \sum_{\pi \in \mathfrak{S}_i} x_{\pi(1)}^{\alpha_1} \cdots x_{\pi(i)}^{\alpha_i},$$

extended linearly to $\mathcal{H}_i[y_1, y_2, \ldots]$. Then $\varpi_{1,\ldots,i} f = 0$ implies that $f = 0$ for $f \in \mathcal{H}_i[y_1, y_2, \ldots]$.

We therefore prove that $U_i = 0$ by proving that $\varpi_{1,\ldots,i} U_i = 0$, for $i = 1, 2, 3$. To determine the action of the symmetrization operator on $A, \ldots, E$ it is convenient to introduce the series $w = w(x)$ as the unique solution of the functional equation

$$w = x e^w \tag{16}$$

in the ring $\mathbb{Q}[[x]]$. By Lagrange's Implicit Function Theorem we have

$$w = \sum_{n\geq 1} \frac{n^{n-1}}{n!} x^n.$$

Now let $w_i = w(x_i)$ and $w_i^{(j)} = (x_i\partial/\partial x_i)^j w_i$. Then, from (16),

$$w_i^{(1)} = \frac{w_i}{1-w_i}, \quad w_i^{(2)} = \frac{w_i}{(1-w_i)^3}, \quad w_i^{(3)} = \frac{w_i+2w_i^2}{(1-w_i)^5}. \tag{17}$$

The action of the symmetrizing operator on $A, \ldots, E$ and their products with ψ_i can be determined in terms of these as follows.

It is readily seen that

$$\varpi_1(\psi_m) \quad = \quad w_1^{(m+1)}, m \geq -1.$$

For $\varpi_1(A)$, using (17), we have

$$\varpi_1(A) = \sum_{k\geq 1} \frac{x_1^k}{k}[x_1^k]\left(w_1^{(2)}\right)^2 = \int_0^{x_1}\left(w_1^{(2)}\right)^2\frac{dx_1}{x_1} = \int_0^{w_1}\frac{w_1}{(1-w_1)^5}dw_1$$

so, by rearrangement

$$\varpi_1(A) \quad = \quad \frac{1}{12}\left((1-w_1)w_1^{(3)} + w_1 w_1^{(2)} - w_1^{(1)}\right).$$

Trivially,

$$\varpi_1(B) = w_1^{(3)} w_1.$$

Next, $\varpi_{1,2}(C)$ is the symmetrization of

$$\sum_{i,j,m\geq 1}\frac{i a_i a_j a_m}{j(j+m)} x_1^{i+j} x_2^m - \frac{1}{2}\sum_{i,j\geq 1}(i+j)a_{i+j}x_1^i x_2^j$$

with respect to x_1 and x_2. Now

$$\begin{aligned}\sum_{i,j,m\geq 1}\frac{i a_i a_j a_m}{j(j+m)} x_1^{i+j} x_2^m &= w_1^{(3)}\sum_{m\geq 1} a_m x_2^m \sum_{j\geq 1}\frac{a_j}{j}\frac{x_1^j}{j+m} \\ &= w_1^{(3)}\sum_{m\geq 1} a_m x_2^m \frac{1}{x_1^m}\int_0^{x_1} w_1^{(1)} x_1^{m-1}dx_1.\end{aligned}$$

But, from (17), and integrating by parts, we obtain

$$\int_0^{x_1} w_1^{(1)} x_1^{m-1} dx_1 = \int_0^{w_1} w_1^m e^{-mw_1} dw_1 = \frac{1}{a_m}\left(1 - x_1^m\sum_{i=1}^m \frac{m^{m-i}}{(m-i)!}\frac{1}{w_1^i}\right) - \frac{x_1^m}{m}.$$

Thus

$$\begin{aligned}\sum_{i,j,m\geq 1}\frac{i a_i a_j a_m}{j(j+m)} x_1^{i+j} x_2^m &= w_1^{(3)}\left(\frac{x_2}{x_1-x_2} - \sum_{m\geq 1} x_2^m \sum_{i=1}^m \frac{m^{m-i}}{(m-i)!}\frac{1}{w_1^i} - w_2^{(1)}\right) \\ &= w_1^{(3)}\left(\frac{x_2}{x_1-x_2} - \sum_{m\geq 1} x_2^m [t^m] e^{mt}\left(\left(1-\frac{t}{w_1}\right)^{-1} - 1\right) - w_2^{(1)}\right) \\ &= w_1^{(3)}\left(\frac{x_2}{x_1-x_2} - \frac{w_2}{w_1-w_2}\frac{1}{1-w_2} - w_2^{(1)}\right)\end{aligned}$$

by the Lagrange Implicit Function Theorem. Moreover, it is easily seen that

$$\sum_{i,j\geq 1}(i+j)a_{i+j}x_1^i x_2^j = \frac{x_2 w_1^{(3)} - x_1 w_2^{(3)}}{x_1 - x_2}.$$

