MAPS, SYMMETRIC FUNCTIONS, MATRIX INTEGRALS, & MATHEMATICAL PHYSICS

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Abstract. This is a collection of deliberately informal Notes on two talks that I gave in Alan Edelman's Seminar on Matrix Integrals, and material devolving from the talks. The aim is to show how the ideas in the title are interrelated. There is a considerable amount of detail hidden below the surface, but no proofs are included. The Notes are intended to give a bird's eye view of the subject matter that is difficult to acquire from the research literature. Details may be obtained from the original papers that are on my web page at Waterloo. I have not made any attempt to cite sources. The work includes some of my own in collaboration with Andrews, Goulden, Harer, Vakil and Visentin, and makes use of ideas of Gessel, Hanlon, 't Hooft, Itzykson, Perry, Stanley, Stembridge, Tutte, Viennot, Zagier, Zuber and others. I have also included a small amount of combinatorial background since this may not be familiar to everyone.

Applications to the counting of ramified covers of the sphere (the Hurwitz Problem) and the determination of the virtual Euler characteristic of the moduli space of curves, which are questions in algebraic geometry, are referred to tangentially, since they can be approached by a natural extension of the approach described here.

Some explicit "algebraic computations" are given at the end, requiring both techniques from matrix integration and hypergeometric transformations. The possibility of obtaining structural information about maps and the structures they themselves encode from such computations, makes this aspect of considerable importance in the theory as a whole.

Where it has been feasible, comments have been made to assist an intuitive understanding of a number of points.

1. Maps

Throughout these Notes a *surface* is assumed to be a 2-manifold without boundary. It is *orientable* if the clockwise and anticlockwise senses around each vertex can be assigned consistently: otherwise it is *non-orientable.* We shall refer to these collectively as *locally orientable.* A *map* $\mathfrak{m}$ is an embedding of a graph (*loops* and *multiple edges* allowed) in a surface so that the deletion of edges decomposes the surface into a union of regions homeomorphic to open discs. The open discs correspond to the *faces* of the map. Such an embedding is called a *2-cell* embedding. Two maps $\mathfrak{a}$ and $\mathfrak{b}$ are *equivalent* ($\mathfrak{a} \sim \mathfrak{b}$) if there is a diffeomorphism of the surface that maps vertices to vertices, edges to edges.

Informally, a map is a graph drawn on a surface in such a way that edges have no self-intersections, and meet only at vertices. Moreover, $\mathfrak{a} \sim \mathfrak{b}$ if one can be transformed into the other by the smooth operations of "rubber sheet geometry,"

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that do not tear the surface. It is clear that the defining property of a map is therefore the order in which edges are incident around each vertex.

1.1. **Example.** A map with one vertex is called a *monopole*. Consider a monopole $\mathfrak{a}$ with two edges labelled a and b, and a vertex v. The map $\mathfrak{a}$ is such that, in an anticlockwise tour of v, these labels are encountered in the order $aabb$. This map is therefore in the sphere. The embedding is a two-cell embedding since deletion of a and b decomposes the sphere into three discs. The same map (the cyclic order of incidence of edges with the vertex must be the same) cannot be embedded in the torus, since deletion of a and b would then decompose the torus into a disc and a cylinder, so it is not a 2-cell embedding.

The map $\mathfrak{b}$ with the cyclic order $abab$ of edges incident with a vertex w cannot be embedded in the sphere, since a and b would have a point of intersection other than w. It can be embedded in the torus since deletion of a and b from the torus decomposes the torus into a single disc. The embedding is therefore a two-cell embedding. Moreover, $\mathfrak{a}$ and $\mathfrak{b}$ are inequivalent since the cyclic ordering $aabb$ cannot be transformed into $abab$ by a smooth operation on the surface.

1.2. **A root of a map.** A map is *rooted* by selecting a mutually incident vertex, edge a face triple. This configuration can be indicated by orienting a unique edge of the map, with the convention that the tail of this edge is the root vertex and the face on the left of the oriented edge is the root face. Such a map is called a *rooted map*. Two rooted maps are equivalent if they are equivalent as maps, and the root of one is mapped onto the root of the other. The only automorphism of a rooted map is the trivial automorphism.

The class of rooted maps is a natural one, and it appears in many applications of the theory of maps.

1.3. **Invariants.** A *map invariant* of a map is a property of a map that is invariant under smooth deformations of the surface. The *vertex distribution* $\delta_v(\mathfrak{m})$ of $\mathfrak{m}$ is the partition listing the vertex degrees. The *face distribution* $\delta_f(\mathfrak{m})$ of $\mathfrak{m}$ is the partition listing the face degrees (the numbers of edges incident with a face). Both of these are map invariants.

1.4. **Generating series (Genus series).** Let $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{i} = (i_1, i_2, \ldots)$ and $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} x_2^{i_2} \cdots$, where $\mathbf{i}$ has only a finite number of non-zero components. In the case of orientable surfaces this is denoted by $M_O(u, \mathbf{x}, \mathbf{y}, z)$, where

$$M_O(u, \mathbf{x}, \mathbf{y}, z) = \sum_{\mathfrak{m}} \mathbf{x}^{\delta_v(\mathfrak{m})} \mathbf{y}^{\delta_f(\mathfrak{m})} z^{\#_e(\mathfrak{m})} = \sum_{\mathbf{i},\mathbf{j},k \geq 0} m_{g,\mathbf{i},\mathbf{j},k} \mathbf{x}^{\mathbf{i}} \mathbf{y}^{\mathbf{j}} z^k u^g,$$

and $m_{g,\mathbf{i},\mathbf{j},k}$ is the number of rooted maps with vertex distribution $\mathbf{i}$, face distribution $\mathbf{j}$ and n edges, in a genus g orientable surface. The sum in the middle constituent is over all rooted maps in orientable surfaces. Note that $M_O \in \mathbb{Q}[u, \mathbf{x}, \mathbf{y}][[z]]$, since there is only a finite number of maps with a specified number of edges. The u is in fact redundant, since genus is recoverable from $\mathbf{i}, \mathbf{j}$ and k by the Euler-Poincaré formula. However, it is will convenient to include it to simplify the presentation of some of the later results. We shall say that x_i *marks* the occurrence of a vertex of degree i, and similarly for other map invariants. M_O is the series whose significance, properties and presentation in terms of classical symmetric functions (Schur, zonal and Jack) and in terms of matrix integrals (Gaussian unitary ensemble (GUE) and Gaussian orthogonal ensemble (GOE)) that is considered in these Notes.

In algebraic combinatorics we are more interested in finding properties of M_O, and the corresponding series M for maps in locally orientable surfaces, than in explicit expressions for its coefficients. However, in mathematical physics, there has been considerable interest in obtaining asymptotic expressions associated with $m_{g,\mathbf{i},\mathbf{j},k}$ for increasing genera since these provide successive approximations to the "partition function" defined in the next section. For example, the first approximation is called the "planar approximation."

1.5. **Notational conventions.** Throughout, symbols subscripted by O refer to orientable surfaces, while the corresponding unsubscripted symbol refers to locally orientable surfaces. M_O and M denote the genus series for rooted maps in orientable and locally orientable surfaces, respectively. Similarly, H_O, H are the corresponding series for hypermaps (defined Section 4.1). To avoid the proliferation of notation, we shall abuse mathematical notation slightly by extending and adjusting the arguments of these series to accommodate the exigencies of the context. So, for example, $M_O(u,x,y,z)$ will denote $M_O(u,\mathbf{x},\mathbf{y},z)$ at $x_1=x_2=\cdots=x$ and $y_1=y_2=\cdots=y$, if only the number of vertices and faces are required rather than the distributions. In like vein, $M_O(u,\mathbf{x},\mathbf{y},z\colon b)$ will denote $M_O(u,\mathbf{x},\mathbf{y},z)$ with an additional indeterminate b that marks another invariant of maps.

2. Physics background

Consider a quark-antiquark pair, regarded as being inseparably linked. This pair sweeps out a 2-dimensional surface in space-time, that can be regarded as a meson. We call these surfaces *ribbons*. The interaction of mesons is represented by the meeting of ribbons at a vertex. At a vertex of degree k there is a coupling constant $k!u_k$. 't Hooft showed that the partition function for this system, with N species of quarks is

$$Z_N(u_2,u_3,\ldots\colon\beta)\propto\int_{\mathcal{V}_N}e^{-\beta\,\mathrm{trace}U(\mathbf{M})}d\mathbf{M}$$

where $\beta\geq 0$, $\mathcal{V}_N$ is the set of all $N\times N$ Hermitian complex matrices, $U(z)=u_2z^2+u_3z^3+\cdots$, and the u_i are weights.

Let $U(\xi)=\sum_{k\geq 1}\frac{1}{k}\sqrt{z}^k=\log\left(1-\sqrt{z}\xi\right)^{-1}$ and $U_4(\xi)=\frac{1}{4}z^2\xi^4$. Then the partition functions for the Penner model and the ϕ^4-model of 2-dimensional quantum gravity are

$$\mathcal{Z}_N(U,\beta)=\frac{\int_{\mathcal{V}_N}e^{-\frac{1}{2}\mathrm{tr}\left(\mathbf{M}^2\right)+\beta\,\mathrm{trace}U(\mathbf{M})}d\mathbf{M}}{\int_{\mathcal{V}_N}e^{-\frac{1}{2}\mathrm{tr}(\mathbf{M}^2)}d\mathbf{M}}$$

and

$$\mathcal{Z}_N(U_4,\beta)=\frac{\int_{\mathcal{V}_N}e^{-\frac{1}{2}\mathrm{tr}\left(\mathbf{M}^2\right)+\frac{1}{4}\beta z^2\mathrm{tr}\left(M^4\right)}d\mathbf{M}}{\int_{\mathcal{V}_N}e^{-\frac{1}{2}\mathrm{tr}(\mathbf{M}^2)}d\mathbf{M}}.$$

't Hooft showed that $\log\mathcal{Z}_N$ is a certain limit of the counting series for *ribbon graphs*. We may regard a ribbon graph as a map, since the embedding surface is uniquely reconstructible from it my sewing discs to the boundaries formed by the edges of ribbons. To make contact with the physics it is necessary to let $N\to\infty$ such that the "area" of the surface is constant, where "area" is discretized as $a(\mathfrak{m})=\frac{1}{2}\sum_{v\in V(\mathfrak{m})}\left(d_{\mathfrak{m}}(v)-2\right)$, $V(\mathfrak{m})$ is the vertex set of $\mathfrak{m}$ and $d_{\mathfrak{m}}(v)$ is the degree of the vertex v in $\mathfrak{m}$. This is called the *double-scaling limit.*

In the ϕ^4-model the ribbon graphs have vertices of degree four only, while in the Penner model all maps are permitted. In fact, technically, vertices of degree two are excluded since interior points of an edge are regarded as topologically equivalent, and vertices of degree one are excluded in the "vacuum case." However, such vertices are easy to exclude from all maps, so we shall not worry about this condition.

