The number of ramified coverings of the sphere by the double torus, and a general form for higher genera*

I.P.Goulden† and D.M.Jackson‡

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Abstract

An explicit expression is obtained for the generating series for the number of ramified coverings of the sphere by the double torus, with elementary branch points and prescribed ramification type over infinity. Thus we are able to prove a conjecture of Graber and Pandharipande, giving a linear recurrence equation for the number of these coverings with no ramification over infinity. The general form of this series is conjectured for the number of these coverings by a surface of arbitrary genus that is at least two.

1 Introduction and Background

Let $\mu_m^{(g)}(\alpha)$ be the number of almost simple ramified coverings of $\mathbb{S}^2$ by X with ramification type α where X is a compact connected Riemann surface of genus g, and α is a partition with m parts. The problem of determining an (explicit) expression for $\mu_m^{(g)}(\alpha)$ is called the *Hurwitz Enumeration Problem*, a brief account of which is given in [2]. The terminology and notation used here will be consistent with the latter paper.

There appears to be a natural topological distinction between the low genera instances of the problem, namely for $g \leq 1$, on the one hand, and the higher genera instances, namely for $g \geq 2$, on the other hand. In this paper we address the higher genera case of the Hurwitz Enumeration Problem. The distinction between the low genera and the high genera cases manifests itself in this paper in the fact that a general form can be given for the higher genera case, but that does not specialize to the low genera case. In this paper we prove an explicit result for g=2, the double torus, which then leads to a proof of the conjectured recurrence equation of Graber and Pandharipande [9] for $\mu_m^{(2)}(1^m)$, corresponding to the case of simple ramification. Moreover, we conjecture an explicit result for $g=3$. We also give a conjecture for the general form of the generating series for $\mu_m^{(g)}(\alpha)$ for arbitrary $g \geq 2$.

Let $\mathcal{C}_\alpha$ be the conjugacy class of the symmetric group $\mathfrak{S}_n$ on n symbols indexed by the partition α of n. Let $\vartheta(\alpha) = \prod_{i\geq 1} i^{m_i} m_i!$, where α has m_i parts equal to i where $i \geq 1$. We write $\alpha \vdash n$ to indicate that $\alpha = (\alpha_1, \ldots, \alpha_r)$ is a partition of n and $\alpha \models n$ to indicate that α is a partition of n with no part equal to one. The length, r, of α is denoted by $l(\alpha)$.

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†Dept. of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada

‡Dept. of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada

Let the generating series for $\mu_m^{(g)}(\alpha)$ be defined by

$$F_g(x,\mathbf{p}) = \sum_{m,n\geq 1} \sum_{\substack{\alpha\vdash n \\ l(\alpha)=m}} \frac{\mu_m^{(g)}(\alpha)}{(n+m+2g-2)!} p_\alpha x^n. \tag{1}$$

where p_i, for $i \geq 1$, are indeterminates and $p_\alpha = p_{\alpha_1}p_{\alpha_2}\cdots$. Let

$$\psi_i(x,\mathbf{p}) = \sum_{n\geq 1} n^{i-1} a_n p_n x^n \tag{2}$$

where $a_n = n^n/(n-1)!$ for $n \geq 1$. Then, as we have shown in previous work ([3],[2]) F_0 and F_1 can be expressed succinctly in terms of $\psi_i \equiv \psi_i(s,\mathbf{p})$ where $s \equiv s(x,\mathbf{p})$ is the unique solution of the functional equation

$$s = xe^{\psi_0(s,\mathbf{p})}. \tag{3}$$

From [3] (Prop. 3.1) and [2] (Thm 4.2), the expressions for F_0 and F_1 in terms of the ψ_i are given by

$$\left(x\frac{\partial}{\partial x}\right)^2 F_0(x,\mathbf{p}) = \psi_0, \tag{4}$$

$$F_1(x,\mathbf{p}) = \frac{1}{24}\left(\log(1-\psi_1)^{-1} - \psi_0\right). \tag{5}$$

In this paper we prove the following explicit expression for F_2.

Theorem 1.1

$$F_2(x,\mathbf{p}) = \frac{1}{5760}\left(\frac{Q_3}{(1-\psi_1)^3} + \frac{Q_4}{(1-\psi_1)^4} + \frac{Q_5}{(1-\psi_1)^5}\right), \tag{6}$$

where $Q_3 = 5\psi_4 - 12\psi_3 + 7\psi_2$, $\quad Q_4 = 29\psi_2\psi_3 - 25\psi_2^2$, $\quad$ *and* $\quad Q_5 = 28\psi_2^3$.

