A combinatorial relationship between Eulerian maps and hypermaps in orientable surfaces

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Abstract

There is a remarkable relationship between the genus series for rooted maps and rooted quadrangulations that has been obtained by a character theoretic argument. Hitherto no combinatorial explanation for this has been given. In this paper we generalise this relationship to a larger set of maps for which such a relationship can be found and we ascertain some of the properties that a putative bijection must possess. Several examples of images of sets under this bijection are given. We give them in detail since they may contain useful information about the bijection.

1 Introduction

1.1 Background

A *map* is an embedding of a graph in an orientable surface such that the deletion of the edges decomposes the surface into regions homeomorphic to open discs. The generating series

$$M(u,x,y,z) = \sum_{g,i,j,k\geq 0} m_{g,i,j,k} u^g x^i y^j z^k$$

for the number $m_{g,i,j,k}$ of rooted maps with i vertices, j faces, k edges and genus g satisfies the following [7] remarkable functional relationship.

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Theorem 1.1. *Let M_4 be the genus series for the set of rooted face-4-regular maps. Then*

$$M_4(u^2, x, y, z) = \mathsf{bis}_u\, M(4u^2, x, x+u, z^2 y),$$

where $\mathsf{bis}_u f$ denotes the even bisection $\frac{1}{2}(f(u) + f(-u))$ of the formal power series f.

Maps are classical combinatorial structures that have been studied extensively, particularly for the sphere [11]. Those with even vertex degree are called *Eulerian* since the associated graph is Eulerian. M, later generalised to include vertex-degree and face-degree distributions, is called the *genus series* and we remark that genus, which is determined by numbers of vertices, edges and faces, is included for symmetry.

Theorem 1.1 implies a bijective correspondence between the set $\mathcal{M}_4$ of rooted quadrangulations (face-4-regular maps) and a subset of the set $\mathcal{M}$ of all rooted maps. The purpose of this paper is to generalise this result to a larger set containing $\mathcal{M}_4$, to obtain information about the elementwise action of this bijection and to give a combinatorial characterisation of the image set.

The functional relationship stated in Theorem 1.1 can be used to establish a relationship between two models in two dimensional quantum gravity (the ϕ^4-model and the Penner model), an account of which is given in [6]. The genus series itself also appears in other contexts, including the study of the Euler characteristic of the moduli spaces of algebraic curves [2] and ramified coverings of the sphere [3], where further knowledge of its combinatorial properties may be of value.

1.2 Preliminaries

Let $\xi(\mathfrak{m}) = (g, i, j, k)$, the above map invariants associated with $\mathfrak{m} \in \mathcal{M}$. Let

$$\widetilde{M}(u, x, y, z) = \mathsf{bis}_u\, M(4u^2, x, x+u, z^2 y).$$

We suppose $\widetilde{M}$ to be the generating series for a set $\widetilde{\mathcal{M}}$ of maps derived from $\mathcal{M}$, with respect to $\widetilde{\xi}(\widetilde{\mathfrak{m}}) = (\widetilde{g}, \widetilde{i}, \widetilde{j}, \widetilde{k})$, where the latter are map invariants associated with $\widetilde{\mathfrak{m}} \in \widetilde{\mathcal{M}}$. Theorem 1.1 implies that there exist bijections $f\colon \mathcal{M}_4 \longrightarrow \widetilde{\mathcal{M}}$ that are ξ-preserving (that is, $\xi = \widetilde{\xi}\, f$), and we assume that among these there is a natural one, to be denoted by Ξ. A bijection is *natural* if it admits an elementwise action having a combinatorial description.

The genus series Φ for a set of rooted maps is said to have the *factorisation property* if

1. $\Phi = cu^2 z \dfrac{\partial}{\partial z} \log F$ where $c \in \mathbb{Q}$ and F is a formal power series in z;

2. $F = F_1F_2$ where $c_1u^2z\frac{\partial}{\partial z}\log F_1$ and $c_2u^2z\frac{\partial}{\partial z}\log F_2$ are again genus series for a class of rooted maps, computed at (perhaps) displaced arguments.

Since Theorem 1.1 is a consequence of the fact that M_4 has the factorisation property, the purpose of this paper is, in this terminology, to determine: **I)** a subset $\mathcal{A}$ of $\mathcal{M}$ that is maximal with respect to the factorisation property (*subject to the techniques employed herein we find that* $\mathcal{A} = \mathcal{M}_{\mathcal{E}}$, *the set of all rooted maps with even face-degree (duals of Eulerian maps)*); **II)** a bijection Ω acting on $\mathcal{A}$ that restricts on $\mathcal{M}$ to Ξ, and is natural (*we show that* Ω *is a partial medial construction*); **III)** a characterisation of $\Omega(\mathcal{A})$ (*we show that* $\Omega(\mathcal{A}) = \widetilde{\mathcal{H}}$, *the set of all rooted bipartite maps with a distinguished even subgraph*); **IV)** a most refined weight function α, for which Ω is α-preserving and a characterisation of map invariants associated with $\alpha\Omega^{-1}$ (*we show that* α *and* $\alpha\Omega^{-1}$ *encode the number of vertices, face degree-sequence and number of edges*).

We surmise that Ω, and therefore Ξ, are expressible in terms of elementary operations on the map, such as the excising of discs, gluing of handles, the adjoining of vertices, and the insertion of edges. Although our knowledge of Ω is far from complete, the constraints imposed by weight preservation have enabled us to provide some partial but explicit information about the elementwise action of Ω. Several examples of sets of maps that are related under Ω are given, and an explanation is given of the element action in the case of the sphere. Any explanation of the element action of Ω in general must be consistent with these sets. We hope that, by obtaining further information on the margins, it will be possible to obtain more complete information.

2 The genus series for maps and hypermaps

If $\mathcal{S}$ is a set of combinatorial objects and $\omega: \mathcal{S} \longrightarrow \{0,1,2,\ldots\}^p$, with p a positive integer, then ω is called a *weight function* on $\mathcal{S}$. We write $\omega = \omega_1 \otimes \omega_2 \otimes \cdots$, where $\omega_j(\sigma) = i_j$, so the tensor product of weight functions corresponds combinatorially to refinement of the weight function. Let $\tau\omega = \omega_1 + \omega_2 + \cdots$. Let $\gamma(\mathfrak{m}), \nu_i(\mathfrak{m}), \phi_i(\mathfrak{m}), \eta_i(\mathfrak{m})$ denote the genus, numbers of vertices, faces (or hyperfaces) and hyperedges of degree i in $\mathfrak{m} \in \mathcal{M}$, and let $\varepsilon(\mathfrak{m})$ denote the number of edges of $\mathfrak{m}$. These weight functions will be central to our argument, and others will be constructed from these. Let $\nu = \nu_1 \otimes \nu_2 \otimes \cdots$. Thus ν is the vertex-degree distribution and is a refinement of $\tau\nu$, which is the number of vertices. The functions $\phi, \tau\phi, \eta, \tau\eta$ are similarly defined.

The generating series for $\mathcal{S}$ with respect to ω is

$$[(\mathcal{S},\omega)]\,(\mathbf{x}) = \sum_{\sigma\in\mathcal{S}} x_1^{\omega_1(\sigma)}\cdots x_n^{\omega_n(\sigma)}.$$

When the indeterminates as arguments are generic and are understood from the context, they are omitted. We will denote both $[(\mathcal{S},\omega)]\,(\mathbf{x})$ and $[(\mathcal{S},\omega\otimes\omega')]\,(\mathbf{x},\mathbf{y})$ by S, since the distinction will be resolved by context. $[(\mathcal{S},\tau\omega)](x)$ is derivable from $[(\mathcal{S},\omega)](\mathbf{x})$ by setting $x_1 = x_2 = \cdots = x$. In this notation Theorem 1.1 asserts that $[(\mathcal{M}_4,\xi)] = \left[\left(\widetilde{\mathcal{M}},\tilde{\xi}\right)\right]$ where $\xi = \gamma\otimes\tau\nu\otimes\tau\phi\otimes\varepsilon$.