Theorem 2.1 (J, Perry, Visentin)**.**

$$\mathcal{Z}_{2N}(U_4, 1) = \mathcal{Z}_N\left(U, N - \frac{1}{2}\right)\mathcal{Z}_N\left(U, N + \frac{1}{2}\right).$$

Z_N and $\mathcal{Z}_N$ are the same object, but with slightly different notational conventions because of the scaling by β.

3. Maps in orientable surfaces

Partitions will be used extensively. We write $\theta \vdash n$ to signify that θ is a partition of n. The number of parts of θ is denoted by $l(\theta)$ and is called the *length* of θ, and the *weight* n of θ is denoted by $|\theta|$. The set of all partitions, with the null partition adjoined, is denoted by $\mathcal{P}$.

3.1. **Encoding a map in an orientable surface.** Label the root edge of $\mathfrak{m}$ with 1. Label the remaining $n-1$ edges from 2 to n. Orient each of the non-root edges (the root edge already has an orientation). Let i^+ denote the tail of the edge labelled i and i^- its head, for $i = 1, \ldots, n$. Let ν be the permutation in $\mathfrak{S}_{2n}$ obtained by listing the labels at ends of edges incident with each vertex. By the Embedding Theorem for orientable surfaces, this permutation uniquely specifies the map. It is called a *rotation system.* The attachment of such labels and orientations is called a *decoration* of the map. The power of the mapping from this set of decorated maps with n edges to rooted maps with n edges is

$$\left[2^{n-1}(n-1)!\colon 1\right].$$

This is because the root edge receives the label 1 and an orientation and, since the automorphisms of the map are trivial, the other $n-1$ edges may be labelled in $(n-1)!$ different ways and oriented in 2^{n-1} different ways.

Let $\varepsilon = (1^+, 1^-)\cdots(n^+, n^-)$. The important property of this encoding is that the cycles in the disjoint cycle decomposition of the permutation

$$\phi = \nu\varepsilon$$

list the names of ends of edges that appear in faces, and hence encode faces. The statement of the Embedding Theorem dates back to at least the 1880's, for it appears in the work of Heffter.

Intuitively, the algorithm that underlies this encoding can be thought of as a special case of the maze threading algorithm. To escape from a maze, put your hand on the wall, walk to the next corner while remaining in contact with the wall (an application of ε), and turn anticlockwise as far as possible (an application of ν). Repeat the procedure until you either escape from the maze or are trapped in a cycle. In the case of maps, the interior of any face is a maze, and the maze threading algorithm cycles on the face.

3.2. **The construction of all maps.** The idea now is to apply this construction in reverse to all permutations $\pi \in \mathfrak{S}_{2n}$. This is carried out as follows. The cycles of π correspond to vertices with marked points of attachment for ends of edges. The edges are represented by the elements of ε, and are attached to vertices in accordance with the marked points of attachment. The resulting structure may not necessarily be connected since the group generated by ε and π may not act transitively on $\{1^+, 1^-, \cdots, n^+, n^-\}$. This structure is therefore a disjoint union of maps, and is called a *premap*. Maps may be recovered through the logarithm since they are the connected components of premaps. More will be said later about the appearance of the logarithm.

3.3. **The general strategy.** The strategy, both for orientable surfaces and for locally orientable surfaces, will be to count decorated premaps through the reverse use of an embedding theorem (there is one for locally orientable surfaces), then to restrict algebraically to decorated maps by factoring out the connected premaps, and then algebraically factoring out the decoration. To make this work we need an algebra that preserves the combinatorial information that we desire.

3.4. **The class algebra of $\mathfrak{S}_n$.** The algebra that is appropriate for the orientable case is the class algebra of the group algebra $\mathbb{C}\mathfrak{S}_n$ of $\mathfrak{S}_n$ over $\mathbb{C}$. The group algebra $\mathbb{C}\mathfrak{S}_n$ is a freely additive algebra, with product taken from the group $\mathfrak{S}_n$. The class algebra of $\mathfrak{S}_n$ is the quotient algebra of $\mathbb{C}\mathfrak{S}_n$ generated by $\mathsf{K}_\theta = \sum_{g \in \mathcal{C}_\theta} g$, for $\theta \vdash n$, where $\mathcal{C}_\theta$ is the conjugacy class of $\mathfrak{S}_n$ indexed by θ. It is a commutative algebra. The following lemma makes the crucial bridge between the combinatorics and the class algebra.

Lemma 3.1. *Let $\alpha, \beta, \theta \vdash N$. The the number of ways of expressing an arbitrary element $c \in \mathcal{C}_\theta$ as $c = ab$ where $(a, b) \in \mathcal{C}_\alpha \times \mathcal{C}_\beta$ is*

$$[\mathsf{K}_\theta]\,\mathsf{K}_\alpha \mathsf{K}_\beta.$$

3.5. **Connexion coefficients.** The connexion coefficients $a^\lambda_{\mu,\nu}$ of the class algebra are defined by

$$a^\lambda_{\mu,\nu} = [\mathsf{K}_\lambda]\,\mathsf{K}_\mu \mathsf{K}_\nu.$$

These need to be determined explicitly. One way of thinking about this is to look for elements whose multiplication table is trivial. The obvious elements to look for are orthogonal idempotents (elements a, b such that $ab = 0$ if $a \neq b$ and $a^2 = a$). A theorem from algebra asserts that there is a complete set of these in present case.

3.5.1. *Orthogonal idempotents.* Now $\{\mathsf{K}_\theta : \theta \vdash n\}$ spans the centre of $\mathbb{C}\mathfrak{S}_n$. Let

$$\mathsf{F}_\alpha = \frac{1}{n!} f^\alpha \sum_{\theta \vdash n} \chi^\alpha_\theta \mathsf{K}_\theta$$

for $\alpha \vdash n$. Then

$$\mathsf{F}_\alpha \mathsf{F}_\beta = \mathsf{F}_\alpha \delta_{\alpha,\beta},$$

and $\{\mathsf{F}_\theta : \theta \vdash n\}$. Moreover, by the orthogonality of the characters, the inverse relation is

$$\mathsf{K}_\alpha = h^\alpha \sum_{\theta \vdash n} \frac{1}{f^\theta} \chi^\theta_\alpha \mathsf{F}_\theta,$$

where $h^{\alpha} = |C_{\alpha}|$, χ_{θ}^{α} is the evaluation on $\mathcal{C}_{\theta}$ of the character χ^{α} of the irreducible representation of $\mathfrak{S}_n$ associated with the conjugacy class $\mathcal{C}_{\alpha}$, and $f^{\alpha} = \chi_{[1^n]}^{\alpha}$, the degree of this representation.

3.5.2. *Schur functions.* Let $h_k(\mathbf{x})$ be the complete symmetric function of degree k. It is defined by the generating series $\sum_{k\geq 0} h_k(\mathbf{x})\, t^k = \prod_{i\geq 0}(1-tx_i)^{-1}$, and is extended multiplicatively to partitions by $h_{\alpha} = h_{\alpha_1} h_{\alpha_2}\cdots$. Recall that the generating series for these characters is the Schur function s_{α} and is (Jacobi-Trudi identity)

$$s_{\alpha} = \det\left[h_{\alpha_i - i + j}\right],$$

and that

$$s_{\alpha} = \sum_{\theta\vdash n} g(\theta)\,\chi_{\theta}^{\alpha} p_{\theta}$$

where $g(\theta) = \prod_i i^{m_i(\theta)} m_i(\theta)!$. These expressions enable us to work within the ring of symmetric functions.

3.5.3. *A construction for the Schur functions.* There are several ways of defining Schur functions. The following is a way that is not circular in the present context.. The Schur function $s_{\alpha}(\mathbf{x})$ can be defined combinatorially as the generating series for column-strict plane partitions of shape α. The Gessel-Viennot correspondence shows that these are in bijective correspondence with the set of ordered self-avoiding $l(\alpha)$-paths on the integer sublattice of $\mathbb{R}^2$, on the step set $\{\downarrow, \rightarrow\}$, that start at $l(\alpha)$ origins and end at $l(\alpha)$ termini. The positions of the origins are fixed (independent of α) and termini are controlled by α. The y-coordinate i of the origin of each $\rightarrow$ is marked by an x_i. These paths are enumerated by $\det\left[h_{\alpha_i-i+j}\right]$, where $\operatorname{per}\left[h_{\alpha_i-i+j}\right]$ counts all ordered $l(\alpha)$-paths from the same set of origins and termini. A sign-reversing involution is used to establish this. The generating series for the characters of irreducible representations comes from representation theory.

3.5.4. *Generating series.* The generating series for the connexion coefficients $a_{\mu,\nu}^{\lambda}$, for the class algebra is given by the following lemma.

Lemma 3.2.

$$\sum_{\lambda,\mu,\nu\vdash n} \frac{|\mathcal{C}_{\lambda}|}{|\lambda|!} a_{\mu,\nu}^{\lambda} p_{\lambda}(\mathbf{x})\, p_{\mu}(\mathbf{y})\, p_{\nu}(\mathbf{z}) = \sum_{\theta\in n} \frac{|\theta|!}{f^{\theta}} s_{\theta}(\mathbf{x})\, s_{\theta}(\mathbf{y})\, s_{\theta}(\mathbf{z}).$$

3.6. **Enumerative properties of the logarithm.** The following is a brief combinatorial explanation for the appearance of the logarithm in subsequent expressions. Skip this if it is familiar to you.

3.6.1. *The ordinary generating series.* Let $[x^n]\, f$ denotes the coefficient of x^n in f. The coefficient operator $[x^n]$ is a linear functional on $\mathbb{C}[[x]])$. The ordinary generating series for a set $\mathcal{S}$ of unlabelled objects with respect to a non-negative integer valued weight function ω on $\mathcal{S}$ is $\sum_{s\in\mathcal{S}} x^{\omega(s)}$.