The form of this theorem was arrived at through a careful examination and further analysis of the expressions for $\mu_m^{(2)}(\alpha)$ for $m \leq 3$ that appear in the Appendix of [5]. In addition, we make the following conjecture for the general form of F_g.

Conjecture 1.2 *For* $g \geq 2$,

$$F_g(x,\mathbf{p}) = \sum_{d=2g-1}^{5g-5} \frac{1}{(1-\psi_1)^d} \sum_{n=d-1}^{d+g-1} \sum_{\substack{\theta\models n \\ l(\theta)=d-2(g-1)}} K_\theta^{(g)} \psi_\theta.$$

where $K_\theta^{(g)} \in \mathbb{Q}$, *and* $\psi_\theta = \psi_{\theta_1}\psi_{\theta_2}\cdots$.

This expression for $F_g(x,\mathbf{p})$ is a sum of rational functions of the ψ_i's with particularly simple denominators and with numerators of prescribed form. For $g = 3$ we have determined the

$K_\alpha^{(g)}$explicitly, based on this form, with the aid of Maple. The resulting expression for $F_3(x, \mathbf{p})$ is displayed in Appendix A.

For $l(\theta) = 1$ and for all g, $K_\theta^{(g)}$ may be obtained quite readily as follows. From [7],

$$\frac{n\mu_1^{(g)}(n)}{(n+2g-1)!} = a_n \frac{n^{2g-2}}{2^{2g}}[x^{2g}]\left(\frac{\sinh(x)}{x}\right)^{n-1},$$

where $[x^{2g}]$ denotes the operator giving the coefficient of x^{2g} in a formal power series. But, as a polynomial in n,

$$[x^{2g}]\left(\frac{\sinh(x)}{x}\right)^{n-1} = \frac{1}{6^g g!}n^g + \cdots + \frac{2(1-2^{2g-1})B_{2g}}{(2g)!}n^0$$

where B_{2g} is a Bernoulli number. Thus, from the conjectured form, we have

$$F_g = \frac{\frac{1}{24^g g!}\psi_{3g-2} + \cdots + \frac{(1-2^{2g-1})B_{2g}}{2^{2g-1}(2g)!}\psi_{2g-2}}{(1-\psi)^{2g-1}} + \cdots + \frac{A_g\psi_2^{3g-3}}{(1-\psi)^{5g-5}}.$$

where A_g is a rational number.

2 Proof of the main result

We use the approach that has been developed in [3] and extended in [2] that makes use of Hurwitz's encoding [6] of the problem as a transitive ordered factorization of a permutation with prescribed cycle type into transpositions. It has already been shown [5] by a cut-and-join analysis of the action of a transposition on the cycle structure of a arbitrary permutation in $\mathfrak{S}_n$, that if

$$F = \widetilde{F}_0 + z\widetilde{F}_1 + z^2\widetilde{F}_2 + \cdots$$

where $\widetilde{F}_g$ is obtained from F_g by multiplying the summand by $u^{n+m+2g-2}$, then F satisfies the partial differential equation

$$\frac{\partial F}{\partial u} = \frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}z\frac{\partial^2 F}{\partial p_i \partial p_j} + ijp_{i+j}\frac{\partial F}{\partial p_i}\frac{\partial F}{\partial p_j} + (i+j)p_ip_j\frac{\partial F}{\partial p_{i+j}}\right). \tag{7}$$

The techniques developed in [3] and in [2] enable us to confirm whether a series satisfies the partial differential equation induced from this by considering only terms of given degree in z, thus grading by genus, but we do not yet possess a method for constructing the solution of such an equation in closed form. The next result gives the linear first order partial differential equation for F_2 that is induced by restricting (7) in this way to terms of degree exactly two in z.