To make this paper self-contained we restate results from [7, 8] that give expressions for the genus series $[(\mathcal{M},\gamma\otimes\nu\otimes\phi\otimes\varepsilon)]$ and $[(\mathcal{H},\gamma\otimes\nu\otimes\phi\otimes\eta\otimes\varepsilon)]$, where $\mathcal{H}$ is the set of all rooted hypermaps. The weight function associated with $\mathcal{M}$ is the maximal refinement of ξ. Let χ^θ_α denote the character of the irreducible representation of $\mathfrak{S}_n$ indexed by the partition θ of n at an arbitrary element of the conjugacy class of $\mathfrak{S}_n$ indexed by the partition α of n. We write $\theta\vdash n$ to indicate that θ is a partition of n. Let $f^\theta = \chi^\theta_{[1^n]}$ denote the degree of this representation. Let $x_\theta = x_{\theta_1}x_{\theta_2}\cdots$ and h^θ be the size of the conjugacy class indexed by θ. Then $h^\theta = |\theta|!\cdot g^\theta$ where $g^\theta = 1/\left(\prod_{i\geq 1} i^{m_i(\theta)}m_i(\theta)!\right)$ with $m_i(\theta)$ equal to the number of occurrences of i in θ. Let $l(\theta) = \sum_{i\geq1} m_i(\theta)$, the *length* of θ. The genus series for rooted maps and hypermaps can now be stated.

Theorem 2.1. *Let* $M = [(\mathcal{M},\gamma\otimes\nu\otimes\phi\otimes\varepsilon)](u,\mathbf{x},\mathbf{y},z)$. *Then*

$$M(u^2,\mathbf{x},\mathbf{y},z) = 2u^2 z\frac{\partial}{\partial z}\log Z(\mathbf{x}u^{-1},\mathbf{y}u^{-1},zu)$$

where

$$Z(\mathbf{x},\mathbf{y},z) = \sum_{n\geq0}\frac{z^n}{n!(2n)!2^n}\sum_{\theta\vdash 2n}\frac{1}{f^\theta}\chi^\theta_{[2^n]}\left(\sum_{\nu\vdash 2n} h^\nu\chi^\theta_\nu x_\nu\right)\left(\sum_{\phi\vdash 2n} h^\phi\chi^\theta_\phi y_\phi\right).$$

Theorem 2.2. *Let* $H = [(\mathcal{H},\gamma\otimes\nu\otimes\phi\otimes\eta\otimes\varepsilon)](u,\mathbf{x},\mathbf{y}^{(1)},\mathbf{y}^{(2)},z)$. *Then*

$$H(u^2,\mathbf{x},\mathbf{y}^{(1)},\mathbf{y}^{(2)},z) = u^2 z\frac{\partial}{\partial z}\log Y(\mathbf{x}u^{-1},\mathbf{y}^{(1)}u^{-1},\mathbf{y}^{(2)}u^{-1},zu)$$

where

$$Y(\mathbf{x},\mathbf{y}^{(1)},\mathbf{y}^{(2)},z) = \sum_{n\geq0}\frac{z^n}{(n!)^2}\sum_{\theta\vdash n}\frac{1}{f^\theta}\left(\sum_{\nu\vdash n} h^\nu\chi^\theta_\nu x_\nu\right)\left(\sum_{\phi\vdash n} h^\phi\chi^\theta_\phi y^{(1)}_\phi\right)\left(\sum_{\varepsilon\vdash n} h^\varepsilon\chi^\theta_\varepsilon y^{(2)}_\varepsilon\right).$$

Specialisation to vertex-degree and face-degree restricted subsets of rooted maps is by setting the appropriate indeterminates to zero. Let $\mathcal{N} = \{1,2,\ldots\}$ and let $\mathcal{A}\subseteq\mathcal{N}$. Let $\mathbf{1}_\mathcal{A}$ be the indicator vector for $\mathcal{A}$, and denote $\mathbf{1}_\mathcal{N}$ by $\mathbf{1}$. Then the genus series for rooted maps with the property that the degree of every face is an element of $\mathcal{A}$ is obtained from Theorem 2.1 as $M(u,x\mathbf{1},y\mathbf{1}_\mathcal{A},z)$. For instance, the genus series M_4 for quadrangulations is obtained by selecting $\mathcal{A}=\{4\}$. Similarly, the genus series for rooted hypermaps having the property that the degree of every hyperedge is an element of $\mathcal{A}$ can be obtained from Theorem 2.2 as $H(u,x\mathbf{1},y^{(1)}\mathbf{1},y^{(2)}\mathbf{1}_\mathcal{A},z)$.

We note that further refinement of the genus series is not possible. The reason for this is that the embedding theorem [4] is used, that encodes a map as a permutation which acts transitively on a canonical fixed-point free involution. The cycle structure of the former is the vertex-degree distribution, the number of cycles in the latter is the number of edges, and the cycle structure of the product is the face-degree distribution of the map. Cycle structure is a natural index for conjugacy classes of the symmetric group, and the above theorems are a consequence of computations in the centre of the group algebra of the symmetric group, whose elements are the formal sums over each conjugacy class. The only combinatorial information that can be retained is therefore information derivable from the natural indices of the elements of this subalgebra, and this is precisely class type. This confines us to the weight functions $\nu, \phi, \eta, \varepsilon$ and weights deducible from these. It is in this sense that $\gamma \otimes \nu \otimes \phi \otimes \varepsilon$ and $\gamma \otimes \nu \otimes \phi \otimes \eta \otimes \varepsilon$ are the most refined weights that we can attach to $\mathcal{M}$ and $\mathcal{H}$.

3 Factorisation of character sums

From the concluding comments of the previous section it is clear that the expressions

$$A^\theta(x) = \sum_{\phi \vdash n} h^\phi \chi^\theta_\phi x_\phi \Big|_{\mathbf{x}=x\mathbf{1}}, \quad B^\theta_{\mathcal{A}}(x) = \sum_{\nu \vdash n} g^\nu \chi^\theta_\nu x_\nu \Big|_{\mathbf{x}=x\mathbf{1}_{\mathcal{A}}}, \tag{1}$$

where $\theta \vdash n$, will play an important role. Of course, $A^\theta(x)$ is equivalent to $n! B^\theta_{\mathcal{N}}(x)$, but these separate definitions prove to be convenient. We note that $A^\theta(x)$ is the principal specialisation of the Schur function s_θ. The expressions $A^\theta(x)$ and $B^\theta_{\mathcal{A}}(x)$ can be factored in an remarkable way which depends on the following two factorization lemmas, both of which are implicit in the results of [7].

Lemma 3.1. *Let $\mathcal{P}$ be the set of all partitions of all nonnegative integers (including the empty partition of 0) and let $\mathcal{P}_2$ be the set of all partitions of an even nonnegative integer. Let q_r be a variable which is defined for all integers r. If $q_r = 0$ for all odd integers r, then there exists a set of partitions $\widehat{\mathcal{P}_2} \subseteq \mathcal{P}_2$ and a bijection $\Delta : \widehat{\mathcal{P}_2} \to \mathcal{P} \times \mathcal{P} : \theta \mapsto (\theta', \theta'')$, such that*

$$w_\theta \left\| q_{\theta_i + i - j} \right\|_{l(\theta) \times l(\theta)} = \begin{cases} \left\| q_{2(\theta'_i + i - j)} \right\| \cdot \left\| q_{2(\theta''_i + i - j)} \right\|, & \text{for } \theta \in \widehat{\mathcal{P}_2}; \\ 0, & \text{for } \theta \notin \widehat{\mathcal{P}_2} \end{cases}$$

where $w_\theta = \pm 1$.

