Transitive factorisations in the symmetric group, and combinatorial aspects of singularity theory

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July 22, 1997

Abstract

We consider the determination of the number $c_k(\alpha)$ of ordered factorisations of an arbitrary permutation on n symbols, with cycle distribution α, into k-cycles such that the factorisations have minimal length and such that the group generated by the factors acts transitively on the n symbols. The case $k = 2$ corresponds to the celebrated result of Hurwitz on the number of topologically distinct holomorphic functions on the 2-sphere that preserve a given number of elementary branch point singularities. In this case the monodromy group is the full symmetric group. For $k = 3$, the monodromy group is the alternating group, and this is another case that, in principle, is of considerable interest.

We conjecture an explicit form, for arbitrary k, for the generating series for $c_k(\alpha)$, and prove that it holds for factorisations of permutations with one, two and three cycles (α is a partition with at most three parts). The generating series is naturally expressed in terms of the symmetric functions dual to the those introduced by Macdonald for the "top" connection coefficients in the class algebra of the symmetric group.

Our approach is to determine a differential equation for the generating series from a combinatorial analysis of the creation and annihilation of cycles in products under the minimality condition.

1. Introduction

1.1. Background

This paper has two goals. The first is to provide some general techniques to assist in the solution of the type of permutation factorisation questions, with transitivity and minimality conditions, that originate in the classical study of holomorphic mappings and branched coverings of Riemann surfaces. Thus, we are concerned with certain *combinatorial* questions that are encountered in aspects of *singularity theory*. The appearance of such questions has long been recognized, and the reader is directed to Arnold [1], for example, for further instances.

Very briefly, the classical construction concerns rational mappings from a Riemann surface to the sphere. Let α be the partition formed by the orders of the poles of this mapping.

The poles are mapped to the point at infinity. Each factor in an ordered factorisation is associated with a distinguished branch point, and it specifies the sheet transitions imposed in a closed tour of the branch point, starting from an arbitrarily chosen base point on the codomain of the mapping. In the generic case, the sheet transitions are transposition (2-cycles). The concatenation of the tours for each branch point, from the same base point, in the designated order, gives a sheet transition that is the product of the sheet transitions for each branch point. But this sheet transition is a permutation with α as its cycle-type. The transitivity condition ensures that the ramified covering is connected, so the resulting Riemann surface is a ramified covering of a sphere. The minimality condition ensures that the covering surface is a sphere also. The monodromy group is the group freely generated by the sheet transitions.

The particular class of permutation factorisation questions that we shall consider in this paper involve as factors only k-cycles, for some fixed, but arbitrary, value of k. The results that we are able to obtain are thus extensions of Hurwitz's [12] result with transpositions as factors, which arose in the singularity theory context described above.

The second goal is to investigate the possibility of determining analogues of Macdonald's "top" symmetric functions that will be appropriate for accommodating the transitivity condition. (It will be recalled that Macdonald's top symmetric functions are associated in a fundamental way with minimal ordered factorisations.) The possibility of this connection arises from the fact that there is a striking common element between the results of this paper on transitive minimal ordered factorisations, and Macdonald's symmetric functions. This common element is the functional equation

$$w = xe^{w^{k-1}}, \tag{1}$$

that arises in both settings when k-cycles are factors, for apparently different reasons. The nature of this possible connection is explored more fully in Section 1.5.

We will refer to these two contexts again, as the ideas in this paper are developed. However, for the most part we now regard ordered factorisations as discrete structures and we treat them by combinatorial techniques. Throughout, we work in the appropriate ring of formal power series. Thus, for example, the functional equation (1) has a unique solution for formal power series in x. Although we have not completely attained the two goals, we have provided a substantial amount of methodology for the first, and concrete evidence for the second. We hope that the results are substantial enough to provoke others to explore further.

1.2. Minimal ordered factorisations

Let $\kappa(\pi)$ denote the number of cycles in $\pi \in \mathfrak{S}_n$. There is an obvious restriction on $\kappa(\pi)$ under permutation multiplication.

Proposition 1.1. *Let $\pi, \pi' \in \mathfrak{S}_n$. Then*

$$(n - \kappa(\pi)) + (n - \kappa(\pi')) \geq (n - \kappa(\pi\pi')).$$

If $(\sigma_1, \ldots, \sigma_j) \in \mathfrak{S}_n^j$ and $\sigma_1 \cdots \sigma_j = \pi$, then $(\sigma_1, \ldots, \sigma_j)$ is called an *ordered factorisation* of π. Immediately from Proposition 1.1, we obtain the inequality

$$\sum_{i=1}^{j}(n - \kappa(\sigma_i)) \geq n - \kappa(\pi). \tag{2}$$

In the case of equality, we call $(\sigma_1, \ldots, \sigma_j) \in \mathfrak{S}_n^j$ a *minimal* ordered factorisation of π.

Such factorisations have an elegant theory and many enumerative applications (see, for example, Goulden and Jackson [9]), including permissible commutation of adjacent factors. In particular, [9] contains an explicit construction for a set of symmetric functions (Macdonald's top symmetric functions) that we shall return to in Section 1.5 of the Introduction. Now we turn to the topic of the present paper.

1.3. Minimal transitive ordered factorisations

We write $\alpha \vdash n$ to indicate that α is a partition of n, and $\mathcal{C}_\alpha$ for the conjugacy class of $\mathfrak{S}_n$ indexed by α. Let $l(\alpha)$ denote the number of parts in α. If $\pi \in \mathcal{C}_\alpha$ then $\kappa(\pi) = l(\alpha)$. An ordered factorisation $(\sigma_1, \ldots, \sigma_j)$ is said to be *transitive* if the subgroup of $\mathfrak{S}_n$ generated by the factors acts transitively on $\{1, \ldots, n\}$. The case where each of the factors is in $\mathcal{C}_{[k,1^{n-k}]}$, and is therefore a pure k-cycle, is of particular interest. A transitive ordered factorisation of $\pi \in \mathcal{C}_\alpha$ with the minimal choice of j consistent with the other conditions is said to be *minimal.* In this case, $j = \mu_k(\alpha)$, where

$$\mu_k(\alpha) = \frac{n + l(\alpha) - 2}{k - 1},$$

as we shall prove in Proposition 2.1. For example, when $k = 3$,

$$(247)(586)(479)(136)(235) = (1386)(254)(79), \tag{3}$$

and $((247), (568), (479), (136), (235))$ is a minimal transitive ordered factorisation of the permutation $(136)(2854)(79)$, into 3-cycles with 5 factors (minimality holds in this example since $\mu_3([4,3,2]) = 5$).