Lemma 2.1 *The series $f = F_2$ satisfies the partial differential equation*

$$T_0 f - T_1 = 0 \tag{8}$$

where

$$\begin{aligned} T_0 &= x\frac{\partial}{\partial x} + \sum_{i\geq 1} p_i\frac{\partial}{\partial p_i} + 2 - \sum_{i,j\geq 1} ijp_{i+j}\frac{\partial F_0}{\partial p_i}\frac{\partial}{\partial p_j} - \frac{1}{2}\sum_{i,j\geq 1}(i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}, \\ T_1 &= \frac{1}{2}\sum_{i,j\geq 1} ijp_{i+j}\frac{\partial F_1}{\partial p_i}\frac{\partial F_1}{\partial p_j} + \frac{1}{2}\sum_{i,j\geq 1} ijp_{i+j}\frac{\partial^2 F_1}{\partial p_i \partial p_j}. \end{aligned}$$

Proof: Clearly,

$$u\frac{\partial}{\partial u}[z^2]\,F = \left(x\frac{\partial}{\partial x} + \sum_{i\geq 1} p_i \frac{\partial}{\partial p_i} + 2\right)[z^2]\,F,$$

and the result follows by applying $[z^2]$ to (7). $\square$

Lemma 2.1 gives a linear partial differential equation for F_2 with coefficients that involve the known series F_0 and F_1 (see (4) and (5)). The proof of Theorem 1.1 consists of showing that the expression for F_2 given in (6) does indeed satisfy this linear partial differential equation. To establish this, extensive use will be made of a number of results in [2] that were used in the determination of the generating series F_1, the case of the torus. These enable us to reduce $T_0F_2 - T_1$ to a polynomial in a new set of variables, and this is shown to be identically zero, with the aid of Maple to carry out the substantial amount of routine simplification. The details of this part of the proof are suppressed, but enough information is retained to permit it to be reproduced.

In particular we require the following results from [2], which are included for completeness. They follow from (2), (3), (4) and (5), with the aid of Lagrange's Implicit Function Theorem.

$$x\frac{\partial s}{\partial x} = \frac{1}{1-\psi_1}, \qquad \frac{\partial s}{\partial p_k} = \frac{1}{k}\frac{a_k s^{k+1}}{1-\psi_1}, \tag{9}$$

$$x\frac{\partial \psi_i}{\partial x} = \frac{\psi_{i+1}}{1-\psi_1}, \qquad \frac{\partial \psi_i}{\partial p_k} = k^{i-1}a_k s^k + \frac{a_k}{k}\frac{\psi_{i+1}s^k}{1-\psi_1}, \tag{10}$$

and

$$\frac{\partial F_0}{\partial p_k} = \frac{a_k}{k^3}s^k - \frac{a_k}{k^2}\sum_{r\geq 1} a_r p_r \frac{s^{k+r}}{k+r}, \tag{11}$$

$$\frac{\partial F_1}{\partial p_k} = \frac{1}{24}\frac{a_k}{k}s^k\left(\frac{1-k}{1-\psi_1} + \frac{\psi_2}{(1-\psi_1)^2}\right). \tag{12}$$

It follows immediately from (12) that

$$\frac{\partial^2 F_1}{\partial p_i \partial p_j} = \frac{1}{24}\frac{a_i s^i}{i}\frac{a_j s^j}{j}\left(\frac{i^2+j^2+ij-i-j}{(1-\psi_1)^2} + \frac{2(i+j)\psi_2 - \psi_2 + \psi_3}{(1-\psi_1)^3} + \frac{2\psi_2^2}{(1-\psi_1)^4}\right). \tag{13}$$

These account for all of the terms in the partial differential equation (8) that do not involve F_2.

Proof: **[Theorem 1.1]** Let G_2 denote the series written explicitly on the right hand side of (6). To prove Theorem 1.1 it is sufficient to show that G_2 is a solution of (8), since G_2, with constant term equal to zero, clearly satisfies the initial condition for F_2. The requisite partial derivatives of G_2 are obtained indirectly from (10) by differentiating with respect to the ψ_i's, giving

$$x\frac{\partial G_2}{\partial x} = \sum_{m\geq 1}\left(\frac{\partial G_2}{\partial \psi_m}\right)\left(x\frac{\partial \psi_m}{\partial x}\right) = \sum_{m\geq 1}\frac{\psi_{m+1}}{1-\psi_1}\frac{\partial G_2}{\partial \psi_m}, \tag{14}$$

$$\frac{\partial G_2}{\partial p_k} = \sum_{m\geq 1}\frac{a_k s^k}{k}\left(k^m + \frac{\psi_{m+1}}{1-\psi_1}\right)\frac{\partial G_2}{\partial \psi_m}. \tag{15}$$