Thus, by symmetrizing the indicated linear combination of these sums, we have

$$\varpi_{1,2}(C) \quad = \quad -w_1^{(3)}w_2^{(1)} - w_1^{(1)}w_2^{(3)} - \frac{w_1^{(3)}w_2^{(1)} - w_1^{(1)}w_2^{(3)}}{w_1 - w_2}.$$

Trivially,

$$\varpi_1(D) = w_1^{(2)}w_1.$$

Finally, $\varpi_{1,2}(E)$ is obtained in a fashion similar to $\varpi_{1,2}(C)$. The expression is

$$\varpi_{1,2}(E) \quad = \quad -w_1^{(2)}w_2^{(1)} - w_1^{(1)}w_2^{(2)} - \frac{w_1^{(2)}w_2^{(1)} - w_1^{(1)}w_2^{(2)}}{w_1 - w_2}.$$

These results may be combined to give expressions for the symmetrizations of U_1, U_2, U_3 as follows.

For the term of degree one,

$$\varpi_1(U_1) \quad = \quad w_1^{(3)} - w_1^{(1)} - \left((1-w_1)w_1^{(3)} + w_1 w_1^{(2)} - w_1^{(1)}\right) - w_1^{(3)}w_1 + w_1^{(2)}w_1.$$

For the term of degree two, after rearrangement,

$$\begin{aligned}\varpi_{1,2}(U_2) \quad = \quad & \left(w_1^{(2)}w_2^{(3)} + w_1^{(3)}w_2^{(2)}\right)(2 - w_1 - w_2) + w_1^{(2)}w_2^{(2)}(w_1 + w_2) \\ & - \frac{w_1^{(3)}w_2^{(1)} - w_1^{(1)}w_2^{(3)}}{w_1 - w_2} + \frac{w_1^{(2)}w_2^{(1)} - w_1^{(1)}w_2^{(2)}}{w_1 - w_2}.\end{aligned}$$

When multiplied by $w_1 - w_2$ and a suitable power of $(1-w_1)^{-1}$ and $(1-w_2)^{-1}$ this becomes a polynomial in w_1 and w_2 that is identically zero.

For the term of degree three, after rearrangement,

$$\begin{aligned}\varpi_{1,2,3}(U_3) \quad = \quad & \frac{1}{w_2 - w_3}\left(w_1^{(2)}\left(w_2^{(3)}w_3^{(1)} - w_2^{(1)}w_3^{(3)}\right) - \left(w_1^{(2)} + w_1^{(3)}\right)\left(w_2^{(2)}w_3^{(1)} - w_2^{(1)}w_3^{(2)}\right)\right) \\ & + \frac{1}{w_1 - w_3}\left(w_2^{(2)}\left(w_1^{(3)}w_3^{(1)} - w_1^{(1)}w_3^{(3)}\right) - \left(w_2^{(2)} + w_2^{(3)}\right)\left(w_1^{(2)}w_3^{(1)} - w_1^{(1)}w_3^{(2)}\right)\right) \\ & + \frac{1}{w_1 - w_2}\left(w_3^{(2)}\left(w_1^{(3)}w_2^{(1)} - w_1^{(1)}w_2^{(3)}\right) - \left(w_3^{(2)} + w_3^{(3)}\right)\left(w_1^{(2)}w_2^{(1)} - w_1^{(1)}w_2^{(2)}\right)\right) \\ & - 2w_1^{(2)}w_2^{(2)}w_3^{(3)}(1 - w_3) - 2w_1^{(2)}w_3^{(2)}w_2^{(3)}(1 - w_2) - 2w_2^{(2)}w_3^{(2)}w_1^{(3)}(1 - w_1) \\ & - 2w_1^{(2)}w_2^{(2)}w_3^{(2)}(w_1 + w_2 + w_3).\end{aligned}$$

When multiplied by $(w_1 - w_2)(w_2 - w_3)(w_1 - w_3)$ and a suitable power of $(1-w_1)^{-1}$, $(1-w_2)^{-1}$ and $(1-w_3)^{-1}$ this becomes a polynomial in w_1, w_2 and w_3 that is identically zero.

It is quickly seen that $\varpi_1(U_1)$ is zero. For both $\varpi_{1,2}(\mathcal{U}_2)$ and $\varpi_{1,2,3}(\mathcal{U}_3)$, however, the polynomial expressions were sufficiently large that it was convenient to use Maple to carry out the routine simplification of this stage.

Thus the symmetrization of $24(1-\psi_1)^2(T_0G_1 - T_1) = 0$ so $T_0G_1 - T_1 = 0$. It follows from Lemma 3.1 that $F_1 = G_1$ and this completes the proof of Theorem 1.1. $\square$

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