Such factorisations are encountered in a number of contexts. These include, for example, the topological classification of polynomials of given degree and a given number of critical values, and the moduli space of covers of the Riemann sphere and properties of the Hurwitz monodromy group, and applications to mathematical physics [2]. The reader is directed to [4, 5, 13] for further background information.

The number of minimal transitive ordered factorisations of an arbitrary but fixed $\pi \in \mathcal{C}_\alpha$ is denoted by $c_k(\alpha)$. Hurwitz [12] determined $c_2(\alpha)$, as a consequence of his study of holomorphic mappings on the sphere (see also Strehl [16], for the proof of an identity that completes Hurwitz's treatment). He showed that

$$c_2(\alpha) = n^{l(\alpha)-3}(n + l(\alpha) - 2)! \prod_{j=1}^{l(\alpha)} \frac{\alpha_j^{\alpha_j}}{(\alpha_j - 1)!}. \tag{4}$$

A shorter and self-contained proof of this result has been given by Goulden and Jackson [8]. The special case $c_2([1^n])$ was derived independently by Crescimanno and Taylor [2]. For related work, in the language of singularity theory, see [15].

The case $k = 3$ is also of considerable interest, for the subgroup generated is the alternating group, and it is therefore hoped that the techniques of this paper will provide information about the number of components in the covering of the alternating group.

1.4. The results and a conjecture

The main conjecture of the paper concerns the form of the generating series for the $c_k(\alpha)$. Let $u, z, p_1, p_2, \ldots$ be indeterminates and let $p_\alpha = p_{\alpha_1} p_{\alpha_2} \ldots$. Then

$$F_k^{(m)}(u, z; p_1, p_2, \ldots) \quad = \sum_{\substack{n \geq 1 \\ k-1 | n+m-2}} \sum_{\substack{\alpha \vdash n \\ l(\alpha) = m}} c_k(\alpha) \, |\mathcal{C}_\alpha| \, p_\alpha \frac{u^{\mu_k(\alpha)}}{\mu_k(\alpha)!} \frac{z^n}{n!}.$$

The series $F_k^{(m)}$ is a formal power series in z with coefficients that are polynomial in $u, p_1, p_2, \ldots$, and we will be working in this ring.

It is more convenient to work with a symmetrised form of the generating series, defined in terms of the following operator ψ_m. If α is a partition with m parts, then

$$\psi_m \left(p_\alpha u^i z^j \right) = \sum_{\sigma \in \mathfrak{S}_m} x_1^{\alpha_{\sigma(1)}} \cdots x_m^{\alpha_{\sigma(m)}}. \tag{5}$$

Now define

$$P_k^{(m)}(x_1, \ldots, x_m) = \psi_m(F_k^{(m)}).$$

In the main conjecture that follows, we let $w_i = w(x_i)$ for $i \geq 1$, and $w(x)$ is the unique power series solution of the functional equation given in (1).

Conjecture 1.2. *For $m \geq 1$,*

$$\left(\sum_{i=1}^m x_i \frac{\partial}{\partial x_i} \right)^{3-m} P_k^{(m)}(x_1, \ldots, x_m) = S_k^{(m)}(w_1, \ldots, w_m) \prod_{i=1}^m x_i \frac{dw_i}{dx_i},$$

where $S_k^{(m)}(w_1, \ldots, w_m)$ is a symmetric polynomial in $w_1, \ldots, w_m$.

The conjectured form for the series $P_k^{(m)}$ therefore involves rational expressions in $w_1, \ldots, w_m$. To see this, differentiate (1) with respect to x, to obtain the rational form

$$x \frac{dw}{dx} = \frac{w}{1 - (k-1) w^{k-1}}. \tag{6}$$

Note that the dependence on k rests in the coefficients of the symmetric polynomial (which we conjecture to be polynomials in k), but more deeply in the functional equation (1). The explicit formal power series for w is actually straightforward, and obtained immediately by Lagrange's Theorem, yielding

$$w(x) = \sum_{m \geq 0} \frac{(1 + (k-1)m)^{m-1}}{m!} x^{1+(k-1)m}. \tag{7}$$

In this paper, we are able to determine explicitly $P_k^{(m)}$ for the cases $m = 1, 2, 3$. These are all of a form that satisfies the above conjecture. The resulting expressions for $S_k^{(m)}$ in these cases are stated below. Let $V(w_1, \ldots, w_m)$ denote the Vandermonde determinant in $w_1, \ldots, w_m$.

Theorem 1.3. $S_k^{(1)}(w_1) = 1$.

Theorem 1.4. $S_k^{(2)}(w_1, w_2) = (w_1^{k-1} - w_2^{k-1})^2 / V(w_1, w_2)^2$.

Theorem 1.5. $S_k^{(3)}(w_1, w_2, w_3) = G^2/V(w_1, w_2, w_3)^2$, *where*

$$\begin{aligned} G &= w_1\left(1-(k-1)w_1^{k-1}\right)(w_3^{k-1}-w_2^{k-1}) + w_2\left(1-(k-1)w_2^{k-1}\right)(w_1^{k-1}-w_3^{k-1}) \\ &\quad + w_3\left(1-(k-1)w_3^{k-1}\right)(w_2^{k-1}-w_1^{k-1}). \end{aligned}$$

The proofs of these results are given in Section 4 of the paper. The method is to solve a partial differential equation for $P_k^{(m)}$ that is obtained in Section 3. This equation is itself deduced by symmetrising a partial differential equation for $F_k^{(m)}$ that is obtained in Section 2. The latter is determined by a combinatorial analysis of minimal permutation multiplication.

The determination of further cases, at present, seems to be intractable, as we discuss in Section 5. The forms obtained above in the first three cases are remarkably simple, although it has not been possible to conjecture a sufficiently precise general form based on this evidence. Although $S_k^{(1)}$, by default, $S_k^{(2)}$ and $S_k^{(3)}$ are perfect squares, we do not believe that this holds in general.

Note that $S_2^{(m)}$ does not restrict to $S_2^{(m-1)}$ through $w_m = 0$, in the cases $m = 2$ and $m = 3$. Also note that if we substitute $k = 2$ in Theorems 1.4 and 1.5 above, then we immediately obtain $S_2^{(2)} = S_2^{(3)} = 1$. In the following result, we demonstrate that this is true when $k = 2$ for arbitrary choice of m, as a direct consequence of Hurwitz's result.

Lemma 1.6. $S_2^{(m)}(w_1, \ldots, w_m) = 1$ *for* $m \geq 1$.