Explicit descriptions of $\Delta(\theta)$ and w_θ are not required because in the proof of Theorem 4.1 the sums are over all partitions $\theta \in \mathcal{P}_2$ and only even powers of w_θ appear. Lemma 3.1

holds in an analogous way for the permanent $\mathrm{per}(q_{\theta_i+i-j})$ with $w_\theta = 1$, although we do not make use of this observation here.

If $\alpha(\theta)$ can be expressed in the form $\alpha_1(\theta')\alpha_2(\theta'')$ where α_1 is independent of θ'' and α_2 is independent of θ', then this is called a Δ-*respecting factorization* of $\alpha(\theta)$. The second lemma gives a general instance of an $\alpha(\theta)$ that has a Δ-respecting factorization.

Lemma 3.2. *Let $\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots$ be indeterminates. Then for all $\theta \in \widehat{\mathcal{P}_2}$*

$$\prod_{i=1}^{l(\theta)}\prod_{j=1}^{\theta_i} f_{j-i} = \left(\prod_{i=1}^{l(\theta')}\prod_{j=1}^{2\theta_i'} f_{j-2i+1}\right)\left(\prod_{i=1}^{l(\theta'')}\prod_{j=1}^{2\theta_i''} f_{j-2i}\right)$$

where $(\theta', \theta'') = \Delta(\theta)$.

The first factorization lemma allows us to obtain a Δ-respecting factorisation, not of $B^\theta_{\mathcal{A}}$ itself, but of $w_\theta B^\theta_{\mathcal{A}}$ when the elements of $\mathcal{A}$ are even. The proof uses properties of symmetric functions that can be found, for example, in [9].

Corollary 3.3. *Let $\mathcal{A} = \{a_1, \ldots, a_k\}$ and define $2\mathcal{A}$ to be the set $\{2a_1, \ldots, 2a_k\}$. Then*

$$w_\theta B^\theta_{2\mathcal{A}}(x) = \begin{cases} B^{\theta'}_{\mathcal{A}}(x/2)B^{\theta''}_{\mathcal{A}}(x/2) & \text{for} \quad \theta \in \widehat{\mathcal{P}_2}; \\ 0 & \text{for} \quad \theta \notin \widehat{\mathcal{P}_2} \end{cases}$$

where $(\theta', \theta'') = \Delta(\theta)$.

Proof: Let s_μ denote the Schur function indexed by $\mu \vdash n$. By a result of Frobenius, $s_\mu = \sum_{\alpha \vdash n} g^\alpha \chi^\mu_\alpha p_\alpha$ where $p_\alpha = p_{\alpha_1}p_{\alpha_2}\cdots$ for $\alpha = (\alpha_1, \alpha_2, \ldots)$, and $p_k = x_1^k + x_2^k + \cdots$ is the power sum symmetric function of degree k. The p_i are algebraically independent, so

$$B^\theta_{\mathcal{A}}(x) = s_\theta\Big|_{\mathbf{p}=x\mathbf{1}_{\mathcal{A}}}.$$

where $\mathbf{p} = (p_1, p_2, \ldots)$. By the Jacobi-Trudi theorem, $s_\mu = \|h_{\mu_i-i+j}\|_{m\times m}$ for $m \geq l(\mu)$ where h_k is the complete symmetric function of degree k, defined by $h_k(\mathbf{x}) = [t^k]\prod_{j\geq 1}(1-tx_j)^{-1}$. It is well known that $h_r = \sum_{\alpha\vdash r} g^\alpha p_\alpha$, so if we let $h^{\mathcal{A}}_r(x) = h_r\Big|_{\mathbf{p}=x\mathbf{1}_{\mathcal{A}}}$, then

$$B^\theta_{\mathcal{A}}(x) = \left\|h^{\mathcal{A}}_{\theta_i+i-j}(x)\right\|. \tag{2}$$

To evaluate $B^\theta_{2\mathcal{A}}(x)$, note that $h^{2\mathcal{A}}_{2r-1}(x) = 0$ for $r \geq 1$. Therefore, from Lemma 3.1, if $\theta \in \widehat{\mathcal{P}_2}$,

$$\begin{aligned} B^\theta_{2\mathcal{A}}(x) &= \left\|h^{2\mathcal{A}}_{\theta_i+i-j}(x)\right\| \\ &= w_\theta \left\|h^{2\mathcal{A}}_{2(\theta_i'+i-j)}(x)\right\| \cdot \left\|h^{2\mathcal{A}}_{2(\theta_i''+i-j)}(x)\right\|. \end{aligned}$$

Writing

$$h_r^{\mathcal{A}}(x) = \sum_{\gamma \in \mathfrak{A}_r} g^{\gamma} x^{l(\gamma)},$$

where $\mathfrak{A}_r$ is the set of all partitions of r having all parts in $\mathcal{A}$, we have $h_{2r}^{2\mathcal{A}}(x) = \sum_{\gamma \in \mathfrak{A}_r} g^{2\gamma} x^{l(\gamma)}$. But $g^{2\gamma} = g^{\gamma}/2^{l(\gamma)}$, so $h_{2r}^{2\mathcal{A}}(x) = \sum_{\gamma \in \mathfrak{A}_r} g^{\gamma}(x/2)^{l(\gamma)}$. It therefore follows that, for $r \geq 1$,

$$h_{2r}^{2\mathcal{A}}(x) = h_r^{\mathcal{A}}(x/2).$$

Thus

$$B_{2\mathcal{A}}^{\theta}(x) \quad = \quad w_{\theta} \left\| h_{\theta_i'+i-j}^{\mathcal{A}}(x/2) \right\| \cdot \left\| h_{\theta_i''+i-j}^{\mathcal{A}}(x/2) \right\|$$

and the result follows from (2). □

The second factorization lemma will now be used to derive a similar result for $A^{\theta}(x)$. In fact, it is A^{θ}/f^{θ} that has a Δ-respecting factorisation. Although $A^{\theta}(x) = |\theta|! B_{\mathcal{N}}^{\theta}(x)$, the above argument cannot be used since the determinant which arises, namely

$$A^{\theta}(x) = |\theta|! \left\| h_{\theta_i+i-j}\big|_{\mathbf{p}=x\mathbf{1}} \right\|, \tag{3}$$

does not have the pattern of zero entries required for Lemma 3.1. Instead, this determinant must be evaluated explicitly.

Corollary 3.4. *Let $\theta \in \mathcal{P}$ and let $(\theta', \theta'') = \Delta(\theta)$. Then*

$$\frac{A^{\theta}(x)}{f^{\theta}} = \frac{4^{|\theta'|}4^{|\theta''|}}{(f^{\theta'})^2 (f^{\theta''})^2} A^{\theta'}\!\left(\frac{x}{2}\right) A^{\theta'}\!\left(\frac{x+1}{2}\right) A^{\theta''}\!\left(\frac{x-1}{2}\right) A^{\theta''}\!\left(\frac{x}{2}\right).$$

Proof: The determinant given in (3) is well known, and it can be shown that

$$A^{\theta}(x) = f^{\theta} \prod_{1 \leq i \leq l(\theta)} (x - i + 1)^{(\theta_i)}, \tag{4}$$

where $(x)^{(k)}$ is the rising factorial $x(x+1)\cdots(x+k-1)$. A proof of this is given in [5], for example. Using Lemma 3.2, with $f_r = x + r$, we now obtain

$$A^{\theta}(x) = f^{\theta} \prod_{1 \leq i \leq l(\theta')} (x - 2i + 2)^{(2\theta_i')} \prod_{1 \leq i \leq l(\theta'')} (x - 2i + 1)^{(2\theta_i'')}.$$

The result now follows by factoring out powers of 2, rearranging the products and using (4) once more. □

We close this section with an observation about f^{θ}. From the degree formula

$$f^{\theta} = n! \frac{\prod_{1 \leq i < j \leq m} (\theta_i - i - \theta_j + j)}{\prod_{j=1}^{m} (\theta_j - j + m)!}$$

where $\theta \vdash n$ and $l(\theta) = m$. Since products occur containing θ_i and θ_j for every subset $\{i, j\}$ of $\{1, .., m\}$, f^{θ} does not possess a Δ-respecting factorisation.