3.6.2. *The exponential generating series.* Let $\mathcal{A}$ and $\mathcal{B}$ be sets of labelled combinatorial objects. For example, $\mathcal{A}$ may be the set $\mathcal{C}$ of all cycles on $\{1,\ldots,n\}$, for $n=0,1,\ldots$. If $a\in\mathcal{A}$ has k labels and $\alpha=\{\alpha_1,\ldots,\alpha_k\}_<$, then $(a)_\alpha$ denotes the object obtained from a by the transformation $k\mapsto\alpha_k$ of labels. Let $\mathcal{A}\star\mathcal{B}$ denote the set of all $\left((a)_\alpha,(b)_\beta\right)$, with $a\in\mathcal{A}$ and $b\in\mathcal{B}$, over all set partitions (α,β) of $\{1,\ldots,n\}$, for $n=0,1,\ldots$. Let $\mathcal{A}\circledast\mathcal{B}$ denote the set of objects $a\circledast(b_1,\ldots,b_k)$, with $b_i\in\mathcal{B}$, where a has k labels, such that there is a unique way of associating b_i with the i-th label of a, for $i=1,\ldots,k$. The operation $\circledast$ is therefore a set theoretic composition with respect to the $\star$-product over disjoint union. Let $A(x)$ and $B(x)$ be the exponential generating series for $\mathcal{A}$ and $\mathcal{B}$, respectively. Recall that the exponential generating series A is defined by $A(x)=\sum_{a\in\mathcal{A}}x^{\lambda(a)}/\lambda(a)!$, where $\lambda(a)$ is the number of labels assigned to a. Then the exponential generating series for $\mathcal{A}\star\mathcal{B}$ and $\mathcal{A}\circledast\mathcal{B}$ are AB and $A\circ B$, respectively. These are counting lemmas.

3.6.3. *Examples.* The following examples illustrate these counting lemmas and the emergence of the logarithm.

Example 3.1. *Let $\mathcal{U}=\{\{1,\ldots,n\}:n=0,1,2,\ldots\}$, and let $\mathcal{L}$ be the set of all permutations on $\{1,\ldots,n\}:n=1,2,\ldots$. Then the disjoint cycle decomposition Ξ can be expressed as*

$$\Xi\colon\mathcal{L}\tilde{\longrightarrow}\mathcal{U}\circledast\mathcal{C}.$$

The exponential generating series for $\mathcal{L}$ is $\sum_{n\geq0}n!\,(x^n/n!)=(1-x)^{-1}$, and the exponential generating series for $\mathcal{U}$ is $\sum_{n\geq0}1\,(x^n/n!)=e^x$, so $(1-x)^{-1}=e^{C(x)}$, whence $C(x)=\log(1-x)^{-1}$. Thus the number of n-cycles is $[x^n/n!]\log(1-x)^{-1}=(n-1)!$.

It is the set theoretic composition of a set with the set $\mathcal{U}$ of all canonical subsets that gives rise to the logarithm.

The following is a classical enumerative question called the Montmort Problem which concerns *derangements* (permutations with no 1-cycles). It is now trivially solved as follows.

Example 3.2. *Let $\mathcal{D}$ be the set of all derangements. By restricting Ξ to $\mathcal{D}$, and by reconstructing the codomain to preserve bijectivity in the restriction, we have*

$$\Xi|_{\mathcal{D}}:\mathcal{D}\tilde{\longrightarrow}\mathcal{U}\circledast\{\mathcal{C}\setminus\{1\}\},$$

since $\{1\}$ is the unique canonical 1-cycle. Then the exponential generating series for $\mathcal{D}$ is $e^{\log(1-x)^{-1}-x}=e^{-x}/(1-x)$, so the number of derangements on n symbols is

$$d_n=\left[\frac{x^n}{n!}\right]\frac{e^{-x}}{1-x}=n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\cdots+(-1)^n\frac{1}{n!}\right).$$

Thus, for example, $\lim_{n\to\infty}d_n/n!=e^{-1}$.

The final example gives the counting series for connected labelled graphs.

Example 3.3. *The generating series for the number of simple (no loops or multiple edges) vertex-labelled graphs with respected to the number of vertices (marked exponentially by x because of the vertex labelling) and the number of edges (marked ordinarily by z) is*

$$\sum_{k\geq0}(1+z)^{\binom{k}{2}}\frac{x^k}{k!}.$$

To see this, note that, given k vertices, there are $\binom{k}{2}$ pairs of vertices which can be either not joined by an edge (contributing $z^0 = 1$) or joined by an edge (contributing z^1). Then the number of connected simple labelled graphs with p vertices and q edges is

$$\left[x^p \frac{z^q}{q!}\right] \log \left(\sum_{k \geq 0} (1+z)^{\binom{k}{2}} \frac{x^k}{k!} \right),$$

since a graph is a disjoint union of its components.

Note the appearance of the logarithm to obtain the prime structures from the composites in the labelled setting.

It is instructive to recognize the additional mathematical machinery that is required to move from graphs to their embedding in surfaces.

3.7. **The genus series.** We may now give an expression for the genus series.

Theorem 3.3 (J,V).

$$M_O(1, \mathbf{x}, \mathbf{y}, z) = \Omega \log R(\mathbf{x}, \mathbf{y}, z), \quad \textit{where}$$

$$R(\mathbf{x}, \mathbf{y}, z) = \sum_{n \geq 0} \frac{z^n}{n!\,(2n)!} \sum_{\nu, \phi \vdash 2n} h^\nu h^\phi \sum_{\theta \vdash 2n} \frac{1}{f^\theta} \chi^\theta_\phi \chi^\theta_\nu \chi^\theta_{[2^n]}$$

and

$$\Omega g(x, y, z) = 2s \frac{\partial}{\partial z} g\left(x, y, \frac{1}{2} z\right).$$

The theorem follows from the encoding of maps described above and the class algebra. R is the generating series for decorated pre-maps. The following observations are made in the context of the general strategy that has been described earlier.

- The restriction from decorated premaps to decorated (connected) maps is through applying log to R.
- Then the restriction from decorated maps to maps is by factoring out the power $\left[2^{n-1}(n-1)!\colon 1\right]$ of the encoding by
  - applying $z\partial/\partial z$ to map $z^n/n!$ onto $z^n/(n-1)!$,
  - dividing each term of degree n in z by regarding the generating series now as an ordinary generating series in z rather than an exponential generating series in z,
  - dividing by 2^{n-1} each term of degree n in z by replacing z by $z/2$ and multiplying by 2 throughout.

In view of the second item we shall refer to Ω as the *undecorating* operator.

3.8. **Quadrangulations.** A *quadrangulation* is a map, each of whose faces has degree four. The following results are required to address the question of finding a relationship between the genus series for quadrangulations and the genus series for all maps in the case when only the numbers of vertices and faces are retained, and not the vertex and face partitions.

Proposition 3.4.

$$\sum_{\alpha \vdash N} h^\alpha \chi^\theta_\alpha x^{l(\alpha)} = f^\theta H_\theta(x)$$

where $H_\theta(x) = \prod_{i=1}^{l(\theta)} (x - i + 1)^{(\theta_i)}$, *and* $(x)^{(k)} = x(x+1)\cdots(x+n-1)$.

Corollary 3.5. *For $x_1 = x_2 = \ldots$, and $y_1 = y_2 = \ldots$, we have*

$$M_O(1, x, y, z) = \Omega \log R(x, y, z), \quad \textit{where}$$
$$R(x, y, z) = \sum_{n \geq 0} \frac{z^n}{n!\,(2n)!} \sum_{\theta \vdash 2n} f^\theta \chi^\theta_{[2^n]} H_\theta(x) H_\theta(y).$$

The next result is a purely technical one.

Lemma 3.6 (J,V)**.** *Let $\mathcal{P}_2$ be the set of all partitions of even non-negative integers. Let q_r be an indeterminate that is zero for odd r. Then there exists a set of partitions $\hat{\mathcal{P}}_2 \subseteq \mathcal{P}$, and a bijection*

$$\Delta\colon \hat{\mathcal{P}}_2 \to \mathcal{P} \times \mathcal{P} \colon \theta \mapsto (\theta', \theta'')$$

such that

$$\det\left[q_{\theta_i+i-j}\right]_{l(\theta)\times l(\theta)} = \begin{cases} \det\left[q_{2\left(\theta'_i+i-j\right)}\right]_{l(\theta)\times l(\theta)} \det\left[q_{2\left(\theta''_i+i-j\right)}\right]_{l(\theta)\times l(\theta)}, & \theta \in \hat{\mathcal{P}}_2, \\ 0, & \theta \notin \hat{\mathcal{P}}_2. \end{cases}$$

Moreover, $H_\theta(x)$ has a factorization that respects Δ.

The above results can be used to prove the following.

Theorem 3.7 (J,V)**.**

$$M_4\left(u^2, x, y, z\right) = \frac{1}{2}\left(M\left(4u^2, x+u, x, z^2y\right) + M\left(4u^2, x-u, x, z^2y\right)\right).$$

This gives the relation between the ϕ^4-model and the Penner model. Recall that we have taken the log, which accounts for this being written additively, rather than multiplicatively. Moreover, in the physics setting, monovalent and bivalent vertices are not allowed. These can be removed algebraically by simple substitution of indeterminates. The details are not important for the purposes of these Notes.

3.9. **The quadrangulation conjecture.** Theorem 3.7 is a very simple relation between the generating series M and M_4, for which there is quite a sophisticated algebraic proof. Its form suggest that there should be a purely combinatorial proof. The left hand side is an even series in u. The right hand side is simply the even bisection of $M\left(4u^2, x+u, x, z^2y\right)$, that is, the expansion of this series with deletion of terms with odd powers of u. The relationship implies the existence of a correspondence between rooted quadrangulations (counted by M_4) and rooted maps, which can be made bijective by suitably decorating the maps to account for multiplicities. The combinatorial problem that presents itself is to find a *natural* bijection between these decorated sets. Implicit in this is the belief that such a bijection would appeal to a significant property of maps. The existence of a natural bijection to account for the above result is called the *Quadrangulation Conjecture.*

Specialization to the sphere is through $u = 0$, and in this case the relation specializes to $M_4(0, x, y, z) = M\left(0, x, x, z^2y\right)$. The construction that accounts for this is well known. It is the *medial construction* of Tutte. This suggest, at the very least, that the medial construction plays a role somehow.