Proof: From (4),

$$\begin{aligned} \left(\sum_{i=1}^m x_i \frac{\partial}{\partial x_i}\right)^{3-m} P_2^{(m)}(x_1, \ldots, x_m) &= \sum_{n \geq 1} \sum_{\substack{\alpha \vdash n \\ l(\alpha)=m}} \frac{|\mathcal{C}_\alpha|}{n!} \left(\prod_{j=1}^m \frac{\alpha_j^{\alpha_j+1}}{\alpha_j!}\right) \sum_{\sigma \in \mathfrak{S}_m} x_1^{\alpha_{\sigma(1)}} \cdots x_m^{\alpha_{\sigma(m)}} \\ &= \frac{1}{m!} \sum_{\alpha_1, \ldots, \alpha_m \geq 1} \left(\prod_{j=1}^m \frac{\alpha_j^{\alpha_j}}{\alpha_j!}\right) \sum_{\sigma \in \mathfrak{S}_m} x_1^{\alpha_{\sigma(1)}} \cdots x_m^{\alpha_{\sigma(m)}} \\ &= \prod_{j=1}^m x_j \frac{dw_j}{dx_j}. \end{aligned}$$

The result now follows. □

We note that, in the case of transpositions, together with Vainshtein, we have recently been able to obtain similar results in the case where there are two more than the minimal number of factors. These correspond to holomorphic mappings from the torus. This work will be reported elsewhere.

1.5. Symmetric functions and minimal ordered factorisations

In [9](see also [14]) an explicit construction is given for symmetric functions u_λ, indexed by $\lambda \vdash n$, that are closely related to minimal ordered factorisations in the symmetric group (note that the term "top" was used for such factorisations in that paper; these are Macdonald's *top symmetric functions*). In particular, the number of minimal ordered factorisations $(\sigma_1, \ldots, \sigma_j)$ of π, where $\sigma_i \in \mathcal{C}_{\beta_i}, i = 1, \ldots, j$, and for each $\pi \in \mathcal{C}_\lambda$, is given by

$$[u_{\lambda-1}]\, u_{\beta_1-1} \cdots u_{\beta_j-1}, \tag{8}$$

where $\beta_i - 1$ is the partition obtained by subtracting one from each part of β_i. Properties that can be developed for u_λ then facilitate the determination of this number. Several examples of their use in enumerative questions is given in [9], together with the enumeration of minimal ordered factorisations up to permissible commutation of adjacent factors.

We now recall the algebraic construction for the symmetric functions u_λ, where $\lambda \vdash n$. Let $H(t; \mathbf{x})$ be the generating series for the complete symmetric functions $h_k(\mathbf{x})$ of degree k in $\mathbf{x} = (x_1, x_2, \ldots)$. Then the functional equation $s = t\, H(t; \mathbf{x})$ has a unique solution $t \equiv t(s, \mathbf{x})$ given by $t = s\, H^\star(s; \mathbf{x})$ where $H^\star(s; \mathbf{x}) = \sum_{j \geq 0} s^j h_j^\star(\mathbf{x})$, and $h_j^\star(\mathbf{x})$ is a symmetric function in $\mathbf{x}$ of total degree j. Let $h_\lambda^\star = h_{\lambda_1}^\star h_{\lambda_2}^\star \cdots$. Then $\{u_\lambda\}$ is defined to be the basis for the symmetric function ring that is dual to the basis $\{h_\lambda^\star\}$ with respect to the inner product for which the monomial and complete symmetric functions are dual (see, e.g. Macdonald [14], for a complete treatment of the required background material).

Thus, for minimal ordered factorisations in which all factors are k-cycles, then in equation (8), we have $u_{\beta_i - 1} = u_{k-1}$ for all $i = 1, \ldots, j$. But, as is shown in [9], $u_{k-1} = -p_{k-1}$, so for minimal ordered factorisations in which all factors are k-cycles, we can restrict attention to a symmetric function algebra in which $p_i = 0$ if $i \neq k-1$. In this case, we have

$$s = tH(t; \mathbf{x}) = \exp\left(\sum_{m \geq 1} \frac{p_m}{m} t^m\right) = t \exp\left(\frac{-p_{k-1}}{k-1} t^{k-1}\right).$$

Thus, if z is substituted for $\frac{p_{k-1}}{k-1}$, in this equation, we obtain

$$t = s e^{z t^{k-1}}.$$

But this is precisely the functional equation (1), whose solution features so centrally in our results for the transitive case above.

We conclude from this that there must be an important relationship between the transitive case of minimal ordered factorisations for which we have obtained partial results in this paper, and minimal ordered factorisations themselves, that have such an elegant theory based on symmetric functions. Although we have been unable to find a direct link between these two classes, we hope that the results of this paper will provide a good starting point for such a direct link, and a similarly elegant theory for the transitive case.

2. The partial differential equation

In this section we determine a partial differential equation for the generating series

$$\Phi^{(k)} = \sum_{m\geq 1} F_k^{(m)}$$

by a case analysis of the creation and annihilation of cycles in products of permutations subject to the minimality condition.

We begin with a discussion of permutation multiplication. First, we prove the expression that has been given in Section 1.3 for $\mu_k(\pi)$.

Proposition 2.1. *Let $\alpha \vdash n$, and let $\pi \in \mathcal{C}_\alpha$. Then $\mu_k(\pi) = \mu_k(\alpha)$, where*

$$\mu_k(\alpha) = \frac{n + l(\alpha) - 2}{k - 1}.$$

Proof: Let $(\sigma_1, \dots, \sigma_j)$ be a minimal transitive ordered factorisation of π into k-cycles. Let π' and π be in the same conjugacy class, so $\pi' = g^{-1}\pi g$ for some $g \in \mathfrak{S}_n$. Then $(g^{-1}\sigma_1 g, \dots, g^{-1}\sigma_j g)$ is a minimal transitive ordered factorisation of π', so $\mu_k(\pi') = \mu_k(\pi)$, and we denote the common value by $\mu_k(\alpha)$ where $\pi \in \mathcal{C}_\alpha$. Now each k-cycle in $\mathfrak{S}_k$ has a minimal transitive ordered factorisation into $\mu_2([k])$ transpositions, so $\mu_2(\alpha) = \mu_2([k])\,\mu_k(\alpha)$. But (Prop. 2.1, [8]), $\mu_2(\alpha) = n + l(\alpha) - 2$, and the result follows. $\square$