4 The existence of Ω

We now give a result that relates Eulerian maps to hypermaps, and therefore generalises Theorem 1.1 in the sense that $\mathcal{M}_4 \subset \mathcal{M}_{\mathcal{E}}$ where $\mathcal{E} = \{2, 4, \ldots\}$.

Theorem 4.1. *Let* $M_{\mathcal{E}} = [(\mathcal{M}_{\mathcal{E}}, \gamma \otimes \tau\nu \otimes \tau\phi \otimes \varepsilon)]\,(u, x, y, z),$

$$H = [(\mathcal{H}, \gamma \otimes \tau\nu \otimes \tau\phi \otimes \tau\eta \otimes \varepsilon)]\,(u, x, y^{(1)}, y^{(2)}, z).$$

Then $M_{\mathcal{E}}$ *and* H *are related by*

$$M_{\mathcal{E}}(u^2, x, y, z) = \mathsf{bis}_u\, H(4u^2, x, x+u, y, z).$$

Proof: If $\theta \vdash 2n$, then from (1), $B^{\theta}_{\{2\}}(z) = \chi^{\theta}_{[2^n]} z^n / 2^n n!$, so from Theorem 2.1

$$M_{\mathcal{E}}(u^2, x, y, z) = 2u^2 z \frac{\partial}{\partial z} \log R(xu^{-1}, yu^{-1}, zu) \tag{5}$$

where

$$R(x, y, z) = \sum_{\theta \in \mathcal{P}_2} \frac{1}{f^{\theta}} A^{\theta}(x) B^{\theta}_{\mathcal{E}}(y) B^{\theta}_{\{2\}}(z).$$

But $B^{\theta}_{\mathcal{E}}(x) = 0$ for $\theta \notin \widehat{\mathcal{P}_2}$, so

$$R(x, y, z) = \sum_{\theta \in \widehat{\mathcal{P}_2}} \frac{1}{f^{\theta}} A^{\theta}(x) B^{\theta}_{\mathcal{E}}(y) B^{\theta}_{\{2\}}(z).$$

Corollary 3.3 may be applied to factorise $B^{\theta}_{\mathcal{E}}(y)$ and $B^{\theta}_{\{2\}}(z)$. With $\Delta(\theta) = (\theta', \theta'')$ we obtain

$$B^{\theta}_{\mathcal{E}}(y) B^{\theta}_{\{2\}}(z) = w^2_{\theta} \left(B^{\theta'}_{\mathcal{N}}(\tfrac{1}{2}y) B^{\theta'}_{\{1\}}(\tfrac{1}{2}z) \right) \left(B^{\theta''}_{\mathcal{N}}(\tfrac{1}{2}y) B^{\theta''}_{\{1\}}(\tfrac{1}{2}z) \right)$$

so, from Corollary 3.4, having noted that $(w^{\theta})^2 = 1$,

$$\frac{1}{f^{\theta}} A^{\theta}(x) B^{\theta}_{\mathcal{E}}(y) B^{\theta}_{\{2\}}(z) = V_{\theta'}\left(\tfrac{1}{2}x, \tfrac{1}{2}(x+1), \tfrac{1}{2}x, 2z\right) V_{\theta''}\left(\tfrac{1}{2}x, \tfrac{1}{2}(x-1), \tfrac{1}{2}x, 2z\right)$$

where

$$V_{\alpha}(x, y, y', z) = \frac{1}{(f^{\alpha})^2} A^{\alpha}(x) A^{\alpha}(y) B^{\alpha}_{\mathcal{N}}(y') B^{\alpha}_{\{1\}}(z).$$

To obtain this we have used the fact that $B^{\alpha}_{\{1\}}(z) = z^n f^{\alpha}/n!$ from (1), since $f^{\alpha} = \chi^{\alpha}_{[1^n]}$, so $4^{|\alpha|} B^{\alpha}_{\{1\}}(z) = B^{\alpha}_{\{1\}}(4z)$, for $\alpha \vdash n$. But $\widehat{\mathcal{P}_2}$ is fixed and therefore independent of the term factored. Thus

$$R(x, y, z) = \sum_{\theta \in \widehat{\mathcal{P}_2}} V_{\theta'}\left(\tfrac{1}{2}x, \tfrac{1}{2}(x+1), \tfrac{1}{2}x, 2z\right) V_{\theta''}\left(\tfrac{1}{2}x, \tfrac{1}{2}(x-1), \tfrac{1}{2}x, 2z\right).$$

But Δ is a bijection, so

$$R(x,y,z)=\sum_{(\theta',\theta'')\in\mathcal{P}\times\mathcal{P}} V_{\theta'}\left(\tfrac{1}{2}x,\tfrac{1}{2}(x+1),\tfrac{1}{2}x,2z\right)V_{\theta''}\left(\tfrac{1}{2}x,\tfrac{1}{2}(x-1),\tfrac{1}{2}x,2z\right).$$

It follows that $R(x,y,z)$ has a factorisation of the form

$$R(x,y,z)=\left(\sum_{\theta\in\mathcal{P}} V_{\theta}\left(\tfrac{1}{2}x,\tfrac{1}{2}(x+1),\tfrac{1}{2}x,2z\right)\right)\left(\sum_{\theta\in\mathcal{P}} V_{\theta}\left(\tfrac{1}{2}x,\tfrac{1}{2}(x-1),\tfrac{1}{2}x,2z\right)\right). \tag{6}$$

But the two factors can be identified as series associated with a class of rooted maps. To see this, we use Theorem 2.2 to obtain

$$H_{\mathcal{A}}(u^2,x,y^{(1)},y^{(2)},z)=u^2z\frac{\partial}{\partial z}\log P(\mathbf{x}u^{-1},\mathbf{y}^{(1)}u^{-1},\mathbf{y}^{(2)}u^{-1},zu)$$

where

$$\begin{aligned}
P(x,y^{(1)},y^{(2)},z) &= \sum_{n\geq 0}\frac{z^n}{n!}\sum_{\theta\vdash n}\frac{1}{f^{\theta}}A^{\theta}(x)A^{\theta}(y^{(1)})B^{\theta}_{\mathcal{A}}(y^{(2)})\\
&= \sum_{\theta\in\mathcal{P}}\frac{1}{(f^{\theta})^2}A^{\theta}(x)A^{\theta}(y^{(1)})B^{\theta}_{\mathcal{A}}(y^{(2)})B^{\theta}_{\{1\}}(z)\\
&= \sum_{\theta\in\mathcal{P}} V_{\theta}\left(x,y^{(1)},y^{(2)},z\right).
\end{aligned}$$

We conclude from (6) that

$$R(x,y,z)=P\left(\tfrac{1}{2}x,\tfrac{1}{2}(x+1),\tfrac{1}{2}y,2z\right)\cdot P\left(\tfrac{1}{2}x,\tfrac{1}{2}(x-1),\tfrac{1}{2}y,2z\right).$$

Therefore, from (5),

$$\begin{aligned}
M_{\mathcal{E}}(u^2,x,y,z) &= 2u^2z\frac{\partial}{\partial z}\log\left(P\left(\frac{x}{2u},\frac{x+u}{2u},\frac{y}{2u},2uz\right)P\left(\frac{x}{2u},\frac{x-u}{2u},\frac{y}{2u},2uz\right)\right)\\
&= \tfrac{1}{2}\left\{H(4u^2,x,x+u,y,z)+H(4u^2,x,x-u,y,z)\right\}
\end{aligned}$$

and the result then follows. $\square$

This result implies the existence of a bijection Ω between $\mathcal{M}_{\mathcal{E}}$ and a set $\widetilde{\mathcal{H}}$ derived from $\mathcal{H}$.