4. Integral presentations of the genus series

One of the easiest ways at this point to obtain an integral representation of the genus series is to use the following result that links the class algebra and matrix integrals.

Lemma 4.1 (J). *Let*

$$\mathcal{I}_\theta(N) = \frac{1}{N!}\sqrt{2\pi}^N \int_{\mathbb{R}^N} V^2(\lambda)\, e^{-p_2/2} s_\theta d\lambda$$

where $V(\lambda)$ *is the Vandermonde determinant. Then*

$$\frac{2^n n!}{\prod_{j=1}^{N-1} j!}\mathcal{I}_\theta(N) = H_\theta(N)\,\chi^\theta_{[2^n]}.$$

Theorem 4.2 (Goulden, J).

$$M(1,\mathbf{x},N,z) = \Omega \log\left(\frac{1}{\sqrt{2\pi}^N \prod_{j=1}^{N-1} j!}\int_{\mathbb{R}^N} V^2(\lambda)\, e^{-p_2/2+\sum_{k\geq 1}\frac{1}{k}\sqrt{z}^k x_k p_k} d\lambda\right).$$

Note that N is an argument of $M(1,\mathbf{x},N,z)$ and is a positive integer. However, we want $M(1,\mathbf{x},y,z)$, where y is an indeterminate. But $M(1,\mathbf{x},y,z) \in \mathbb{Q}[x,y][[z]]$, so $M(1,\mathbf{x},y,z)$ may be recovered from $M(1,\mathbf{x},N,z)$ by first giving $M(1,\mathbf{x},N,z)$ determining an explicit presentation of $M(1,\mathbf{x},N,z)$ in $\mathbb{Q}[x,y][[z]]$ for generic N, and then, invoking the fundamental theorem of algebra, formally replacing N by y.

Note also that the form of $M(1,\mathbf{x},N,z)$ is suggestive of it having come from a matrix integral because of the presence of $V^2(\lambda)$. In fact it does also arise in this way, and there is a way of explaining the matrix integral combinatorially. This will be done later.

4.1. Generalization.

A *hypermap* is a face 2-colourable map (edges separate faces of different colours). Let $H_O(\mathbf{x},\mathbf{y},\mathbf{z},t)$, with $t=1$, denote the genus series for such maps in orientable surfaces. There is a encoding of hypermaps, due to Walsh, that is closely related to the Embedding Theorem, that leads to the following result.

Theorem 4.3 (G, J).

$$H_O(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t) = t\frac{\partial}{\partial t}\log\left(\sum_{\theta\in\mathcal{P}} H_\theta s_\theta(\mathbf{x})\, s_\theta(\mathbf{y})\, s_\theta(\mathbf{z})\, t^{|\theta|}\right).$$

This is a symmetric series in $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$, that avoids the asymmetry of the appearance of $\chi^\theta_{[2^n]}$ in the genus series in Theorem 3.3, and which therefore justifies the introduction of hypermaps. Moreover, the genus series for maps can be recovered from this series by first requiring that the hyperedges have degree 2, so that they are digons (a face bounded by two edges), and then identifying the two edges (digon conflation). This construction can be implemented algebraically by suitable (and simple) conditions on the indeterminates. Note also how closely it is related to the generating series given in Lemma 3.2for the connexion coefficients of the class algebra of the symmetric group. This is the right way to think about it both algebraically and combinatorially.

5. Maps in locally orientable surfaces

We now consider maps in non-orientable surfaces (projective plane, Klein Bottle,...). An indirect way for counting such maps is first to count maps in locally orientable surfaces and then to subtract the maps that are in orientable surfaces. The latter are counted by the methods that have been given already. The reason for this indirectness will become apparent both in the case of symmetric functions and in the case of matrix integrals.. In the case of symmetric functions, the reason is quite stark: there are two algebras that look after the orientable and the locally orientable case. In the case of the matrix integrals, the reason is more subtle, and is deferred to Section 6.2.

The idea now is to obtain an encoding scheme, to see whether there is a convenient algebra that contains the information that we want, and then to see if there is a corresponding matrix integral. In short, the general strategy adopted for orientable surfaces will serve as a guide, although the details are expected to be significantly different.

5.1. **Encoding a map as a corner system.** The following is a straightforward adaptation of Tutte's encoding for maps in locally orientable surfaces. Each edge of a map has two ends, and each end and two sides. Thus, in a map with n edges there are $4n$ *side-end positions*. A map in a locally orientable surface is rooted by selecting a side-end position. To encode a map, assign 1 to the root side-end position. Assign the remaining $4n-1$ labels to the other side end positions. Let $\mathcal{F}_{2n}$ denote the set of all perfect matchings on $\{1,\ldots,4n\}$. Let $F_e \in \mathcal{F}_{2n}$ consist of pairs of labels at the same end of edges. Let $F_s \in \mathcal{F}_{2n}$ consist of pairs of labels at the same side of edges. Let $F_c \in \mathcal{F}_{2n}$ consist of pairs of labels at the same corner made by a pair of incident edges. For $f, g \in \mathcal{F}_n$, let $f \boxplus g$ denote the graph on $\{1,\ldots,4n\}$ whose edge set is the union of the edge sets of f and g. This is called the superposition f and g, obtained by. The triple (F_e, F_s, F_c) is said to be connected if $F_e \boxplus F_s \boxplus F_c$ is connected. We shall call (F_e, F_s, F_c) a *corner system* for a map. Clearly, we can choose F_e and F_s canonically, so then F_c is the only matching that is allowed to vary. The power of the mapping between connected triples (F_e, F_s, F_c) on $4n$ symbols and maps with n edges in a locally orientable surfaces is

$$[(4n-1)!\colon 1].$$

If $f, g \in \mathcal{F}_{2n}$, then $f \boxplus g$ is a graph on $\{1,\ldots,4n\}$ as vertex set, with components that are cycles since each vertex has degree two. The cycle type of $f \boxplus g$ is necessarily of the form 2θ, where $\theta \vdash 2n$, since every vertex is incident with an edge of f and an edge of g. Let $\Lambda(f,g) = \theta$. If follows that if (F_e, F_s, F_c) is a corner system for a map $\mathfrak{m}$, then $\Lambda(F_e, F_c)$ is the vertex-distribution of $\mathfrak{m}$, $\Lambda(F_s, F_c)$ is the face-distribution of $\mathfrak{m}$, and $\Lambda(F_e, F_s) = [4^n]$ since each edge of $\mathfrak{m}$ corresponds to a 4-cycle (each edge has four side-end positions).

5.2. **The construction of all maps.** We may now carry out this construction in reverse, by generating each matching F_c on $\{1,\ldots,4n\}$ and then constructing a structure through superposition with the aid of the canonical matchings F_e and F_s. The resulting structure is not necessarily connected since the group generated by F_e, F_s and F_c, regarded as fixed point free involutions in $\mathfrak{S}_{4n}$ for this purpose, does not necessarily act transitively on $\{1,\ldots,4n\}$. The structure is therefore a decorated premap.

The restriction to rooted maps from decorated premaps may be accomplished algebraically, as it was in the case of maps in orientable surfaces, by factoring out the decorated maps (connected) by means of the logarithm, and then applying the undecorating operator and converting to ordinary generating series in the edge indeterminate z to factor out the power of the encoding mapping. This will not be elaborated upon further.

5.3. **The double coset algebra.** The hyperoctahedral group $\mathfrak{B}_n$ is a subgroup of $\mathfrak{S}_{2n}$ that fixes a specified matching. It is therefore abstractly isomorphic to $\mathfrak{S}_2 \wr \mathfrak{S}_n$ (wreath product). Then $|\mathfrak{B}_n| = 2^n n!$. The double cosets of $\mathfrak{S}_{2n}$ by $\mathfrak{B}_n$ (that is, the members of $\mathfrak{B}_n \diagdown \mathfrak{S}_{2n} \diagup \mathfrak{B}_n$) have partitions of n as a natural index, and can be generated by the chain construction. Let $\mathcal{D}_\theta$ be a double coset. Each element of $\mathcal{D}_\theta$ is a permutation with an even number of cycles. The cycles are paired since each vertex contributes two cycles to the permutation. The two cycles correspond to the two senses in which the alternate labels on the leading side-end positions at a vertex may be read off in a circular tour of the vertex. These cycles are, of course, vertex disjoint since the two sets of labels on the leading side-end positions are mutually disjoint.

Intuitively, this is reasonable since circulation around two different vertices cannot be compared in non-orientable surfaces, so both must senses must be conveyed by each vertex.

Let $\mathsf{D}_\theta = \sum_{g \in \mathcal{D}_\theta} g$. Then $\{\mathsf{D}_\theta : \theta \vdash n\}$ generate a commutative subalgebra called the *double coset algebra*. Let

$$b_{\mu,\nu}^{\lambda} = [\mathsf{D}_\lambda]\, \mathsf{D}_\mu \mathsf{D}_\nu,$$

the connexion coefficients of this algebra. An account of this has been given by Hanlon, Stanley and Stembridge.

Lemma 5.1. *Let $f_1, f_2 \in \mathcal{F}_n$ be fixed matchings such that $\Lambda(f_1, f_2) = \lambda \vdash n$. Then, for $\mu, \nu \vdash n$,*

$$b_{\mu,\nu}^{\lambda} = 2^n n!\, |\{f_3 \in \mathcal{F}_n : \Lambda(f_1, f_3) = \mu, \Lambda(f_2, f_3) = \nu\}|.$$

The point to observe is that there is a clear connexion with counting maps in locally orientable surfaces. To see this, regard f_1 and f_2 as F_e and F_s, chosen canonically through a labelling convention. Then regard f_3 as F_c, a corner system for a map $\mathfrak{m}$. From the above construction, the vertex distribution of the map corresponding to F_c is therefore μ and the face distribution is ν.