Next we give a combinatorial characterisation of minimal transitive ordered factorisations. The following lemma characterises the relationship between σ_1 and $\sigma_2 \cdots \sigma_j$ for a minimal transitive ordered factorisation $(\sigma_1, \dots, \sigma_j)$ of $\pi \in \mathfrak{S}_n$ into k-cycles. Some terminology will be useful. The multi-graph $\mathcal{D}_{\sigma_1,\dots,\sigma_j}$ has vertex-set $\{1, \dots, n\}$, and edges consisting of the edges of the k-cycles in the factorisation. Let $\mathcal{V}_1, \dots, \mathcal{V}_l$ be the vertex-sets of the connected components of $\mathcal{D}_{\sigma_2,\dots,\sigma_j}$, so $\{\mathcal{V}_1, \dots, \mathcal{V}_l\}$ is a partition of $\{1, \dots, n\}$ into nonempty subsets. For $i = 1, \dots, l$, let α_i consist of all $t \in \{2, \dots, j\}$ such that all of the k elements on σ_t belong to $\mathcal{V}_i$, so $\{\alpha_1, \dots, \alpha_l\}$ is a partition of $\{2, \dots, j\}$. Suppose $\alpha_i = \{\alpha_{i1}, \dots, \alpha_{is_i}\}$, with $\alpha_{i1} < \cdots < \alpha_{is_i}$, and $\sigma_{\alpha_{i1}} \cdots \sigma_{\alpha_{is_i}} = \pi_i$, for $i = 1, \dots, l$. Then clearly, by construction, $(\sigma_{\alpha_{i1}}, \cdots, \sigma_{\alpha_{is_i}})$ is a minimal transitive ordered factorisation of π_i, for $i = 1, \dots, l$, and we have

$$\pi = \sigma_1 \pi_1 \cdots \pi_l. \tag{9}$$

For example, in the minimal transitive factorisation given in (3), we have $l = 2$, with $\mathcal{V}_1 = \{1, 2, 3, 5, 6, 8\}$, and $\mathcal{V}_2 = \{4, 7, 9\}$; $\alpha_1 = \{2, 4, 5\}$, and $\alpha_2 = \{3\}$; $\pi_1 = (1386)(25)$, and $\pi_2 = (479)$.

For $\pi \in \mathfrak{S}_n$ and $\mathcal{A} \subseteq \{1, \dots, n\}$, the $\mathcal{A}$-*restriction* of π is the permutation on $\mathcal{A}$ obtained by deleting the elements not in $\mathcal{A}$ from the cycles of π. For example, if $\pi = (1538)(27469)$ and $\mathcal{A} = \{1, 4, 6, 7, 8\}$, then the $\mathcal{A}$- restriction of π is $(18)(467)$.

Lemma 2.2. *Let $(\sigma_1, \dots, \sigma_j)$ be a minimal transitive ordered factorisation of $\pi \in \mathfrak{S}_n$ into k-cycles, and let $\pi_1, \dots, \pi_l$ be constructed as above. Then*

1. *σ_1 has at least one element in common with each of $\pi_1, \dots, \pi_l$.*

2. *The elements of σ_1 in common with π_i lie on a single cycle of π_i, for $i = 1, \dots, l$.*

3. *Let $\mathcal{U}$ denote the k-subset of $\{1,\ldots,n\}$ consisting of the elements on the k-cycle σ_1. Let γ denote the $\mathcal{U}$-restriction of σ_1, and let τ denote the $\mathcal{U}$-restriction of π. If $\rho=\gamma\tau$, then $(k-\kappa(\rho))+(k-\kappa(\tau))=k-\kappa(\gamma)$, so (ρ,τ^{-1}) is a minimal ordered factorisation of γ.*

Proof: Since $(\sigma_1,\ldots,\sigma_j)$ is a transitive factorisation of π, then $\mathcal{D}_{\sigma_1,\ldots,\sigma_l}$ is connected. Thus the single k-cycle in D_{σ_1} has at least one vertex in each of the connected components of $\mathcal{D}_{\sigma_2,\ldots,\sigma_l}$, and this establishes part 1.

Now, from (9) and the fact that $(\sigma_{\alpha_{i1}},\cdots,\sigma_{\alpha_{is_i}})$ is a minimal transitive ordered factorisation of π_i, for $i=1,\ldots,l$, we have

$$\mu(\pi)=1+\mu(\pi_1)+\cdots+\mu(\pi_l). \tag{10}$$

But, from Proposition 2.1

$$\mu(\pi)=\frac{n+\kappa(\pi)-2}{k-1} \qquad \text{and} \qquad \mu(\pi_i)=\frac{|\mathcal{V}_i|+\kappa(\pi_i)-2}{k-1},$$

for $i=1,\ldots,l$. Thus, substituting these values for the $\mu's$ into (10) we obtain

$$n+\kappa(\pi)-2=k-1+\sum_{i=1}^{l}(|\mathcal{V}_i|+\kappa(\pi_i)-2).$$

But $n=\sum_{i=1}^{l}|\mathcal{V}_i|$, and substituting this into the above gives

$$\kappa(\pi)-\sum_{i=1}^{l}\kappa(\pi_i)=k+1-2l. \tag{11}$$

Now let ρ_i be the $\mathcal{U}$-restriction of π_i, for $i=1,\ldots,l$, so $\tau=\pi_1\cdots\pi_l$. Then

$$\kappa(\pi)-\sum_{i=1}^{l}\kappa(\pi_i)=\kappa(\rho)-\sum_{i=1}^{l}\kappa(\rho_i)=\kappa(\rho)-\kappa(\tau),$$

so from (11) we have

$$\kappa(\rho)-\kappa(\tau)=k+1-2l. \tag{12}$$

On the other hand, since γ,ρ and τ act on a k-set and $\rho\tau^{-1}=\gamma$ we have from Proposition 1.1 that $(k-\kappa(\rho))+(k-\kappa(\tau))\geq(k-\kappa(\gamma))$. But $\kappa(\gamma)=1$, so $\kappa(\rho)+\kappa(\tau)\leq k+1$, and in addition, from part 1 we have $\kappa(\tau)\geq l$. It follows that $\kappa(\rho)-\kappa(\tau)\leq k+1-2\kappa(\tau)\leq k+1-2l$. Combining this with (12) gives $\kappa(\tau)=l$. Together with part 1, this establishes part 2.

Part 3 follows immediately from $\kappa(\tau)=l$, $\kappa(\gamma)=1$ and (12). □

We now use this characterisation as a construction for deriving a partial differential equation for $\Phi^{(k)}$ with arbitrary k. In the interests of succinctness, we suppress the occurrences of k in $\Phi^{(k)}$ and $P_k^{(m)}$. From Lemma 2.2(.3), the terms in the equation are in one-to-one correspondence with minimal ordered factorisations of a k-cycle. These factorisations are themselves in one-to-one correspondence with a particular class of trees, as was shown in [10], and described as follows: Let $\mathcal{B}^{(k)}$ be the set of all plane two-coloured (black, white) trees with k edges, with the indices $i_1, \ldots, i_k$ assigned to different edges in a canonical way. Let $\mathfrak{t}$ be such a tree. For $v \in \mathfrak{t}$ let $\omega(v)$ be the sum of the indices of edges incident with v. Let $\widehat{\mathfrak{t}}$ denote the tree obtained from $\mathfrak{t}$ by deleting monovalent white vertices. Let $\mathrm{aut}\,(\widehat{\mathfrak{t}})$ denote the automorphism group of $\widehat{\mathfrak{t}}$ with the convention that if $\widehat{\mathfrak{t}}$ is an isolated black vertex, then $\mathrm{aut}\,(\widehat{\mathfrak{t}})$ is the cyclic group on k symbols.