By inspection, each summand of $T_0G_2 - T_1$ may be expressed through (11), (12), (13), (14) and (15) entirely in terms of q_i's where $q_i = s^i p_i$ for $i \geq 1$. The q_i's are algebraically independent, and the dependency on s is perfectly subsumed. Now, in terms of these q_i's, let

$$M_{k,l} = \sum_{i,j\geq 1} q_{i+j} a_i a_j i^k j^l, \tag{16}$$

so $M_{k,l} = M_{l,k}$, and $M_{k,l}$ is homogeneous of degree one in the q_i's. Let

$$N_k = \frac{1}{2}\sum_{i,j\geq 1}\left(q_i q_j a_{i+j}(i+j)^k - 2q_{i+j}a_j j^k \frac{a_i}{i}\sum_{r\geq 1} q_r \frac{a_r}{i+r}\right), \tag{17}$$

so N_k is homogeneous of degree two in the q_i's. Then, from (12) and (13)

$$T_1 = \frac{1}{5760}\sum_{i=2}^{4}\frac{S_i}{(1-\psi_1)^i}, \tag{18}$$

where

$$\begin{aligned} S_2 &= 240M_{2,0} + 125M_{1,1} - 250M_{1,0} + 5M_{0,0}, \\ S_3 &= \psi_2\,(490M_{1,0} - 130M_{0,0}) + 120\psi_3 M_{0,0}, \\ S_4 &= 245M_{0,0}, \end{aligned}$$

and, from (11), (14), (15), (16) and (17),

$$T_0G_2 = 2G_2 + \sum_{m\geq 1}\left(\frac{\psi_{m+1}}{1-\psi_1}(1+\psi_0 - M_{-2,0} - N_0) + \psi_m - M_{-2,m} - N_m\right)\frac{\partial G_2}{\partial \psi_m}. \tag{19}$$

But

$$\frac{\partial G_2}{\partial \psi_m} = \frac{1}{5760}\sum_l \frac{R_{l,m}}{(1-\psi_1)^l},$$

where $R_{l,1} = (l-1)Q_{l-1}$ for $l = 4,5,6$, $R_{3,2} = 7$, $R_{4,2} = 29\psi_3 - 50\psi_2$, $R_{5,2} = 84\psi_2^2$, $R_{3,3} = -12$, $R_{4,3} = 29\psi_2$, $R_{3,4} = 5$ and $R_{l,m}$ is zero otherwise. Substituting these into (19), and combining with (18), we obtain the following explicit expression for the left hand side of the partial differential equation (8):

$$\begin{aligned} 5760(T_0G_2 - T_1) &= \sum_{i=2}^{4}\frac{S_i}{(1-\psi_1)^i} + \sum_{i=3}^{5}\frac{2Q_i}{(1-\psi_1)^i} \\ &\quad + \sum_{i=4}^{7}\frac{1}{(1-\psi_1)^i}\sum_m \psi_{m+1}\,(1+\psi_0 - M_{-2,0} - N_0)\,R_{i-1,m} \\ &\quad + \sum_{i=3}^{6}\frac{1}{(1-\psi_1)^i}\sum_m (\psi_m - M_{-2,m} - N_m)\,R_{i,m}. \end{aligned} \tag{20}$$

Then $5760(T_0G_2 - T_1)(1-\psi_1)^7$ is a polynomial of degree 6 in the q_i's, with constant term equal to zero. Let C_i be the contribution from terms of (homogeneous) total degree i in the q_i's for $i = 1, \ldots, 6$. The explicit expressions for these are given in Appendix B.

To complete the proof of Theorem 1.1 we show that each of these contributions is identically zero. It is convenient to introduce the symmetrization operator $\varpi_{1,\ldots,i}$ on the ring $\mathcal{H}_i[q_1, q_2, \ldots]$ of homogeneous polynomials of total degree i in $q_1, q_2, \ldots$, defined by

$$\varpi_{1,\ldots,i}\left(q_{\alpha_1}\cdots q_{\alpha_i}\right) = \sum_{\pi\in\mathfrak{S}_i} x_{\pi(1)}^{\alpha_1}\cdots x_{\pi(i)}^{\alpha_i}$$

and extended linearly to the whole of the ring. Since $\varpi_{1,\ldots,i}f = 0$ implies that $f = 0$ for $f \in \mathcal{H}_i[q_1, q_2, \ldots]$, it is sufficient to show that $\varpi_{1,\ldots,i}C_i = 0$ for $i = 1, \ldots, 6$.