5.4. **Connexion coefficients.**

5.4.1. *Orthogonal idempotents.* Let

$$\phi^\lambda(\mu) = \sum_{\sigma \in \mathcal{D}_\mu} \chi^{2\lambda}(\sigma).$$

Then

$$\mathsf{E}_\mu = \frac{1}{H_{2\mu}} \sum_{\nu \vdash n} \frac{1}{|\mathcal{D}_\nu|} \phi^\mu(\nu)\, \mathsf{D}_\nu,$$

for $\mu \vdash n$ have the property that $\mathsf{E}_\mu \mathsf{E}_\nu = \mathsf{E}_\mu \delta_{\mu,\nu}$. They are therefore orthogonal idempotents. Moreover,

$$\mathsf{D}_\mu = \sum_{\nu \vdash n} \phi^\nu(\mu)\, \mathsf{E}_\nu.$$

It follows from these that

$$b_{\mu,\nu}^{\lambda}=\frac{1}{|\mathcal{D}_\lambda|}\sum_{\theta\vdash n}\frac{1}{H_{2\theta}}\phi^{\theta}(\mu)\,\phi^{\theta}(\nu)\,\phi^{\theta}(\lambda)\,.$$

5.4.2. *Zonal polynomials.* There are several definitions of the zonal polynomials. The one that is convenient here is the following. Let $\langle\cdot,\cdot\rangle_2$ be the inner product defined on the ring of symmetric functions over $\mathbb{C}$ by

$$\langle p_\lambda, p_\mu\rangle_\alpha=\frac{|\mathcal{C}_\lambda|}{|\lambda|!}\alpha^{l(\lambda)}\delta_{\lambda,\mu},$$

where α is an indeterminate. The *zonal polynomials* Z_λ, where $\lambda\vdash n$, are characterized by the orthogonality condition $\langle Z_\lambda, Z_\mu\rangle_2=\delta_{\lambda,\mu}$ for $\lambda\neq\mu$, the triangularity condition $[m_\mu]\,Z_\lambda=0$ if $\lambda\prec\nu$ ($\prec$: reverse lexicographic ordering), and normalization $\left[m_{[1^n]}\right]Z_\lambda=n!$,

It can be shown that

$$Z_\lambda=\frac{1}{|\mathcal{D}_\lambda|}\sum_{\mu\vdash n}\phi^{\lambda}(\mu)\,p_\mu$$

is the generating series for ϕ^λ with respect to the power sum basis,and that the inverse relation is

$$p_\lambda=\frac{1}{|\mathcal{D}_\lambda|}\sum_{\theta\vdash n}\frac{|\mathcal{B}_n|}{H_{2\theta}}\phi^{\theta}(\mu)\,Z_\theta.$$

5.4.3. *Generating series.* The generating series for the connexion coefficients of the double coset algebra is given in the next result.

Lemma 5.2.

$$\sum_{\lambda,\mu,\nu\vdash n}\frac{|\mathcal{C}_\lambda|}{n!^2 2^{n+l(\lambda)}}b_{\mu,\nu}^{\lambda}p_\lambda(\mathbf{x})\,p_\mu(\mathbf{y})\,p_\nu(\mathbf{z})=\sum_{\theta\vdash n}\frac{1}{H_{2\theta}}Z_\theta(\mathbf{x})\,Z_\theta(\mathbf{y})\,Z_\theta(\mathbf{z})\,.$$

5.5. **The genus series for hypermaps in locally orientable surfaces.** Let $H(\mathbf{x},\mathbf{y},\mathbf{z},1)$ denote the genus series for hypermaps. The following lemma is a consequence of the encoding of maps in locally orientable surfaces by corner systems.

Lemma 5.3 (GJ)**.**

$$H(\mathbf{x},\mathbf{y},\mathbf{z},t)=2t\frac{\partial}{\partial t}\log\left(1+\sum_{n\geq1}\frac{t^n}{n!}\sum_{\lambda,\mu,\nu\vdash n}\frac{|\mathcal{C}_\lambda|}{2^{l(\lambda)}}\frac{b_{\mu,\nu}^{\lambda}}{2^n n!}\mathbf{x}_\lambda\mathbf{y}_\mu\mathbf{z}_\nu\right).$$

The following theorem gives a presentation of this series in terms of zonal polynomials.

Theorem 5.4 (GJ)**.**

$$H(p(\mathbf{x}),p(\mathbf{y}),p(\mathbf{z}),t)=2t\frac{\partial}{\partial t}\log\left(\sum_{\theta\in\mathcal{P}}\frac{1}{H_{2\theta}}Z_\theta(\mathbf{x})\,Z_\theta(\mathbf{y})\,Z_\theta(\mathbf{z})\,t^{|\theta|}\right).$$

5.6. **Jack symmetric functions.** The genus series for maps in orientable and locally orientable surfaces can be put into a common form with the aid of the Jack symmetric functions $J_\theta(x:\alpha)$. These are characterized in the same way that the zonal polynomials are, but for generic α, rather than $\alpha = 2$. (Schur functions are characterized in the same way, but with $\alpha = 1$). The coefficients of the Jack symmetric functions are known to be polynomial in α. Accounts of their properties have been given by Macdonald and by Stanley.

The common form is

$$H(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t:b) = \alpha t \frac{\partial}{\partial t} \log \left(\sum_{\theta \in \mathcal{P}} \frac{t^{|\theta|}}{\langle J_\theta, J_\theta \rangle_\alpha} J_\theta(\mathbf{x}) J_\theta(\mathbf{y}) J_\theta(\mathbf{z}) \right),$$

where $b = 1+\alpha$,and $J_\theta(\mathbf{x}:\alpha)$ has been denoted temporarily by $J_\theta(\mathbf{x})$ for brevity. Then

$$\begin{aligned} H_O(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t) &= H(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t:0), \\ H(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t) &= H(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t:1) \end{aligned}$$

are the genus series for hypermaps in orientable and locally orientable surfaces, respectively.

5.7. **The b-conjecture.** From computations with Maple it appears, to the limits of the machine, that $H(\mathbf{x}, \mathbf{y}, \mathbf{z}, t:b)$ is formal power series in t with coefficients that are polynomials in x_i, y_i, z_i for $i = 1, 2, \ldots$, with coefficients that are polynomials in b with *non-negative integer coefficients*. It has been conjectured [GJ] that there is an invariant of rooted maps that is marked algebraically by the indeterminate b. In fact, an algorithm has been found that accounts for two infinite classes of maps, in the sense that the power of b is correctly assigned. However, as yet, no combinatorial algorithm is known that acts on rooted maps and computes the appropriate power of b. The conjecture is called the *b-Conjecture*.

It has not yet been proved that $H(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t:b)$ has coefficients that are polynomial in b. Of course, a proof of the b-Conjecture would establish this combinatorially.

5.8. **Comments on the moduli space of curves.** There is also an important connexion with the moduli spaces of real and complex algebraic curves. The contours of the Strebel derivative induce a cell decomposition on the moduli space of complex algebraic curves, and this can be represented combinatorially by a polygon with identification of edges. This is then a monopole (a map with only one vertex), and the virtual Euler characteristic of the moduli space can be expressed as an alternating weighted sum of the numbers of monopoles. This converts a geometrical question into a question in algebraic combinatorics, in which the theory of maps can be applied. It turns out that the algebraic theory that has been described above leads to character sums that are difficult to deal with. However, the integral presentation of the genus series is more effective in this context, since the Sel'berg integration theorems can be applied.

The b-conjecture can be interpreted in terms of the moduli space of curves. In this context it is therefore conjectured [G,Harer,J] that there is a line bundle associated with the moduli space of complex curves that provides a natural interpolation between the moduli space of complex curves and the moduli space of real curves.

I shall not discuss this topic further, since it is going too far afield. However, more can be said about the integral presentation of the genus series.

6. Integral presentation of the genus series

There is an entirely different decomposition for maps in orientable surfaces.

6.1. **The ribbon and socket decomposition.** Let $\mathfrak{m}$ be a map in an orientable surface. Now cut the surface along lines in the surface drawn parallel to each edge and distance ϵ from it. The connected object that remains when the discs are discarded is called a *ribbon graph* (the term *fat graph* is also used in physics; no one seems to have suggested the term "inspissated graph" as an alternative to these!). The surface can be recovered by sewing discs to the circuits formed by the boundaries of ribbons. The original map $\mathfrak{m}$ is recovered by drawing a line along the midline of each of the ribbons, to meet at vertices that are represented by the intersection of ribbons.

The neighbourhood of each vertex is a disc with ends of ribbons marked upon it. The degree of a vertex is the number of ribbon ends (we shall call them *sockets*) that the disc has marked upon it. When these discs are deleted, the ribbon graph decomposes into a collection of indistinguishable ribbons. These can be though of as corresponding to the edges of the original maps.

Now consider the construction of a map with given vertex distribution. That is to say, discs are given that have this distribution of sockets among discs, and enough ribbons are given to connect the sockets. A premap can be formed by connecting pairs of sockets by ribbons (the structure is not necessarily connected). This is a simple combinatorial construction. The problem is to encode it algebraically, or analytically, in such a way that the number of faces and the vertex distribution can be recovered (it is difficult to preserve both vertex and face distributions).

6.2. **Connector functions.** Suppose that the surface is orientable. Now consider two sockets. Label the two sides of a socket by integers p and q so that the direction $\overrightarrow{pq}$ across the mouth of the socket is anticlockwise. Label the other socket with r and s in a similar way. Then a ribbon joining these to sockets corresponds to the function $\psi_O\left((p,q),(r,s)\right)$ taking the value one. Clearly, the boundary of the ribbon must join p to s and q to r, for otherwise the ribbon would have a twist, so the corresponding edge would pass through a cross cap, which is not possible in an orientable surface. Thus the function

$$\psi_O\left((p,q),(r,s)\right)=\delta_{p,s}\delta_{q,r}$$

specifies the correct joining of sockets. In the locally orientable case, the connector function is

$$\psi\left((p,q),(r,s)\right)=\delta_{p,s}\delta_{q,r}+\delta_{p,r}\delta_{q,s},$$

in which the additional term $\delta_{p,r}\delta_{q,s}$ permits the ribbon to be attached with a twist.

The intuition behind this is the following. For ψ_O, only one mode of connecting sockets is allowed, so this gives the orientable case. For ψ, two modes are allowed, so one represents "no twist" and the other a "twist." In fact, it is not possible to identify which of the two terms in ψ represents a twist. The immediate question this raises is how can the maps in non-orientable surfaces be recognized.. However, this question has already be resolved at the beginning of Section 5

6.3. **A combinatorial sum.** We may now express M_O as a combinatorial sum involving delta functions. Let $\kappa(\pi)$ denote the number of cycles in a permutation π.