Theorem 2.3. *Let $\mathbf{i} = (i_1, \ldots, i_k)$ where $i_1, \ldots, i_k \geq 1$. Then Φ satisfies the nonlinear, inhomogeneous partial differential equation*

$$\sum_{\mathbf{i} \geq \mathbf{1}} \sum_{\mathfrak{t} \in \mathcal{B}^{(k)}} \frac{1}{|\mathrm{aut}\,(\widehat{\mathfrak{t}})|} \left(\prod_{\mathsf{v} \in \mathcal{V}_{black}(\mathfrak{t})} p_{\omega(\mathsf{v})} \prod_{\mathsf{w} \in \mathcal{V}_{white}(\mathfrak{t})} \omega(\mathsf{w}) \frac{\partial \Phi}{\partial p_{\omega(\mathsf{w})}} \right) = \frac{\partial \Phi}{\partial u}, \tag{13}$$

with the convention that empty sums are zero and empty products are equal to one.

Proof: From Lemma 2.2(.3), (ρ, τ^{-1}) is a minimal ordered factorisation in $\mathfrak{S}_k$ of the k-cycle γ. Thus, from [10] Theorem 2.1, (ρ, τ^{-1}) uniquely encodes an edge-rooted 2-coloured plane tree $\mathfrak{t}$ with k edges, such that the black vertex-degrees are given by the cycle-type of τ^{-1}, and the white vertex-degrees are given by the cycle-type of ρ.

We now observe that, in the product $\gamma\tau$, cycles with length equal to the degree of each of the black vertices are annihilated, and combined to form cycles of length equal to the degree of a white vertex. This observation permits us to reconstruct the cycle distribution of π from σ_1 and the cycle distributions of $\pi_1, \ldots, \pi_l$.

The tree $\mathfrak{t}$ can be regarded as the boundary of a polygon. As the boundary of $\mathfrak{t}$ is traversed, each edge is encountered twice, once in the direction from its black vertex towards its white vertex, and once in the direction from its white vertex towards its black vertex. The indexed symbols $i_1, \ldots, i_k$ are assigned to the edges of $\mathfrak{t}$, starting from the root-edge, as each edge is encountered in the direction from its black vertex towards its white vertex. Moreover, i_j is the number of elements in $\{1, \ldots, n\}$ that separate two elements in σ_1 on cycles in π_i.

In this encoding the degree of a black vertex is the number of elements of σ_1 that are incident with π_i and the number of black vertices is $\kappa(\tau)$. This indicates which cycles in $\pi_1, \ldots, \pi_l$ are annihilated in premultiplication by σ_1 and which cycles are created. It is necessary only to keep track of the lengths of these cycles.

The contribution from cycles that are created is therefore

$$\prod_{\mathsf{v} \in \mathcal{V}_{black}(\mathfrak{t})} p_{\omega(\mathsf{v})}.$$

The contribution from cycles that are annihilated is

$$\prod_{\mathsf{w} \in \mathcal{V}_{white}(\mathfrak{t})} \omega(\mathsf{w}) \frac{\partial \Phi}{\partial p_{\omega(\mathsf{w})}}.$$

To see this, select one of the cycles ρ_i. Next select an element on it. Then mark off the cycle into a number of contiguous segments equal to the degree of the corresponding black vertex in $\mathfrak{t}$. However, this overcounts by a factor of $1/|\mathrm{aut}\,(\widehat{\mathfrak{t}})|$. Thus summing we have

$$\sum_{\mathbf{i}\geq 1} \sum_{\mathfrak{t}\in\mathcal{B}^{(k)}} \frac{1}{|\mathrm{aut}\,(\widehat{\mathfrak{t}})|} \left(\prod_{\mathsf{v}\in\mathcal{V}_{black}(\mathfrak{t})} p_{\omega(\mathsf{v})} \prod_{\mathsf{w}\in\mathcal{V}_{white}(\mathfrak{t})} \omega(\mathsf{w}) \frac{\partial\Phi}{\partial p_{\omega(\mathsf{w})}} \right).$$

But this is equal to the generating series for minimal transitive ordered factorisations with the leftmost factor deleted. But this is $\partial\Phi/\partial u$. The result now follows. □

Note that, if p_i is the power sum symmetric function of degree k in an infinite set of ground variables, then $j\partial/\partial p_j = p_j^\star$, where $p_j^\star$ is the adjoint of premultiplication by p_j (see, e.g., [14] for details). The partial differential equation therefore can be rewritten in the following form, that exhibits the symmetry between black and white vertices, as

$$\sum_{\mathbf{i}\geq 1} \sum_{\mathfrak{t}\in\mathcal{B}^{(k)}} \frac{1}{|\mathrm{aut}\,(\widehat{\mathfrak{t}})|} \left(\prod_{\mathsf{v}\in\mathcal{V}_{black}(\mathfrak{t})} p_{\omega(\mathsf{v})} \right) \left(\prod_{\mathsf{w}\in\mathcal{V}_{white}(\mathfrak{t})} p^\star_{\omega(\mathsf{w})}\Phi \right) = \frac{\partial\Phi}{\partial u}.$$

It will be useful to list explicitly the first few trees on the left hand side of (13) in the arbitrary case, graded by the number of black vertices in $\widehat{\mathfrak{t}}$, to find the equations for the low order terms of Φ, in the p's. We consider below all of the trees with at most three black vertices.

First tree: Let $\widehat{\mathfrak{t}_1}$ be the tree in $\mathcal{B}_k$ consisting of one black vertex joined to k white vertices. Assign $i_1,\ldots,i_k$ to the edges. Then $\widehat{\mathfrak{t}_1}$ is the tree consisting of an isolated black vertex. By the convention on automorphisms, $\mathrm{aut}\,(\mathfrak{t}_1) = k$.
Second tree: Let $\widehat{\mathfrak{t}_2}$ be the tree in $\mathcal{B}_k$ consisting of a path $\widehat{\mathfrak{t}_2}$ with two black vertices and one white vertex. Attach i_{k-1} and i_k to the two edges incident with the white vertex. Now join r white vertices to one of the black vertices, and attach $i_1,\ldots,i_r$ to the edges. Join $k-r-2$ white vertices to the other black vertex, and attach labels $i_{r+1},\ldots,i_{k-2}$ to the edges. The resulting tree $\mathfrak{t}_2$ therefore has k edges, and it is readily seen that $\widehat{\mathfrak{t}_2}$ is the tree obtained from $\mathfrak{t}_2$ by removing monovalent white vertices. Moreover, $|\mathrm{aut}\,(\widehat{\mathfrak{t}_2})| = 2$.