5 Some properties of a bijection Ω

We are now in a position to make some observations on the additional information about the bijection Ω that can be obtained from Theorem 4.1.

Since the proof of Theorem 1.1 appeals to properties of particular characters of irreducible representations of the symmetric group, for which there is apparently no combinatorial interpretation, the proof itself has provided us with no combinatorial information about the elementwise action of Ω. However, for the sphere, Theorem 1.1 specialises to

$$M_4(0,x,y,z) = M(0,x,x,z^2y). \tag{7}$$

On the right hand side the appearance of x in place of the face indeterminate y suggests that the construction involves the insertion of a vertex in each face. Moreover, the appearance of z^2y in place of the edge indeterminate z suggests that the construction replaces an edge by a face and two edges. These two pieces of information are consistent with the *medial construction*, and indeed the latter is a bijection between quadrangulations in the sphere and all maps in the sphere. This is a classical result [10], and implies the relationship (7) between generating series. Thus the restriction of Ω to the set of rooted quadrangulations in the sphere is precisely the medial construction.

However, the medial construction is reversible only for bipartite quadrangulations, and quadrangulations are necessarily bipartite only for the sphere. We conclude that, while the medial construction establishes Theorem 1.1 for the sphere, it does not extend to higher genera, at least not without modification, and is therefore not identified as Ω. Thus, on the one hand, we wish to determine features of the medial construction that are consistent with the generalisation of Theorem 1.1 that is induced by Ω; on the other hand, we wish to derive more detailed information about Ω by considering its induced action on vertices of specified degree, for example the bivalent vertices.

5.1 Weight preservation under Ω

We begin with the preservation of face-degree under the action of Ω. This is consistent with Theorem 4.1, and we now establish it. The following factorization lemma is required.

Lemma 5.1. *For $\theta \vdash n$, let*

$$B^\theta_{\mathcal{E}}(\mathbf{y}) = \sum_{\phi\vdash n} g^\phi \chi^\theta_\phi y_\phi \Big|_{y_1=y_3=\cdots=0}, \quad \text{and} \quad B^\theta(\mathbf{y}) = \sum_{\nu\vdash n} g^\nu \chi^\theta_\nu y_\nu.$$

Then

$$w_\theta B^\theta_{\mathcal{E}}(\mathbf{y}) = \begin{cases} B^{\theta'}(\mathbf{y}/2) B^{\theta''}(\mathbf{y}/2)\Big|_{y_i\mapsto y_{2i}, i=1,2,\ldots} & \text{if } \theta \in \widehat{\mathcal{P}_2}; \\ 0 & \text{if } \theta \notin \widehat{\mathcal{P}_2}. \end{cases}$$

Proof: Using the same approach as in the proof of Corollary 3.3, if we let

$$h_r(\mathbf{y}) = h_r\big|_{p_i=y_i, i\geq 1} \quad \text{and} \quad \widetilde{h}_r(\mathbf{y}) = h_r\Big|_{\substack{p_{2i}=y_{2i}, i\geq 1 \\ p_1=p_3=\cdots=0}},$$

since the power sum symmetric functions are algebraically independent, then

$$B^\theta_{\mathcal{E}}(\mathbf{y}) = \left\| \widetilde{h}_{\theta_i+i-j}(\mathbf{y}) \right\| \quad \text{and} \quad B^\theta(\mathbf{y}) = \left\| h_{\theta_i+i-j}(\mathbf{y}) \right\|. \tag{8}$$

Now $\widetilde{h}_{2r-1}(\mathbf{y}) = 0$ for $r \geq 1$. Therefore from Lemma 3.1, if $\theta \in \widehat{\mathcal{P}_2}$ then

$$B^\theta_{\mathcal{E}}(\mathbf{y}) = w_\theta \left\| \widetilde{h}_{2(\theta'_i+i-j)}(\mathbf{y}) \right\| \cdot \left\| \widetilde{h}_{2(\theta''_i+i-j)}(\mathbf{y}) \right\|. \tag{9}$$

Moreover, $\widetilde{h}_{2r}(\mathbf{y}) = \sum_{\gamma\vdash r} g^{2\gamma} y_{2\gamma}$. But $g^{2\gamma} = g^\gamma / 2^{l(\gamma)}$, so $\widetilde{h}_{2r}(\mathbf{y}) = \sum_{\gamma\vdash r} g^\gamma y_\gamma \Big|_{y_i \mapsto y_{2i}/2}$. It therefore follows that for $r \geq 1$,

$$\widetilde{h}_{2r}(\mathbf{y}) = h_r(\mathbf{y}/2)\Big|_{y_i \mapsto y_{2i}, i=1,2,\ldots}.$$

Therefore from (9)

$$B^\theta_{\mathcal{E}}(\mathbf{y}) = w_\theta \left\| h_{\theta'_i+i-j}(\mathbf{y}/2)\Big|_{y_i \mapsto y_{2i}, i=1,2,\ldots} \right\| \cdot \left\| h_{\theta''_i+i-j}(\mathbf{y}/2)\Big|_{y_i \mapsto y_{2i}, i=1,2,\ldots} \right\|$$

and the result follows from (8). □

The next theorem is an immediate consequence of this lemma, and refines the weight function that appears in Theorem 4.1.

Theorem 5.2. *The genus series* $M_{\mathcal{E}} = \left[(\mathcal{M}_{\mathcal{E}}, \gamma \otimes \tau\nu \otimes \phi \otimes \varepsilon)\right](u, x, \mathbf{y}, z),$

$$H = \left[(\mathcal{H}, \gamma \otimes \tau\nu \otimes \tau\phi \otimes \eta \otimes \varepsilon)\right](u, x, y^{(1)}, \mathbf{y}^{(2)}, z)$$

are related by

$$M_{\mathcal{E}}(u^2, x, \mathbf{y}, z) = \mathsf{bis}_u\, H(4u^2, x, x+u, \mathbf{y}, z)\Big|_{y_i \mapsto y_{2i}, i=1,2,\ldots}.$$

It follows that

$$\left[(\mathcal{M}_{\mathcal{E}}, \psi)\right](u^2, x, \mathbf{y}, z) = \left[\left(\widetilde{\mathcal{H}}, \widetilde{\psi}\right)\right](u^2, x, \mathbf{y}, z)\Big|_{y_i \mapsto y_{2i}, i=1,2,\ldots}$$

where $\psi = \gamma \otimes \tau\nu \otimes \phi \otimes \varepsilon$. $\widetilde{\mathcal{H}}$ is a set that is constructed from $\mathcal{H}$, and is to be characterised, and $\widetilde{\psi}$ is a weight function on $\widetilde{\mathcal{H}}$. This implies that Ω is ψ-respecting.