Lemma 6.1. *Let $\nu \in \mathcal{C}_\alpha$. Let*

$$A_\alpha(y) = \begin{cases} \sum_{\pi \in \nu \mathcal{C}_{[2^n]}} y^{\kappa(\pi)} & \text{if } |\alpha| \text{ is even,} \\ 0 & \text{if } |\alpha| \text{ is odd.} \end{cases}$$

Then

$$M_O(u, \mathbf{x}, \mathbf{y}, \mathbf{z}) = \Omega_u \log \left(\sum_{k \geq 0} \frac{2^k z^k}{(2k)!} \sum_{\nu \vdash 2k} h^\nu A_\nu(y) \mathbf{x}_\nu \right).$$

The proof is by the Embedding Theorem for maps in orientable surfaces. It is reasonably straightforward to see that the following result holds for $A_\alpha(y)$.

Lemma 6.2. *Let N be a positive integer. Let $\nu \in \mathcal{C}_\alpha$, $\alpha \vdash 2n$, ν fixed. Then*

$$A_\alpha(N) = \sum_{1 \leq i_1, \ldots, i_{2n} \leq N} \sum_{\rho \in C_{[2^n]}} \prod_{j=1}^{2n} \psi_O\left(\left(i_j, i_{\nu(j)}\right), \left(i_{\rho(j)}, i_{\nu\rho(j)}\right)\right).$$

The proof uses the simple, yet ingenious, observation that, for $\pi \in \mathfrak{S}_n$, $\kappa(\pi)$ may be computed by means of

$$N^{\kappa(\pi)} = \sum_{1 \leq i_1, \ldots, i_n \leq N} \delta_{i_1, i_{\pi(1)}} \cdots \delta_{i_n, i_{\pi(n)}}.$$

It is to be noted that the "marker" for $\kappa(\pi)$ is N, a positive integer, so it appears that $\kappa(\pi)$ is not recoverable. However, it is recoverable, by means of the fundamental theorem of algebra, and this will be mentioned later.

6.4. **Construction of connector functions.** To simplify the sum of finite products that appears in $A_\alpha(N)$, we attempt to transform the sum into an integral. We will then address the evaluation of such integrals. This transformation involves finding an integral presentation of a discontinuous function, namely the delta function. We start with $\delta_{p,q}$, where p and q are integers and $1 \leq p, q \leq N$. It is easily seen that

$$\delta_{p,q} = \frac{\int_{\mathbb{R}^N} x_p x_q e^{-p_2(\mathbf{x})/2} d\mathbf{x}}{\int_{\mathbb{R}^N} e^{-p_2(\mathbf{x})/2} d\mathbf{x}},$$

by straightforward evaluation, where $\mathbf{x} = (x_1, \ldots, x_N)$ here. If $p \neq q$, then the integrand of the numerator is an odd function of x_p (and x_q) so the numerator is zero, while the denominator is finite. On the other hand, if $p = q$, the integral evaluates to

$$\frac{\int_{\mathbb{R}} x_p e^{-x_p^2/2} dx_p}{\int_{\mathbb{R}} e^{-x_p^2/2} dx_p} = 1.$$

We now generalize this device, by double indexing the indeterminates, to obtain an integral presentation of the connector function ψ_O. The expression is

$$\psi_O((p,q), (r,s)) = \frac{\int_{\mathcal{V}_N} m_{p,q} m_{r,s} e^{-\frac{1}{2}\mathrm{tr}\left(\mathbf{M}^2\right)} d\mathbf{M}}{\int_{\mathcal{V}_N} e^{-\frac{1}{2}\mathrm{tr}(\mathbf{M}^2)} d\mathbf{M}},$$

where it is to be recalled that $\mathcal{V}_N$ is the set of $N \times N$ Hermitian complex matrices. Similarly, for locally orientable surfaces,

$$\psi\left((p,q),(r,s)\right) = \frac{\int_{\mathcal{W}_N} m_{p,q} m_{r,s} e^{-\frac{1}{2}\mathrm{tr}\left(\mathbf{M}^2\right)} d\mathbf{M}}{\int_{\mathcal{W}_N} e^{-\frac{1}{2}\mathrm{tr}(\mathbf{M}^2)} d\mathbf{M}},$$

where $\mathcal{W}_N$ is the set of $N \times N$ real symmetric matrices.

There are in fact degrees of freedom in the choice of classes of $N \times N$ complex matrices $\mathbf{M}$ that may be used in the construction of the connector functions. The general class of such matrices is found by considering the maximal coordinatizing set of the matrices, and then writing down the algebraic conditions necessary for obtaining a representation. In this under determined system, that includes Hermitian complex matrices and real symmetric matrices, the natural choices are $\mathcal{V}_N$ and $\mathcal{W}_N$ since it is to these that the Weyl integration theorems can be applied, subject to the condition of invariance of the integrand under the adjoint action of the associated groups (unitary and orthogonal). These integration theorems transform the matrix integrals into $\mathbb{R}^N$-integrals by diagonalization.

6.5. **Integral presentation of the genus series.** Combining these results we have

$$A_\alpha(N) = \left\langle \prod_{j\geq 1} \left(\mathrm{tr}\left(\mathbf{M}^j\right)^{a_j}\right) \right\rangle_{\mathcal{V}_N},$$

where $\alpha = [1^{a_1}, 2^{a_2}, \ldots]$ and

$$\langle f(\mathbf{M})\rangle_{\mathcal{V}_N} = \frac{\int_{\mathcal{V}_N} f(\mathbf{M})\, e^{-\frac{1}{2}\mathrm{tr}\left(\mathbf{M}^2\right)} d\mathbf{M}}{\int_{\mathcal{V}_N} e^{-\frac{1}{2}\mathrm{tr}(\mathbf{M}^2)} d\mathbf{M}}.$$

Theorem 6.3.

$$M_O(1,\mathbf{x},N,z) = 2z\frac{\partial}{\partial z}\log\left\langle e^{\sum_{k\geq 1}\frac{1}{k}x_k\sqrt{2z}^k\mathrm{tr}\left(\mathbf{M}^k\right)}\right\rangle_{\mathcal{V}_N}.$$

There is a comparable theorem for locally orientable surfaces.

Theorem 6.4 (GJ). *Let*

$$\langle f(\mathbf{M})\rangle_{\mathcal{W}_N} = \frac{\int_{\mathcal{W}_N} f(\mathbf{M})\, e^{-\frac{1}{4}\mathrm{tr}\left(\mathbf{M}^2\right)} d\mathbf{M}}{\int_{\mathcal{W}_N} e^{-\frac{1}{4}\mathrm{tr}(\mathbf{M}^2)} d\mathbf{M}}.$$

Then

$$M(1,\mathbf{x},N,z) = 4z\frac{\partial}{\partial z}\log\left\langle e^{\frac{1}{2}\sum_{k\geq 1}\frac{1}{k}x_k\sqrt{2z}^k\mathrm{tr}\left(\mathbf{M}^k\right)}\right\rangle_{\mathcal{W}_N}.$$

In both of these theorems, the face marker is N, which is necessarily a positive integer, and this seems to destroy the value of these results. However, we know combinatorially that both series are in $\mathbb{Q}[\mathbf{x}, N][[z]]$ so, provided the integration is carried out for general N and the series are given an explicit presentation in this ring, we can formally replace N by the indeterminate y, again by the fundamental theorem of algebra, thereby constructing the genus series.

6.6. **Diagonalization.** An integral presentation of the genus series for maps in orientable surfaces has already been given. It can be rederived from the above theorem by means of the Weyl integration theorem, or its first cousin. The result, together with one for locally orientable surfaces, is given in the next corollary.

Corollary 6.5.

$$\begin{aligned} M_O(1,\mathbf{x},N,z) &= 2z\frac{\partial}{\partial z}\log\left(\frac{\int_{\mathbb{R}^N}|V(\lambda)|^2 e^{\sum_{k\geq 1}\frac{1}{k}\sqrt{z}^k x_k p_k(\lambda)}e^{-\frac{1}{2}p_2(\lambda)}d\lambda}{\int_{\mathbb{R}^N}|V(\lambda)|^2 e^{-\frac{1}{2}p_2(\lambda)}d\lambda}\right), \\ M(1,\mathbf{x},N,z) &= 4z\frac{\partial}{\partial z}\log\left(\frac{\int_{\mathbb{R}^N}|V(\lambda)|\, e^{\frac{1}{2}\sum_{k\geq 1}\frac{1}{k}\sqrt{z}^k x_k p_k(\lambda)}e^{-\frac{1}{4}p_2(\lambda)}d\lambda}{\int_{\mathbb{R}^N}|V(\lambda)|\, e^{-\frac{1}{4}p_2(\lambda)}d\lambda}\right). \end{aligned}$$

Proof. The diagonalizing groups are the unitary group and the orthogonal group. The Jacobians of the diagonalizing transformations are $|V(\lambda)|^2$ and $|V(\lambda)|$, respectively. The integrands involve only $\operatorname{tr}(\mathbf{M}^k)$, which is invariant under the adjoint action of the diagonalizing groups, so the Weyl theorems may be applied. The contribution from the volumes of the diagonalizing groups cancels through the normalization. □

6.7. **A generalization.** A joint generalization for $M_O(\mathbf{x},N,z)$ and $M(\mathbf{x},N,z)$ is

$$M(1,\mathbf{x},N,z:\gamma) = \frac{2}{\gamma}z\frac{\partial}{\partial z}\log\left(\frac{\int_{\mathbb{R}^N}|V(\lambda)|^{2\gamma} e^{\gamma\sum_{k\geq 1}\frac{1}{k}\sqrt{z}^k x_k p_k(\lambda)}e^{-\frac{\gamma}{2}p_2(\lambda)}d\lambda}{\int_{\mathbb{R}^N}|V(\lambda)|^{2\gamma} e^{-\frac{\gamma}{2}p_2(\lambda)}d\lambda}\right).$$

This is a series for which we already have a symmetric function presentation in terms of Jack symmetric functions which arose in connexion with the b-conjecture.

No matrix model for this expression is known. If there were one, we would be able to construct a connector function, and then use this to deduce the role played by b from its appearance in the connector function. This is because we know the combinatorics of inserting ribbons into sockets in discs.