For the trees with three black vertices, we give only the tree from which monovalent white vertices have been removed. The summation variables $i_1,\ldots,i_k$ are attached to edges in the way described in the previous cases.
Third tree: Let $\widehat{\mathfrak{t}_3}$ be the tree consisting of one white vertex to which three black vertices are joined. Then $|\mathrm{aut}\,(\widehat{\mathfrak{t}_3})| = 3$. Note that, in this case, the path separates into two sets the additional white vertices that are joined to the black vertex in the middle of the path.
Fourth tree: Let $\widehat{\mathfrak{t}_4}$ be the path consisting of three black vertex and two white vertices. Then $|\mathrm{aut}\,(\widehat{\mathfrak{t}_4})| = 2$.

The partial differential equations for minimal transitive ordered factorisations into 2-cycles, and into 3-cycles, can be written down explicitly from the terms that have been given. Let $\Phi_j \equiv j\partial\Phi/\partial p_j$, for$j\geq 1$.

When $k=2$ the only trees with two edges correspond to $\widehat{\mathfrak{t}_1}$ and$\widehat{\mathfrak{t}_2}$, so in this case (13) becomes

$$\tfrac{1}{2} \sum_{i_1,i_2\geq 1} \left(\Phi^{(2)}_{i_1}\Phi^{(2)}_{i_2} p_{i_1+i_2} + \Phi^{(2)}_{i_1+i_2} p_{i_1}p_{i_2} \right) = \frac{\partial\Phi^{(2)}}{\partial u}. \tag{14}$$

This is the equation given in [8], where we demonstrated that a series conjectured from numerical computations satisfied the equation uniquely.

When $k=3$ the only trees with three edges correspond to $\widehat{\mathfrak{t}_1}$, $\widehat{\mathfrak{t}_2}$ and $\widehat{\mathfrak{t}_3}$, so

$$\sum_{i_1,i_2,i_3\geq 1}\left(\tfrac{1}{3}\Phi^{(3)}_{i_1}\Phi^{(3)}_{i_2}\Phi^{(3)}_{i_3}p_{i_1+i_2+i_3}+\tfrac{1}{3}\Phi^{(3)}_{i_1+i_2+i_3}p_{i_1}p_{i_2}p_{i_3}+\Phi^{(3)}_{i_1}\Phi^{(3)}_{i_2+i_3}p_{i_1+i_2}p_{i_3}\right)=\frac{\partial\Phi^{(3)}}{\partial u}. \tag{15}$$

We do not know of any method for solving this equation for $\Phi^{(3)}$ explicitly, and have not been able to conjecture the solution from numerical computations, as we could for $k=2$. However, as we show in the next section, we are able to determine the low order terms of $\Phi^{(k)}$ in the p's, for arbitrary k.

3. Restriction of the differential equation by grading

In this section we determine a partial differential equation for $P_k^{(m)}$ that can be used recursively to construct $P_k^{(m)}$ for all $m\geq 1$. Our method is to apply the symmetrisation operator ψ_m to the partial differential equation (13) given in Theorem 2.3. Some notation is needed for this purpose. Let

$$h_i^+(x_1,\ldots,x_k)=\sum_{\substack{j_1,\ldots,j_k\geq 1\\ j_1+\cdots+j_k=i}}x_1^{j_1}\cdots x_k^{j_k},$$

for $i\geq 1$, and $h_0^+(x_1,\ldots,x_k)=1$. Now let $H^+(t;x_1,\ldots,x_k)=\sum_{i\geq 0}h_i^+(x_1,\ldots,x_k)t^i$. If $a(t)$ and $b(t)$ are the generating series for $\{a_i\}$ and $\{b_i\}$, let $a\circ b$ denote the summation $\sum_{i\geq 0}a_ib_i$. This is essentially the *umbral* composition of a and b with respect to t, which will be the only indeterminate used in this paper for umbral composition.

We begin by showing in two particular examples how the action of ψ_m, defined in (5), can be expressed conveniently in terms of umbral composition. These will suffice to indicate the general procedure. The first example is the application of ψ_2 to the partial differential equation (14). As a preliminary, apply ψ_2 to the final term on the left hand side, which yields

$$\sum_{i_1,i_2\geq 1}x_1^{i_1}x_2^{i_2}(i_1+i_2)\frac{\partial}{\partial p_{i_1+i_2}}F_2^{(1)}=\sum_{j\geq 1}\left(\sum_{\substack{i_1,i_2\geq 1\\ i_1+i_2=j}}x_1^{i_1}x_2^{i_2}\right)j\frac{\partial}{\partial p_j}F_2^{(1)}=t\frac{\partial}{\partial t}P_2^{(1)}(t)\circ H^+(t;x_1,x_2).$$

Note that the presence of umbral composition in this expression is explained in an entirely elementary way. The other terms require no explanation, and application of ψ_2 to the partial differential equation (14) yields

$$\tfrac{1}{2}x_1\frac{\partial}{\partial x_1}P_2^{(1)}(x_1)x_1\frac{\partial P_2^{(2)}}{\partial x_1}+\tfrac{1}{2}x_2\frac{\partial}{\partial x_2}P_2^{(1)}(x_2)x_2\frac{\partial P_2^{(2)}}{\partial x_2}+t\frac{\partial}{\partial t}P_2^{(1)}(t)\circ H^+(t;x_1,x_2)$$
$$=\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right)P_2^{(2)}.$$

Now, for more variables and more complicated equations the main complication is the proliferation of terms that arises from adding the contributions from permuting the variables.

To organize this we introduce another symmetrisation operator, Ξ_m, defined on power series in $x_1, \ldots, x_m$ by

$$\Xi_m f(x_1, \ldots, x_m) = \sum_{\sigma \in \mathfrak{S}_m}{}' f(x_{\sigma(1)}, \ldots, x_{\sigma(m)}),$$

where the "$'$" indicates that the summation is over distinct terms.