5.2 The identification of a natural image set for Ω

We now address the question of identifying a combinatorial set $\widetilde{\mathcal{H}}$ that is related to maps, that has a natural description and has a weight function $\widetilde{\psi}$, to be determined, such that

$$\left[\left(\widetilde{\mathcal{H}}, \widetilde{\psi}\right)\right](u^2, x, \mathbf{y}, z) = \mathsf{bis}_u\, H(4u^2, x, x+u, \mathbf{y}, z)\Big|_{y_i \mapsto y_{2i}, i=1,2,\ldots}. \tag{10}$$

This is a consequence of Theorem 5.2. We wish to transform the translated arguments in H into the generic list of arguments by attaching combinatorial markers (decoration) to a map. It is then a subsequent (and final) task to transform the decorated rooted map into a characterizable class of rooted maps. The following definitions are needed.

Definition 5.3. *Let $\mathfrak{s}$ be a submap of a map $\mathfrak{m}$ which contains all the vertices of $\mathfrak{m}$, has the same genus as $\mathfrak{m}$ and has precisely 1 face. Let $\mathfrak{t}$ be a spanning tree of $\mathfrak{s}$. The set $\mathcal{S}$ of edges which are in $\mathfrak{s}$ but not in the spanning tree $\mathfrak{t}$ is called a* strut set *for $\mathfrak{m}$.*

If $\mathfrak{m}$ has genus 0, then $\mathfrak{s}$ is a spanning tree and a strut set must be empty. If $\mathfrak{m}$ has higher genus, then a spanning tree is not a 2-cell embedding of the surface, so the struts provide a minimal set of edges which can be added to a spanning tree to get a 2-cell embedding of the surface. It is always possible to make a canonical choice for $\mathfrak{s}$, and then $\mathfrak{t}$ [12]. Hence we can choose a canonical set of struts for any map.

Definition 5.4. *Let $\mathcal{S}$ be a canonical set of struts for a hypermap $\mathfrak{m}$. A* decorated map *$\tilde{\mathfrak{m}}$ is a hypermap $\mathfrak{m}$ with a distinguished set of struts chosen from $\mathcal{S}$ and a distinguished even subset of hyperfaces.*

Theorem 5.5. *Let $\rho(\tilde{\mathfrak{m}})$ be the number of distinguished hyperfaces of a decorated map $\tilde{\mathfrak{m}}$, and let $\widetilde{\mathcal{H}}$ be the set of all such decorated maps. Then*

$$[(\mathcal{M}_{\mathcal{E}}, \gamma \otimes \tau\nu \otimes \phi \otimes \varepsilon)]\,(u^2, x, \mathbf{y}, z) = \left[\left(\widetilde{\mathcal{H}}, \widetilde{\psi}\right)\right](u^2, x, \mathbf{y}, z)$$

where $\widetilde{\psi} = (\gamma + \frac{\rho}{2}) \otimes (\tau\nu + \tau\phi - \rho) \otimes 2\eta \otimes \varepsilon$.

Proof: In order to obtain a description of a weight function $\widetilde{\psi}$ which satisfies (10), consider the term in $\mathsf{bis}_u\, H(4u^2, x, x+u, \mathbf{y}, z)$ associated with a hypermap $\mathfrak{m}$, namely

$$\mathsf{bis}_u\, 4^{\gamma} u^{2\gamma} x^{\tau\nu} (x+u)^{\tau\phi} y_{2\eta} z^{\varepsilon} = 2^{2\gamma} \sum_{i \geq 0} \binom{\tau\phi}{2i} u^{2\gamma+2i} x^{\tau\nu+\tau\phi-2i} y_{2\eta} z^{\varepsilon},$$

where $\gamma = \gamma(\mathfrak{m})$, $\tau\nu = \tau\nu(\mathfrak{m})$, $\tau\phi = \tau\phi(\mathfrak{m})$, $\eta = \eta(\mathfrak{m})$, and $\varepsilon = \varepsilon(\mathfrak{m})$. We now associate each monomial on the right hand side with a decorated map. To account for the factor of $2^{2\gamma}$, we will specify a canonical set of 2γ edges of $\mathfrak{m}$ from which we may choose subsets. Towards this end, observe that a map with $2\gamma + \tau\nu - 1$ edges has 1 face (by the Euler-Poincaré Formula). Since a spanning tree has $\tau\nu - 1$ edges, $\mathcal{S}$ is a set of 2γ edges and there are therefore $2^{2\gamma}$ ways to select a subset of these edges. To account for the factor of $\binom{\tau\phi}{2i}$, we select an even number of hyperfaces to distinguish in some way. The result follows from Theorem 5.2 by comparing the coefficients in the term associated with $\mathfrak{m}$ above. $\square$

This thereom tells us that the set $\widetilde{\mathcal{H}}$ characterises $\Omega(\mathcal{M}_{\mathcal{E}})$ as promised in point **III** of the Introduction. (It takes care of point **IV**, as well.) It should be noted that there are other sets which could serve to characterise $\Omega(\mathcal{M}_{\mathcal{E}})$. For example, the dual of a hypermap is a rooted bipartite map, and with an appropriate choice of weight functions it is possible to use an argument similar to the one above to characterise $\Omega(\mathcal{M}_{\mathcal{E}})$ as the set of all rooted bipartite maps with a distinguished even subgraph. Although this seems like a fairly natural set of maps, we have chosen to work with $\widetilde{\mathcal{H}}$ in the rest of the paper because we believe the notion of a decorated map has additional structure which may help one to obtain an elementwise description of Ω.

6 Vertex information

6.1 Action of Ω on bivalent vertices

As we have seen from Theorem 5.2, Ω is $\gamma \otimes \tau\nu \otimes \phi \otimes \varepsilon$-preserving. However, it is easily seen from the appropriate proofs that it is not $\gamma \otimes \nu \otimes \phi \otimes \varepsilon$-preserving. Nevertheless, we wish to determine what happens to specific vertex degrees under the action of Ω. This is difficult to ascertain in general but, as a first step in this direction, we trace vertices of degree two.

Corollary 6.1. *Let* $H = [(\mathcal{H}, \gamma \otimes \nu_2 \otimes \tau\nu \otimes \tau\phi \otimes \tau\eta \otimes \varepsilon)]\,(u, t, x, y^{(1)}, y^{(2)}, z)$ *and let* $\bar{t} = t - 1$. *Then*

$$\mathsf{bis}_u\, H(4u^2, t, x, x+u, y, z) = (1 - \bar{t}xz)\, M_{\mathcal{E}}\Big(u^2, x, y, \frac{z}{1 - \bar{t}xz}\Big) + \frac{\bar{t}x^2yz}{1 - \bar{t}xz}.$$

Proof: Let $H_{\bar{2}}(u^2, x, y^{(1)}, y^{(2)}, z)$ be the genus series for the set $\mathcal{H}_{\bar{2}}$ of rooted hypermaps with no vertices of degree 2. We begin by establishing a relationship between $H_{\bar{2}}$ and the genus series $H(u^2, 1, x, y^{(1)}, y^{(2)}, z)$ for all rooted hypermaps.