By comparing the two expressions for $M(\mathbf{x},N,z:\gamma)$, one given in Section 5.6 in terms of Jack symmetric functions and the other given above, we have the following conjecture [GJ]. Let

$$\langle f(\lambda)\rangle_\gamma = \frac{\int_{\mathbb{R}^N} f(\lambda)\,|V(\lambda)|^{2\gamma} e^{-p_2(\lambda)/2}d\lambda}{\int_{\mathbb{R}^N}|V(\lambda)|^{2\gamma} e^{-p_2(\lambda)/2}d\lambda}.$$

Then

$$\left\langle J_\theta\left(\lambda:\gamma^{-1}\right)\right\rangle_\gamma = J_\theta\left(1_N:\gamma^{-1}\right)\left([p_2^m]\,J_\theta\right),$$

where 1_N is the vector of 1's with N components. This was proved by Okounkov.

7. An explicit calculation for the genus series for monopoles and dipoles

This section is concerned almost exclusively with the use of the above theory to carry out "algebraic computations" to obtain explicit forms from the matrix integrals. A good deal of additional material, from the theory of hypergeometric functions, for example, has to be imported for this purpose. Clearly, some computations may be easier with the characters, but this will not be examined here. The evaluation of matrix integrals is the subject of extensive study, and there are results and techniques for calculating some of them. These include the use of orthogonal polynomials and, in particular, the Hermite polynomials. There is a useful technique of integrating over variables of odd index first that is due to Mehta. The computations that are given are for monopoles. It has already been observed that these are of relevance to geometry, but this case also serves as a manageable example.

The computations lead to questions about deriving a natural basis of the genus series (for monopoles, in this case). Some fairly arcane hypergeometric transformations will be required even for the case of monopoles. The nature of this natural basis, and the reasons for seeking it, are developed in Section 7.3.

7.1. **Hermite polynomials.** Only the material used in these Notes is given here. The Hermite polynomials $H_n(x)$ are orthogonal with respect to the measure $e^{-x^2}dx$ on $\mathbb{R}$ and their standardization is $[x^n]\,H_n(x) = 2^n$. The orthogonality relation is

$$\int_R H_m(x)\,H_n(x)\,e^{-x^2}dx = 2^n n!\sqrt{\pi}\delta_{m,n}.$$

Two standard properties of the Hermite polynomials that will be needed are

$$\frac{1}{2^r r!}H_r^2(x) = \sum_{k=0}^{r} \frac{H_{2k}(x)}{2^k k!^2 (r-k)!} \tag{1}$$

and

$$\int_{\mathbb{R}} x^{2n} H_{2k}(x)\,e^{-x^2}dx = \begin{cases} 2^{-2(n-k)}\frac{(2n)!}{(n-k)!}\sqrt{\pi}, & \text{if } n \geq k, \\ 0, & \text{otherwise.}\end{cases} \tag{2}$$

7.2. **The genus series for monopoles.**

Corollary 7.1. *The number $m_g(n)$ of monopoles in an orientable surface of genus g on n edges is given by*

$$m_g(n) = \left[u^{n+1-2g}\right] h^{[2^n]} \sum_{k=1}^{n+1} \binom{u}{k}\binom{n}{k-1} 2^{k-1},$$

Proof. Now $m_g(n) = \left[u^{2g} x_{2n} z^n\right] M\left(u^2, \mathbf{x}, 1, z\right)$since the number of faces is forced in a monopole by specifying the genus and the number of edges. Then, from Corollary 6.5, $m_g(n) = \left[u^{n+1-2g}\right] A_{[2n]}(u)$ where

$$\begin{aligned} A_{[2n]}(N) &= \frac{1}{2n}\frac{\int_{\mathbb{R}^N} |V(\lambda)|^2 p_{2n}(\lambda)\,e^{-\frac{1}{2}p_2(\lambda)}d\lambda}{\int_{\mathbb{R}^N} |V(\lambda)|^2 e^{-\frac{1}{2}p_2(\lambda)}d\lambda} \\ &= \frac{2^n}{2n}\frac{\int_{\mathbb{R}^N} |V(\lambda)|^2 p_{2n}(\lambda)\,e^{-p_2(\lambda)}d\lambda}{\int_{\mathbb{R}^N} |V(\lambda)|^2 e^{-p_2(\lambda)}d\lambda}. \end{aligned}$$

But $\sqrt{2}^{N(N-1)}V(\lambda) = \det\left[H_{k-1}(\lambda_j)\right]_{N\times N}$, by elementary row operations on the Vandermonde matrix. Then, for the denominator, using orthogonality of the Hermite polynomials and the expansion of the determinants, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |V(\lambda)|^2 e^{-p_2(\lambda)} d\lambda &= \frac{1}{2^{N(N-1)}}\int_{\mathbb{R}^N} \det\left[H_{k-1}(\lambda_j)\right]^2_{N\times N} e^{-p_2(\lambda)} d\lambda \\ &= \frac{1}{2^{N(N-1)}}\sum_{\pi\in\mathfrak{S}_N}\int_{\mathbb{R}^N}\left(\prod_{k=1}^N H_{\pi(k)-1}(\lambda_k)\, e^{-\lambda_k^2}\right) d\lambda \\ &= \frac{N!}{2^{N(N-1)}}\left(\prod_{k=1}^N \int_{\mathbb{R}} H_{\pi(k)-1}(x)\, dx\right) \\ &= \frac{N!}{2^{N(N-1)}}\prod_{k=0}^{N-1} 2^k k!.\end{aligned}$$

Let

$$c_{N,n} = \frac{\sqrt{2}^{N(N-1)} 2^n}{\sqrt{\pi}^N \prod_{j=1}^N j!}.$$

Then, expanding the numerator in a similar way, we have

$$\begin{aligned}A_{[2n]}(N) &= \frac{c_{N,n}}{2^{N^2-N}}\int_{\mathbb{R}^N} \det\left[H_{k-1}(\lambda_j)\right]^2_{N\times N} e^{-p_2}\sum_{q=1}^N \lambda_q^{2n} d\lambda \\ &= \frac{c_{N,n}}{2^{N^2-N}}\sum_{q=1}^N \sum_{\sigma,\rho\in\mathfrak{S}_N} \operatorname{sgn}(\sigma\rho) \\ &\quad \cdot\left(\prod_{1\le j\le N, j\ne q} 2^{\sigma(j)-1}(\sigma(j)-1)!\sqrt{\pi}\delta_{\sigma(j)-1,\rho(j)-1}\right) \\ &\quad \cdot\int_{\mathbb{R}} \lambda_q^{2n} H_{\sigma(q)-1}(\lambda_q) H_{\rho(q)-1}(\lambda_q)\, e^{-\lambda_q^2} d\lambda_q,\end{aligned}$$

by orthogonality. For non-zero contributions, $\sigma(j) = \rho(j)$ for all $j \in \{1,\ldots,N\} - \{N\}$. Thus $\sigma(q) = \rho(q)$ since σ and ρ are bijective, whence $\sigma = \rho$. Then

$$\begin{aligned}A_{[2n]}(N) &= \frac{2^n}{\sqrt{\pi}N!}\sum_{q=1}^N\sum_{r=0}^{N-1}\sum_{\sigma\in\mathfrak{S}_N,\sigma(q)=r+1}\int_{\mathbb{R}} x^{2n}\frac{H_r^2(x)}{2^r r!}e^{-x^2}dx \\ &= \frac{2^n}{\sqrt{\pi}}\int_{\mathbb{R}} x^{2n}\sum_{r=0}^{N-1}\frac{1}{2^r r!}H_r^2(x)\, e^{-x^2}dx.\end{aligned}$$

Then, from (1) and (2),

$$A_{[2n]}(N) = \frac{(2n)!}{2^n n!}\sum_{k=1}^{n+1}\binom{N}{k}\binom{n}{k-1}2^{k-1},$$

assuming that $N > n$. But this is a presentation of $A_{[2n]}(N)$ in $\mathbb{Q}[N]$ so, by the fundamental theorem of algebra,

$$A_{[2n]}(u) = \frac{(2n)!}{2^n n!}\sum_{k=1}^{n+1}\binom{u}{k}\binom{n}{k-1}2^{k-1},$$

where u is an indeterminate, and the result follows. $\square$

We note in passing that the set of all perfect matchings $\mathcal{F}_n$ on $2n$ symbols has the decomposition

$$\mathcal{F}_n \tilde{\longrightarrow} \mathcal{U} \circledast \{\{1,2\}\},$$

since $\{1,2\}$ represents a canonical vertex labelled edge, and a matching is simply an unordered vertex disjoint collection of these, so the number of perfect matchings is

$$\left[\frac{x^{2n}}{(2n)!}\right] e^{\frac{1}{2}x^2}.$$

This is highly suggestive of a natural association between Hermite polynomials and perfect matchings and, indeed, some of the classical properties of Hermite polynomials, such as the Mehler formula, have been derived by combinatorial constructions. The Mehler formula was derived in this way by Foata.

7.3. **Degree and parity respecting presentations.** From Corollary 7.1, the generating series for monopoles is

$$F = h^{[2^n]} \sum_{k=1}^{n+1} \binom{u}{k} \binom{n}{k-1} 2^{k-1}.$$

It is evident that the degree of F in u is $n+1$, and that the term with highest exponent in F corresponds to the $g=0$ case (the sphere), for which the exponent $n+1-2g$ in the coefficient extraction is greatest. We say that such a presentation of a series is *degree respecting*. That is to say, the degree of F is manifest from its presentation (for example, $(x+1)^2 - (x-1)^2$ is not a degree respecting presentation of $4x$). We also know that if n is odd, then F is even, and if n is even, then F is odd. Thus the presentation of F is not parity respecting. In general, the presentation of a generating series whose parity is manifest from its presentation is called *parity respecting*. Typically, genus series obtained by means of the algebraic approach described in the earlier sections of these Notes are seldom degree respecting because of identities among the characters of $\mathfrak{S}_{2n}$ that are hard to resolve.

Intuitively, the reason for looking for a parity respecting a degree respecting presentation of a genus series is that a generating series has a lot of redundancy removed from it and whose algebraic properties reflect the topological properties of the maps it counts is more likely to reveal further information.

In the case of k-poles (k vertices, each of the same degree), we seek degree and parity respecting presentations of their generating series in the belief that this will provide more information about a natural basis for these generating series as an indirect approach to the ϕ^4-model. The approach is reasonable because the sum of the generating series for k-poles, over all k, contains the genus series for quadrangulations and it may be possible to retrieve this indirectly as a section of the bigger series.

We now briefly consider these phenomena in the case of monopoles.

7.3.1. *Hypergeometric transformations.* The only transformations that are given are the ones that are used in the analysis of monopoles. Let

$$(x)_k = x\,(x+1)\cdots(x+k-1),$$

the Pochhammer symbol. The question requires the following hypergeometric transformations.