For the second example, illustrating this complication, we apply ψ_4 just to the third term on the left hand side of the partial differential equation (15), to obtain

$$\Xi_4 \left(x_1 \frac{\partial}{\partial x_1} P_3^{(3)}(x_1, x_3, x_4) t \frac{\partial}{\partial t} P_3^{(1)}(t) \circ H^+(t; x_1, x_2) + x_1 \frac{\partial}{\partial x_1} P_3^{(2)}(x_1, x_3) t \frac{\partial}{\partial t} P_3^{(2)}(t, x_4) \circ H^+(t; x_1, x_2) \right.$$
$$\left. + x_1 \frac{\partial}{\partial x_1} P_3^{(1)}(x_1) t \frac{\partial}{\partial t} P_3^{(3)}(t, x_3, x_4) \circ H^+(t; x_1, x_2) \right).$$

Without further discussion, in the following result, we now apply the operator ψ_m to the partial differential equation (13) given in Theorem 2.3 directly, yielding a partial differential equation for $P_k^{(m)}$, for all $m \geq 1$. This equation is in terms of $P_k^{(1)}, \ldots, P_k^{(m-1)}$, for $m \geq 2$, and we will use it recursively, starting with $P_k^{(1)}$, in Section 4. In the statement of the result, let

$$\Psi(z, x_1;) = \sum_{m \geq 1} x_1 \frac{\partial}{\partial x_1} P^{(m)}(x_1;) z^{m-1}.$$

In addition, we adopt the convention that $P^{(i)}(x_1;) P^{(j)}(x_1;)$ denotes $P^{(i)}(x_1, \alpha) P^{(j)}(x_1, \beta)$, for $i + j = n$, where (α, β) is a canonical bipartition of $\{1, \ldots, n\} - \{1, 2\}$ of size $(i-1, j-1)$. Let $\mathcal{N}_{\widehat{\mathfrak{t}}}(w)$ be the neighbour set of $w \in \mathcal{V}_{\widehat{\mathfrak{t}}}$.

Theorem 3.1. *For $m \geq 1$, the partial differential equation for $P^{(m)}$ is*

$$\left[y^k z^m \right] \sum_{\widehat{\mathfrak{t}}} \frac{\Xi_m}{\operatorname{aut}(\widehat{\mathfrak{t}})} \left(\prod_{\mathsf{v} \in \mathcal{V}_{black}(\widehat{\mathfrak{t}})} z \left(\frac{y}{1 - y \Psi(z, x_{\mathsf{v}};)} \right)^{d_{\widehat{\mathfrak{t}}}(\mathsf{v})} \right) \left(\prod_{\mathsf{w} \in \mathcal{V}_{white}(\widehat{\mathfrak{t}})} \Psi(z, t;) \circ H^+(t; \mathcal{N}_{\widehat{\mathfrak{t}}}(\mathsf{w})) \right)$$
$$= \frac{1}{k-1} \left(x_1 \frac{\partial}{\partial x_1} + \cdots + x_m \frac{\partial}{\partial x_m} + m - 2 \right) P^{(m)}.$$

Proof: Let m denote the number of black vertices in $\widehat{\mathfrak{t}}$. Assign the symbols $1, \ldots, m$ arbitrarily to these vertices.

With each black vertex v of $\widehat{\mathfrak{t}}$ associate the expression

$$z \left(\frac{y}{1 - y \Psi(z, x_{\mathsf{v}};)} \right)^{d_{\widehat{\mathfrak{t}}}(\mathsf{v})}.$$

This accounts for the attachment of monovalent white vertices by edges to v.

With each white vertex w of $\widehat{\mathfrak{t}}$ associate the expression

$$\Psi(z, t;) \circ H^+(t; \mathcal{N}_{\widehat{\mathfrak{t}}}(\mathsf{w})).$$

The result follows by taking the product of these expressions. $\square$

The application of the coefficient operator $[y^k z^m]$ is routine but increasingly laborious as m increases. In the next section we will carry this out for $m = 1, 2, 3$. The following result will be needed to give explicit forms for the umbral composition with H^+.

Proposition 3.2. *Let $f(t)$ be a formal power series in t. Then*

$$f(t) \circ H^{+}(t; x_1, \ldots, x_k) = \sum_{i=1}^{k} f(x_i) \prod_{\substack{1<p\leq k \\ p\neq i}} \frac{x_p}{x_i - x_p}.$$

4. Proofs of the supporting theorems

4.1. Proof of Theorem 1.3

Consider the case $m = 1$ in Theorem 3.1. Then since contributions on the left hand side come only from the tree $\widehat{\mathfrak{t}_1}$, we obtain the differential equation

$$\frac{1}{k}\left(x_1 \frac{dP^{(1)}}{dx_1}\right)^k = \frac{1}{k-1}\left(x_1 \frac{d}{dx_1} - 1\right) P^{(1)}$$

for $P^{(1)}$. To solve this equation, differentiate the equation with respect to x_1 and multiply by x_1. Then, with $f = x_1 dP^{(1)}/dx_1$, we obtain

$$f^{k-1} x_1 \frac{df}{dx_1} = \frac{1}{k-1} x_1 \frac{df}{dx_1} - \frac{1}{k-1} f,$$

so, solving for $x_1 \frac{df}{dx_1}$, we have

$$x_1 \frac{df}{dx_1} = \frac{f}{1 - (k-1) f^{k-1}}.$$

It is now straightforward to determine, for formal power series in x, that $f = w_1$, by comparing this differential equation with (6), and using the initial condition $f(0) = 0$. The result follows immediately. □

4.2. Proof of Theorem 1.4

Consider the case $m = 2$ in Theorem 3.1. Now contributions on the left hand side come only from the trees $\widehat{\mathfrak{t}_1}$ and $\widehat{\mathfrak{t}_2}$. Thus, substituting the expression for $P^{(1)}$ from Theorem 1.3, and applying Proposition 3.2 to carry out the umbral compositions, we obtain

$$w_1^{k-1} x_1 \frac{\partial P^{(2)}}{\partial x_1} + w_2^{k-1} x_2 \frac{\partial P^{(2)}}{\partial x_2} + \frac{x_2 w_1 - x_1 w_2}{x_1 - x_2} \frac{w_1^{k-1} - w_2^{k-1}}{w_1 - w_2} = \frac{1}{k-1}\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) P^{(2)}.$$

so, rearranging, we have

$$\frac{1}{k-1}\left((1-(k-1)w_1^{k-1}) x_1 \frac{\partial}{\partial x_1} + (1-(k-1)w_2^{k-1}) x_2 \frac{\partial}{\partial x_2}\right) P^{(2)} = \frac{x_2 w_1 - x_1 w_2}{x_1 - x_2} \frac{w_1^{k-1} - w_2^{k-1}}{w_1 - w_2}.$$

It is now straightforward to verify that

$$P^{(2)}(x_1, x_2) = \log\left(\frac{w_1 - w_2}{x_1 - x_2}\right) - \frac{w_1^k - w_2^k}{w_1 - w_2},$$

by confirming that it satisfies the above differential equation, and the initial condition $P^{(2)}(0,0) = 0$. (Note that the constant term in the expansion of $(w_1 - w_2)/(x_1 - x_2)$ as a formal power series in x_1, x_2 is 1, so the logarithm exists.)