Almost every rooted hypermap can be constructed uniquely from an element of $\mathcal{H}_{\bar{2}}$ by subdividing edges in the following way. First subdivide each nonroot edge. Since subdivision affects neither the genus of the map nor the numbers of hyperedges and hyperfaces, the genus series for these subdivided hypermaps is obtained by applying the substitution $z \mapsto \delta_z(x)$ to $H_{\bar{2}}/z$, where $\delta_z(x) = z/(1 - xz)$. For brevity we will omit the subscript z. Next subdivide the root edge and distinguish an edge in this subdivision. This is accomplished by multiplying the genus series by $z\frac{\partial}{\partial z}(\delta(x))$. The only rooted hypermaps not obtainable by this procedure are rooted cycles, which are genus 0 maps having one hyperedge, one hyperface, n edges and n vertices, for any $n \geq 1$. Then

$$H(u^2, 1, x, y^{(1)}, y^{(2)}, z) = \left(z\frac{\partial}{\partial z}\delta(x)\right)\left(\frac{1}{z}H_{\bar{2}}(u^2, x, y^{(1)}, y^{(2)}, z)\Big|_{z \mapsto \delta(x)}\right) + xy^{(1)}y^{(2)}\delta(x)$$

which can be rearranged to give

$$H_{\bar{2}}\Big(u^2, x, y^{(1)}, y^{(2)}, \delta(x)\Big) = (1 - xz)H(u^2, 1, x, y^{(1)}, y^{(2)}, z) - xy^{(1)}y^{(2)}z.$$

The compositional inverse of $z \mapsto \delta(x)$ is $z \mapsto \delta(-x)$ so

$$H_{\bar{2}}(u^2, x, y^{(1)}, y^{(2)}, z) = \delta(-x)\left(\frac{1}{z}H\left(u^2, 1, x, y^{(1)}, y^{(2)}, \delta(-x)\right) - xy^{(1)}y^{(2)}\right). \tag{11}$$

Using the above analysis now to keep track of vertices of degree 2 (marked by t) we readily obtain

$$\begin{aligned} H(u^2, t, x, y^{(1)}, y^{(2)}, z) &= \left(z\frac{\partial}{\partial z}\delta(tx)\right)\left(\frac{1}{z}H_{\bar{2}}(u^2, x, y^{(1)}, y^{(2)}, z)\Big|_{z\mapsto\delta(tx)}\right) + txy^{(1)}y^{(2)}\delta(tx) \\ &= \delta(\bar{t}x)\left(H\left(u^2, 1, x, y^{(1)}, y^{(2)}, \delta(\bar{t}x)\right) + \bar{t}xy^{(1)}y^{(2)}\right) \end{aligned}$$

after routine simplification using (11). The compositional inverse of $z \mapsto \delta(\bar{t}x)$ is $z \mapsto \delta(-\bar{t}x)$ which can be used to show that

$$H(u^2, 1, x, y^{(1)}, y^{(2)}, z) = \delta(-\bar{t}x)\left(\frac{1}{z}H\left(u^2, t, x, y^{(1)}, y^{(2)}, \delta(-\bar{t}x)\right) - \bar{t}xy^{(1)}y^{(2)}\right)$$

after rearrangement. Substituting the above equation into Theorem 4.1 gives

$$M_{\mathcal{E}}(u^2, x, y, z) = \mathsf{bis}_u\,\delta(-\bar{t}x)\left(\frac{1}{z}H\left(4u^2, t, x, y, x+u, \delta(-\bar{t}x)\right) - \bar{t}xy(x+u)\right).$$

The result is obtained by using the substitution $z \mapsto \delta(\bar{t}x)$ and routine simplification. □

This result is a composition of combinatorially interpretable series H and $M_{\mathcal{E}}$ with the combinatorially interpretable series $1/(1-\bar{t}xz)$, and seems to be simple enough in form to admit a direct combinatorial explanation. On the left hand side t marks the occurrences of vertices of degree 2. However, we have not been able to determine the map invariant that t records on the right hand side under the action of Ω. We believe that an explanation of this will provide marginal information about the combinatorial action of Ω in general.

6.2 Other classes of maps

Although we have been able to deal with Eulerian maps, doubly Eulerian maps present difficulties which stem from the fact, observed earlier, that it is not possible to obtain a Δ-respecting factorization of f^θ. The genus series for doubly Eulerian maps is

$$M_{\mathcal{E}}^{\mathcal{E}}(u^2, x, y, z) = 2u^2 z\frac{\partial}{\partial z}\log E(xu^{-1}, yu^{-1}, zu)$$

where

$$E(x, y, z) = \sum_{\theta\in\mathcal{P}_2}\frac{1}{|\theta|!\,f^\theta}B^\theta_{2\mathcal{N}}(x)B^\theta_{2\mathcal{N}}(y)B^\theta_{\{2\}}(z).$$

Here, a factorization of the sort given for Eulerian maps does not seem possible because of the presence of $1/f^\theta$ with no corresponding A^θ that is required by Corollary 3.4. However, the medial construction seems to give some relationships on the sphere (eg. between Eulerian quadrangulations and all doubly Eulerian maps) for which it is not clear what sort of shape extensions to higher genera (if any) may take. Perhaps a complete combinatorial explanation of Theorem 4.1 would shed some light on this question.

7 Actions of Ω on decorated rooted maps

In this section we list a number of sets of maps between which we know from algebraic considerations that Ω acts bijectively. We therefore seek an elementwise explanation of this specific information.

We will make detailed reference to the collection of polygonal representations of maps on a small number of edges which are given at the end of the section. Maps of genus 0 appear inside square boxes formed by dotted boundary lines. Maps of genus 1 appear inside squares with directed dashed boundary lines where it is to be understood that a torus can be formed by identifying first the top and bottom directed boundary lines of the square, and then the left and right directed boundary lines of the square. Maps $\mathfrak{m}19.1$ through $\mathfrak{m}19.4$ in Table 19 are of genus 2. Here the double torus can be formed by identifying the like-labelled directed boundary lines of the octagon in pairs. To further help the reader get a clear description of the embedding we have also included a drawing of the underlying graph with edges crossing, but with the correct cyclic ordering of half-edges about each vertex.

The collection is organized into tables according to the genus and face degrees of the dually Eulerian maps. Eulerian maps in the same table have the same face partition ϕ and the same genus. The corresponding decorated maps have a hyperedge partition η, where $\phi = 2\eta$. and the genus varies in a known way. The significance of these tables is that each contains a domain $\mathcal{D} \subseteq \widetilde{\mathcal{H}}$ and image set $\mathcal{R} \subseteq \mathcal{M}_{\mathcal{E}}$ under the action of Ω. To reduce the size of the tables, we have not listed the individual rootings of each map, but instead have indicated the number of rootings in the top right corner of each diagram. Thus each map is to be regarded as *canonically rooted.* For example, Table 5 indicates that the 3 rootings of the decorated map $\widetilde{\mathfrak{m}}5.1$a and the 3 rootings of the decorated map $\widetilde{\mathfrak{m}}5.1$b are in bijective correspondence with the 6 rootings of the dually Eulerian map $\mathfrak{m}5.1$.

The following notational conventions are adopted for decorated maps in the presentation of this material. In a decorated map, an even subset of the hyperfaces is distinguished. We use a single $*$ to indicate a *distinguished hyperface* and a single $+$ to indicate a hyperface which has not been distinguished. Where a hyperface is represented by a union of two or more regions in its polygonal representation we put a $\cdot$ in each hyperface region not already containing a $*$ or $+$. Regions corresponding to *hyperedges* are left blank. In effect, regions containing a symbol are of one colour and regions without a symbol are of the other colour. Moreover, the number of hyperfaces is the number of $*$'s plus the number of $+$'s. For example, $\tilde{\mathfrak{m}}19.8$ has 2 hyperfaces of degree 2 and one hyperedge of degree 4. Each hyperface is a union of 2 regions in this representation, one containing an asterisk to indicate that the hyperface is distinguished, and the other containing a dot to indicate that the region is part of a hyperface.

The *struts* in a decorated map are indicated by broken lines of two different sorts. For a *selected* strut the broken line is a thick dashed line, ----. For an *unselected* strut the broken line is a thick dotted line, ········.