(3)

$$ {}_2F_1\left[\begin{matrix} a, & b \\ -, & \kappa - b \end{matrix}\;\middle|\; -1\right] = \frac{\Gamma(\kappa - b)\,\Gamma(\kappa - a)}{\Gamma(\kappa)\,\Gamma(\kappa - a - b)}\, {}_3F_2\left[\begin{matrix} b, & \frac{1}{2}(a+1), & \frac{1}{2}a \\ -, & \frac{1}{2}\kappa, & \frac{1}{2}(\kappa+1) \end{matrix}\;\middle|\; 1\right] $$

given by Whipple, and Sheppard's transformation (a generalization of the Pfaff-Saalschutz formula), namely,

$$ \begin{aligned} {}_3F_2\left[\begin{matrix} -n, & b, & c \\ -, & e, & f \end{matrix}\;\middle|\; 1\right] &= \frac{(e-b)_n\,(f-b)_n}{(e)_n\,(f)_n} \\ &\cdot {}_3F_2\left[\begin{matrix} -n, & b, & -n+b+c+1-e-f \\ -, & -n+b-f+1, & -n+b-e+1 \end{matrix}\;\middle|\; 1\right]. \end{aligned} \tag{4} $$

7.3.2. *A parity and degree respecting form for monopoles.* The following result gives a parity respecting presentation of the generating series for monopoles while still preserving the degree respecting property. The basis that is used, and which emerges naturally from the analysis, is $\left\{(u-j)_{2j+1} : j = 0, 1, \ldots\right\}$. Since

$$ (u-j)_{2j+1} = u \prod_{i=1}^{j} \left(u^2 - i^2\right), $$

it is clear that this is an odd parity basis. The corresponding even parity basis is

$$ \prod_{i=1}^{j} \left(u^2 - i^2\right). $$

The odd parity basis is used in the first statement in the enunciation of the following lemma, while the even parity basis, which is slightly disguised, appears in the second - there has to be a cancellation of u with the coefficient operator.

Lemma 7.2 (Andrews,J). *A parity and degree respecting presentation of the generating series for monopoles is*

$$ h^{[2^n]} \sum_{j=0}^{m} \frac{4^j}{j+1} \binom{m}{j} \frac{(u-j)_{2j+1}}{(2j+1)!}, \quad \text{if } n \text{ is even,} $$

and

$$ h^{[2^n]} \sum_{j=0}^{m} 4^j \binom{m}{j} \frac{(u-j)_{2j+1}}{(2j+1)!}, \quad \text{if } n \text{ is odd.} $$

Proof. Let

$$ G = \frac{1}{h^{[2^n]}} \binom{u}{2m+2}^{-1} F. $$

Then, for $n = 2m+1$,

$$
\begin{aligned}
G &= 2^{2m+1}\,{}_2F_1\left[\begin{matrix} -2m-1, & -2m-2 \\ -, & u-2m-1 \end{matrix}\,\middle|\,\frac{1}{2}\right] \\
&= {}_2F_1\left[\begin{matrix} -2m-1, & u+1 \\ -, & u-2m-1 \end{matrix}\,\middle|\,-1\right] \\
&= \frac{(2u-2m)_{2m+1}}{(u-2m-1)_{2m+1}}\,{}_3F_2\left[\begin{matrix} u+1, & -m, & -m-\frac{1}{2} \\ -, & u-m, & u-m+\frac{1}{2} \end{matrix}\,\middle|\,1\right], \text{ by (3)} \\
&= \frac{(2u)!\,(u-2m-2)!}{(2u-2m-1)!\,(u-1)!}\cdot\frac{(-m-1)_m\left(-m-\frac{1}{2}\right)_m}{(u-m)_m\left(u-m+\frac{1}{2}\right)_m} \\
&\quad \cdot {}_3F_2\left[\begin{matrix} u+1, & -m, & -u-1 \\ -, & 2, & \frac{3}{2} \end{matrix}\,\middle|\,1\right], \text{ by (4)} \\
&= \binom{u}{2m+2}^{-1} u^2\,{}_3F_2\left[\begin{matrix} u+1, & -m, & -u-1 \\ -, & 2, & \frac{3}{2} \end{matrix}\,\middle|\,1\right].
\end{aligned}
$$

This is equivalent to the first result. The proof of the second result is similar. □

7.4. **Dipoles.** There is a similar, but more difficult, analysis for determining a degree and parity respecting presentation for the genus series for dipoles. Only the basic steps and the results will be given, for comparison purposes with the monopole case.

Let $d_g(n)$ be the number of dipoles with n edges in an orientable surface of genus g. Then, from Corollary 6.5,

$$d_g(n) = \frac{1}{n}\left[u^{n-2g}\right]\left(A_{[n^2]}(u) - A^2_{[n]}(u)\right),$$

where

$$A_\nu(N) = \frac{\sqrt{2}^{N^2-N}\,2^n}{\sqrt{\pi}^N\prod_{j=1}^N j!}\int_{\mathbb{R}^N} V^2(\lambda)\,e^{-p_2(\lambda)}p_\nu(\lambda)\,d\lambda.$$

The generating series for $d_g(n)$ is given in the next theorem in a form that comes directly from the matrix integration. Recall that it is necessary to carry out the $\mathbb{R}^N$-integral fully and to give it a presentation as a polynomial in N so that this may be replaced by the face indeterminate y. Clearly, there is little expectation that the result will be in parity respecting form and, indeed, as seen below, it is not. However, it is degree respecting.

Theorem 7.3 (GJ).

$$
\begin{aligned}
d_g(n) &= (n-1)!\left[u^{n-2g}\right]\sum_{j=0}^{\lfloor\frac{1}{2}(n-1)\rfloor}\binom{2j}{j}\sum_{r=0}^{n-2j-1}\binom{u+j+r}{n}\binom{n-2j-1}{r} \\
&\quad\cdot\sum_{k=0}^{\lfloor\frac{1}{2}(n-2j-1)\rfloor}\frac{1}{4^k}\binom{2k}{k}\binom{n}{2k}.
\end{aligned}
$$

The degree and parity respecting presentations of the genus series for dipoles are given in the next two theorems, for the odd and even cases. The same basis arises as in the case of monopoles, suggesting that this is a good basis for k-poles.

Theorem 7.4 (AJ). *Let $n = 2m+1$. Then*

$$\begin{aligned} d_g(n) &= n!\binom{2m}{m}\left[u^{n-2g}\right]\sum_{i=0}^{m}\frac{(u-i)_{2i+1}}{(2i+1)!}4^{i-m}\binom{m}{i} \\ &\quad \cdot\sum_{k=0}^{m-i}(-1)^k\frac{1}{2k+1}\binom{m-i}{m}\binom{2m-k}{m} \end{aligned}$$

Theorem 7.5 (AJ). *Let $n = 2m$. Then*

$$\begin{aligned} d_g(n) &= n!\binom{2m}{m}\left[u^{n-2g-1}\right]\sum_{i=0}^{m}\frac{(u-i)_{2i+1}}{(2i+1)!\,(i+1)}4^{i-m}\binom{m-1}{i} \\ &\quad \cdot\sum_{k=0}^{m-i-1}(-1)^k\frac{1}{2k+1}\binom{m-i-1}{m}\binom{2m-k-1}{m}. \end{aligned}$$

The proofs require the result that

$$\sum_{r=0}^{2m-2j}\binom{2m-2j}{r}\binom{u+j+r}{2m+1}=\sum_{i=0}^{m}\frac{4^{i-j}}{(2i+1)!}\binom{m-j}{i-j}(u-i)_{2i+1}.$$

The proof of this result is to express the left hand side as a limiting case of a ${}_5F_4$, and then to use Dougall's theorem for the ${}_5F_4$. AN alternative proof is to use a relationship (obscure) between Krawtchouk polynomials of even degree and Hahn polynomials. This having been done, $d_g(n)$ can be expressed as a limiting case of a ${}_3F_2$ in the odd case. The even case is much harder.

Clearly, a more general approach is needed for k-poles, but the monopole and dipole cases provide potentially valuable data for testing a more general approach.

7.5. **Closing comments.** Collected below are a few comments, and a few observations about general open problems.

- No one yet has found an algebra associated with the Jack symmetric functions that plays the role that the class algebra of the symmetric group plays for the Schur functions and the double coset algebra plays of the zonal polynomials. No one has found a matrix model that corresponds to this case.
- The quaternionic case seems to be of little interest. The connector function exists, but it does not appear to convey combinatorial information. The diagonalizing group is the symplectic group, and the Jacobian of the diagonalizing transformation is $|V(\lambda)|^4$. This steers us away from the difficulty of the absolute value, but replaces it with a fourth power. The three cases $(\mathbb{C},\mathbb{R},\mathbb{H})$ correspond to the only finite dimensional real division algebras, and, in this sense, the three cases are natural.
- The Quadrangulation Conjecture, the b-Conjecture and its extension to the moduli space of curves remain open.
- Parity and degree respecting bases are know for monopoles and dipoles. Such a basis has not been constructed for k-poles in general.
- There is a natural extension of these ideas to another classical question, the *Hurwitz Enumeration Problem*, and the Hurwitz moduli space. This problem concerns the determination of the number of ramified covers of the sphere by curves of genus g, with elementary branch points, and prescribed

ramification over infinity. The question is to count these up to topological equivalence. There is a combinatorial approach to this through Hurwitz's original combinatorialization of this question as an ordered factorization of a permutation in $\mathfrak{S}_n$ into transpositions such that the transpositions act transitively on $\{1, \ldots, n\}$. The numbers $1, \ldots, n$ label sheets, and the transpositions specify sheet transitions at a branch point.

- The genus of the covering is obtained by the Riemann-Hurwitz formula. It as been shown [G,J,Vakil] that the generating series can be expressed as a rational series in a set of series that satisfy a certain functional equation. The coefficients in the partial fraction decomposition of this rational series are precisely identified as certain intersection numbers (Hodge integrals).
- The Okounkov-Pandharipande encoding encodes a ramified cover as a map with a particular type of edge labelling than encodes the ramification over ∞. A substantial problem in geometry is to find a matrix model (along the lines of the ones described above) for the generating series for "Hurwitz numbers."
- Ramified covers also appear in connexion with the partition function for 2-dimensional quantum chromo-dynamics.

This is probably a good place to stop!