Now let $w \equiv w(x)$ be the unique solution of the functional equation

$$w = xe^w, \tag{21}$$

and let $w_i \equiv w(x_i)$, where x_i is an indeterminate for $i \geq 1$. Let $w_i^{(j)} = x_i(\partial/\partial x_i)^j w_i$. Since $x\partial/\partial x = w/(1-w)\partial/\partial w$, it is a straightforward matter to express $w_i^{(k)}$ as a rational function of w_i. For example,

$$w_i^{(1)} = \frac{w_i}{1-w_i}, \quad w_i^{(2)} = \frac{w_i}{(1-w_i)^3}, \quad w_i^{(3)} = \frac{w_i + 2w_i^2}{(1-w_i)^5}.$$

Moreover, $w_1, w_2, \ldots$ are algebraically independent. Now from [2]

$$\sum_{i,j,r\geq 1} \frac{a_ia_ja_rj^k}{i(i+r)} x_1^{i+j}x_2^r = w_1^{(k+2)}\left(\frac{x_2}{x_1-x_2} - \frac{w_2^{(1)}}{w_1-w_2} - w_2^{(1)}\right),$$

so

$$\begin{aligned}
\varpi_{1,2}N_k &= \frac{w_1^{(k+2)}w_2^{(1)} - w_2^{(k+2)}w_1^{(1)}}{w_1-w_2} + w_1^{(k+2)}w_2^{(1)} + w_2^{(k+2)}w_1^{(1)}, \\
\varpi_1 M_{i,j} &= w_1^{(i+2)}w_1^{(j+2)}, \\
\varpi_1\psi_k &= w_1^{(k+1)}.
\end{aligned}$$

Each of these is a polynomial in W_1 and W_2 where $W_i = 1/(1-w_i)$, so $\varpi_{1,\ldots,i}C_i$ is a polynomial in $W_1, \ldots, W_i$ alone. Then $\varpi_{1,\ldots,i}C_i$ may be obtained from the constituents $\varpi_1\psi_i$, $\varpi_1 M_{l,m}$ and $\varpi_{1,2}N_k$ by distributing the indeterminates $x_1, \ldots, x_i$ as disjoint subsets of arguments for these constituents in all possible ways. We have used Maple to carry out this routine but laborious task, and have thus established that $\varpi_{1,\ldots,i}C_i$ is identically zero as a polynomial in $W_1, \ldots, W_i$ for $i = 1, \ldots, 6$. This completes the proof. □

3 An explicit expression for the number of ramified coverings of the sphere by the double torus

In Theorem 1.1 we have determined the generating series F_2 for the ramification numbers $\mu_m^{(2)}(\alpha)$. In this section we expand this series and thus give an explicit expression for $\mu_m^{(2)}(\alpha)$. The following result is needed, in which $a_\alpha = a_{\alpha_1}a_{\alpha_2}\cdots$ and m_ν is the monomial symmetric function with exponents specified by the parts of the partition ν.

Lemma 3.1 *For $\alpha \vdash n$, $n \geq 1$,*

$$[x^n p_\alpha]\, \frac{1}{1-\psi_1} \prod_{i\geq 1} \frac{\psi_i^{j_i}}{j_i!} = \frac{a_\alpha}{\vartheta(\alpha)} n^{l(\alpha)-l(\nu)} m_\nu(\alpha),$$

where $\nu = (1^{j_1}\, 2^{j_2} \ldots)$.

Proof: Let

$$\Lambda = [x^n]\, \frac{1}{1-\psi_1} \prod_{i\geq 1} \frac{\psi_i^{j_i}}{j_i!}.$$