Finally, apply the operator $x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$ to $P^{(2)}$, and the result follows. □

4.3. Proof of Theorem 1.5

Consider the case $m = 3$ in Theorem 3.1. Then since contributions on the left hand side come only from the trees $\widehat{\mathfrak{t}}_1, \ldots, \widehat{\mathfrak{t}}_4$, having substituted the expression for $P^{(1)}$ from Theorem 1.3, it follows that

$$\begin{aligned}
& \Xi_3 w_1^{k-1} x_1 \frac{\partial}{\partial x_1} P^{(3)} \\
& +\Xi_3 (k-1) w_1^{k-2} \left(x_1 \frac{\partial}{\partial x_1} P^{(2)}(x_1, x_2) \right) \left(x_1 \frac{\partial}{\partial x_1} P^{(2)}(x_1, x_3) \right) \\
& +\Xi_3 \left(\frac{\partial}{\partial w_1} \frac{w_1^{k-1} - w_2^{k-1}}{w_1 - w_2} \right) \left(x_1 \frac{\partial}{\partial x_1} P^{(2)}(x_1, x_3) \right) \left(w(t) \circ H^+(t; x_1, x_2) \right) \\
& +\Xi_3 \left(\frac{w_1^{k-1} - w_2^{k-1}}{w_1 - w_2} \right) \left(t \frac{\partial}{\partial t} P^{(2)}(t, x_3) \circ H^+(t; x_1, x_2) \right) \\
& +\Xi_3 2 h_{k-3}(w_1, w_2, w_3) \left(w(t) \circ H^+(t; x_1, x_2, x_3) \right) \\
& +\Xi_3 \left(\frac{\partial}{\partial w_2} h_{k-3}(w_1, w_2, w_3) \right) \left(w(t) \circ H^+(t; x_1, x_2) \right) \left(w(t) \circ H^+(t; x_2, x_3) \right) \\
= \; & \frac{1}{k-1} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + 1 \right) P^{(3)}.
\end{aligned}$$

The six expressions on the left hand side arise from $\widehat{\mathfrak{t}}_1$, $\widehat{\mathfrak{t}}_1$, $\widehat{\mathfrak{t}}_2$, $\widehat{\mathfrak{t}}_2$, $\widehat{\mathfrak{t}}_3$ and $\widehat{\mathfrak{t}}_4$, respectively. Note that, under the action of Ξ_3, the six expressions on the left hand side expand into $3, 3, 6, 3, 1$ and 3 terms, respectively. Now apply Proposition 3.2 to carry out the umbral compositions, and use the fact that

$$(1 - (k-1) w^{k-1}) x \frac{\partial}{\partial x} = w \frac{\partial}{\partial w} \tag{16}$$

(this latter follows from (6)). Simplifying with the help of Maple, we obtain

$$\begin{aligned}
& \frac{1}{k-1} \left(\sum_{i=1}^{3} w_i \frac{\partial}{\partial w_i} + 1 \right) P^{(3)} = (k-1) \left(w_1^{k-2} A_{12} A_{13} + w_2^{k-2} A_{21} A_{23} + w_3^{k-2} A_{31} A_{32} \right) \\
& + \frac{w_1^{k-1} - w_2^{k-1}}{(w_1 - w_2)^2} (w_2 A_{13} - w_1 A_{23}) + \frac{w_1^{k-1} - w_3^{k-1}}{(w_1 - w_3)^2} (w_3 A_{12} - w_1 A_{32}) \\
& + \frac{w_2^{k-1} - w_3^{k-1}}{(w_2 - w_3)^2} (w_3 A_{21} - w_2 A_{31}),
\end{aligned}$$

where

$$A_{ij} = \frac{w_i w_j}{1 - (k-1) w_i^{k-1}} \frac{w_i^{k-1} - w_j^{k-1}}{(w_i - w_j)^2}.$$

The solution to this equation is given in Theorem 1.5, and has been verified with the aid of Maple, giving the desired result. $\square$

5. Computational comments and conjectures

We have shown in Section 4 that $P^{(1)}, P^{(2)}$ and $P^{(3)}$ can each be obtained as the solutions to first order linear partial differential equations. We believe that $P^{(m)}$, for $m \geq 4$, can be obtained in a similar way as the solution of such an equation. Moreover, we conjecture that the equation for any $m \geq 3$,(obtained from Theorem 3.1, and applying (16) as described for $m = 1, 2, 3$ in Section 4) after multiplying through by $k-1$, is of the form

$$\left(\sum_{i=1}^{m} w_i \frac{\partial}{\partial w_i} + (m-2)\right) P^{(m)} = R_m(w_1, \ldots, w_m),$$

where R_m is a rational function in $w_1, \ldots, w_m$, obtained from $P^{(1)}, \ldots, P^{(m-1)}$. That is, there is no dependency of R_m on $x_1, \ldots, x_m$ except through (7). Now let $Q^{(m)}(t)$ be obtained by substituting tw_i for w_i in $P^{(m)}$ for $1 = 1, \ldots, m$. Then the above partial differential equation is transformed into the first order linear ordinary differential equation

$$\frac{d}{dt}(t^{m-2} Q^{(m)}(t)) = t^{m-3} R_m(tw_1, \ldots, tw_m), \tag{17}$$

which can be solved routinely, in theory. In practice, this is precisely how we obtained $P^{(3)}$, with the aid of Maple, in Section 4 above. However, even in this case, the simplification of the equation was difficult; we provided human help by proving that the rational expression on the right hand side of the equation is independent of the x's, and then replaced each x_i by w_i to evaluate it. This explains how the A_{ij} arise, as $x_i \frac{\partial}{\partial x_i} P^{(2)}(x_i, x_j)$ evaluated at $x_i = w_i$ and $x_j = w_j$.

For $m = 4$, the expressions became too big to be tractable, and we have not found a convenient way of circumventing this. We conjecture that, for each $m \geq 3$, $P^{(m)}$ is a rational function of $w_1, \ldots, w_m$, whose denominator is consistent with Conjecture 1.2, using (6). (Note that for $m = 2$, the right hand side of the equation, as obtained in the Proof of Theorem 1.4, is not a rational function of w_1, w_2 alone, but rather involves x_1, x_2 also.)

Acknowledgements

This work was supported by grants from the Natural Sciences and Engineering Research Council of Canada.

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