The reader's attention is drawn to maps $\tilde{\mathfrak{m}}14.1$, $\tilde{\mathfrak{m}}14.2$ and $\tilde{\mathfrak{m}}14.3$. These appear to be identical. However, recall that these maps are canonically rooted, so once the root edge has been selected there are 3 different choices for the 2 distinguished hyperfaces. A similar observation is true for maps $\tilde{\mathfrak{m}}18.1$ through $\tilde{\mathfrak{m}}18.6$, and elsewhere.

7.1 The action of Ω on the sphere

Some information about constructing a set of maps that is in bijection with the set of decorated rooted maps is obtained by considering the restriction to the sphere. This set, of course, is then the image set of $\mathcal{M}_{\mathcal{E}}$ under the action of Ω.

Let $H_2(u^2, x, y^{(1)}, y^{(2)}, z) = H(u^2, x, y^{(1)}, y^{(2)}\mathbf{1}_{\{2\}}, z)$, the genus series for rooted hypermaps for which every hyperedge is a digon. Note that by replacing these hyperedges of degree 2 by single edges, it is clear that $H_2(u^2, x, y^{(1)}, y^{(2)}, z) = M(u^2, x, y^{(1)}, z^2y^{(2)})$, and this in turn explains how Theorem 4.1 specializes to Theorem 1.1. (In fact, Theorem 4.1 is obtained from Theorem 5.2 by setting $\mathbf{y}^{(2)} = y^{(2)}\mathbf{1}_{\{4\}}$.) In the same sense, we may generalise (7) to

$$M_{\mathcal{E}}(0, x, y, z) = H(0, x, x, y, z), \qquad (12)$$

a result which suggests that there is a bijection between all maps with even face degrees and all hypermaps in the sphere.

Such a bijection is easily exhibited with the following *modified medial construction.* Let $\mathfrak{m}$ be a rooted hypermap and put a new vertex in the interior of each of its hyperfaces. In traversing the boundary of each hyperface, join each vertex encountered to the new vertex put into that hyperface. Now consider the tail v of the root edge of $\mathfrak{m}$, and find the first edge e not in $\mathfrak{m}$ encountered in a direction consistent with the orientation of the sphere. Delete the edges of $\mathfrak{m}$ and direct the edge e so that its tail is v. This is to be the root edge of the resulting map $\mathfrak{m}^*$. Each face of this map, which we will call the *medial* of $\mathfrak{m}$, has degree twice that of a corresponding hyperedge in $\mathfrak{m}$ and so $\mathfrak{m}^* \in \mathcal{M}_{\mathcal{E}}$. This construction μ is reversible and gives the desired bijection. In fact, the restriction of Ω to the sphere is the inverse of μ. Tables 1 through 11 illustrate the medial construction.

7.2 The action of Ω in general

In a sense Ω is a lifting of the medial construction to higher genera and there is no evidence to contradict that a partial medial construction suffices. It is possible that the following type of construction could work: given a rooted hypermap, in each hyperface place either a vertex or a half-handle; in faces where a vertex is placed, perform the medial construction; in faces where a half-handle is placed, the construction is not known.

In certain cases the elementwise action of Ω is forced. Thus, for example, we know that $\tilde{\mathfrak{m}}12.1$ maps to $\mathfrak{m}12.1$ and $\tilde{\mathfrak{m}}13.1$ maps to $\mathfrak{m}13.1$ up to rooting. For maps with 4 or more edges, there are many more possibilities and it seems that the action of Ω will not be simple to describe in general. Our original hope was that the (forced) actions observed in the smaller cases together with the idea of a partial medial construction would somehow induce a reasonable set of actions for the next higher case, and that this programme could be recursed a small number of times until a clear description of the action of Ω in general would become evident. Unfortunately, a careful study of the Tables reveals that some new ideas are still needed.

For the maps in Table 15 it seems reasonable to suggest that $\tilde{\mathfrak{m}}15.1$ be mapped to $\mathfrak{m}15.1$ and $\tilde{\mathfrak{m}}15.2$ be mapped to $\mathfrak{m}15.2$, as natural generalisations of the (forced) mapping of Table 13. In Table 14 struts appear for the first time and now it's not obvious how Ω acts. Perhaps $\tilde{\mathfrak{m}}14.1$ through $\tilde{\mathfrak{m}}14.3$ should be mapped to 3 of the rootings of $\mathfrak{m}14.1$, since we can perform a partial medial construction inside the unselected hyperface and combine this with the (forced) mapping of Table 12. It then seems reasonable that $\tilde{\mathfrak{m}}14.4$ be mapped to $\mathfrak{m}14.2$ as a natural generalisation of the mapping of Table 12. Finally, $\mathfrak{m}14.3$ is the medial of maps $\tilde{\mathfrak{m}}14.5$ through $\tilde{\mathfrak{m}}14.8$, so perhaps $\tilde{\mathfrak{m}}14.5$ maps to $\mathfrak{m}14.3$ since no struts have been selected in $\tilde{\mathfrak{m}}14.5$. This would then force $\tilde{\mathfrak{m}}14.6$ through $\tilde{\mathfrak{m}}14.8$ to map to the remaining 3 rootings of $\mathfrak{m}14.1$.

Up to this point we have been trying to establish some rules that Ω must obey or properties that it must have, but when we proceed to Table 16, we find that we can no longer keep these rules consistent. For example, it seems that $\tilde{\mathfrak{m}}16.2$ should map to $\mathfrak{m}16.2$ because to remain consistent with previous choices, the bivalent vertex in $\tilde{\mathfrak{m}}16.2$ should map to a bivalent vertex. However, performing a partial medial construction in the unselected hyperface of $\tilde{\mathfrak{m}}16.4$ would also create a bivalent vertex, so it too should map to $\mathfrak{m}16.2$. Similar inconsistencies arise in Tables 17–19.

It is our hope that the diagrams in these Tables will eventually help give a clearer understanding of the action of Ω.

References

[1] D. Bessis, C. Itzykson and J.B. Zuber, *Quantum field theory techniques in*

graphical enumeration, Adv. Applied Math. **1** (1980), 109–157.

[2] I.P. Goulden, J.L. Harer and D.M. Jackson, *The virtual Euler characteristic of the moduli spaces of real and complex algebraic curves*, preprint.

[3] I.P. Goulden and D.M. Jackson, *Transitive factorisations into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc. (to appear).

[4] J.L.Gross and T.W.Tucker, *"Topological Graph Theory,"* Wiley Interscience, New York, 1987.

[5] D.M. Jackson, *Some combinatorial problems associated with products of conjugacy classes of the symmetric group,* J. Combinatorial Theory **A 49** (1988) 363–369.

[6] D.M. Jackson, M.J. Perry and T.I. Visentin, *Factorisations for partition functions of random Hermitian matrix models*, Commun. Math. Phys. **179** (1996) 25–60.

[7] D.M. Jackson and T.I. Visentin, *A character theoretic approach to embeddings of rooted maps in an orientable surface of given genus*, Trans. Amer. Math. Soc. **322** (1990), 343–363.

[8] D.M. Jackson and T.I. Visentin, *Character theory and rooted maps in an orientable surface of given genus: face-coloured maps*, Trans. Amer. Math. Soc. **322** (1990), 365–376.

[9] I.G. Macdonald, *"Symmetric Functions and Hall Polynomials,"* Clarendon Press, Oxford, 1995.

[10] W.T. Tutte, *The dissection of equilateral triangles into equilateral triangles*, Proc. Cambridge Philos. Soc. **44** (1948), 463–482.

[11] W.T. Tutte, *A census of planar maps*, Canad. J. Math. **15** (1963), 249–271.

[12] W.T. Tutte, *"Graph Theory,"* Encyclopedia Math. Appl., Addison-Wesley, Reading, Mass., 1984.