Then by Lagrange's Implicit Function Theorem

$$\begin{aligned}\Lambda &= [t^n] \left(\prod_{i\geq 1} \frac{\psi_i(t,\mathbf{p})^{j_i}}{j_i!} \right) e^{n\psi_0(t,\mathbf{p})} \\ &= [t^n] \sum \prod_{i,k\geq 1} \frac{\left(k^{i-1} a_k p_k t^k\right)^{j_{i,k}}}{j_{i,k}!} \prod_{k\geq 1} \frac{\left(nk^{-1} a_k p_k t^k\right)^{d_k}}{d_k!}\end{aligned}$$

where the summation is over $j_{i,k} \geq 0$, for $i, k \geq 1$, and $d_k \geq 0$ for $k \geq 1$, restricted by $\sum_{k\geq 1} j_{i,k} = j_i$, for $i \geq 1$. Thus, if $\alpha = (1^{b_1}\, 2^{b_2}\, \ldots)$, where $\alpha \vdash n$, then

$$[p_\alpha]\Lambda = \frac{a_\alpha}{\vartheta(\alpha)} \sum n^{d_1+d_2+\cdots} \prod_{k\geq 1} \frac{b_k!}{d_k!} \prod_{i\geq 1} \frac{k^{i j_{i,k}}}{j_{i,k}!},$$

where the summation is now further restricted by $\sum_{i\geq 1} j_{i,k} = b_k - d_k$, for $k \geq 1$. Then

$$[p_\alpha]\Lambda = \frac{a_\alpha}{\vartheta(\alpha)} n^{b_1+b_2+\cdots-j_1-j_2-\cdots} \left[y_1^{j_1} y_2^{j_2} \cdots\right] \prod_{k\geq 1} \left(1 + \sum_{i\geq 1} k^i y_i\right)^{b_k}$$

and the result follows. □

Applying Lemma 3.1 to the generating series F_2 given in terms of the ψ_i's in Theorem 1.1, we immediately obtain the following result which gives an explicit expression for $\mu_m^{(2)}(\alpha)$. This expression is a symmetric function in the parts of α, a linear combination of monomial symmetric functions. The explicit expression for $\mu_m^{(1)}(\alpha)$ obtained in [2] is a symmetric function expressed in terms of the elementary symmetric functions $e_k(\alpha)$, where $k \geq 1$. These forms are closely related since $e_k = m_{(1^k)}$.

Corollary 3.2

$$\begin{aligned}\mu_m^{(2)}(\alpha) &= (n+m+2)! \frac{a_\alpha}{\vartheta(\alpha)} \frac{n^m}{5760} \sum_{k\geq 0} \left(\frac{(k+1)!}{n^{k+1}} \left(5m_{(4\,1^k)} - 12m_{(3\,1^k)} + 7m_{(2\,1^k)}\right) \right. \\ &\quad \left. + \frac{(k+2)!}{n^{k+2}} \left(\frac{29}{2} m_{(3\,2\,1^k)} - 25m_{(2^2\,1^k)}\right) + \frac{(k+3)!}{n^{k+3}} 28m_{(2^3\,1^k)} \right).\end{aligned}$$

Proof: Direct from Theorem 1.1 and Lemma 3.1. □

4 A proof of the Graber-Pandharipande recurrence equation

Graber and Pandharipande [9] have conjectured the following recurrence equation for $\mu_n^{(2)}(1^n)$, corresponding to the case of simple ramification:

$$\begin{aligned}\mu_n^{(2)}(1^n) &= n^2\left(\frac{97}{136}n-\frac{20}{17}\right)\mu_n^{(1)}(1^n)+\sum_{j=1}^{n-1}\binom{2n}{2j-2}\mu_j^{(0)}(1^j)\,\mu_{n-j}^{(2)}(1^{n-j})j(n-j)\left(8n-\frac{115}{17}j\right)\\ &+\sum_{j=1}^{n-1}\binom{2n}{2j}\mu_j^{(1)}(1^j)\,\mu_{n-j}^{(1)}(1^{n-j})j(n-j)\left(\frac{11697}{34}j(n-j)-\frac{3899}{68}n^2\right).\end{aligned} \tag{22}$$

This recurrence equation can be proved by Theorem 1.1 as follows. For $g\geq 0$ let f_g be the specialization of F_g with $p_1=1$, and $p_i=0$ for $i>1$. Then

$$f_g=\sum_{n\geq 1}x^n\frac{\mu_n^{(g)}(1^n)}{(2n+2g-2)!},$$

and the recurrence equation (22) immediately gives the differential equation

$$\begin{aligned}\left(4\left(x\frac{\partial}{\partial x}\right)^2+6x\frac{\partial}{\partial x}+2\right)f_2 &= \left(\frac{97}{136}\left(x\frac{\partial}{\partial x}\right)^3-\frac{20}{17}\left(x\frac{\partial}{\partial x}\right)^2\right)f_1+8\left(\left(x\frac{\partial}{\partial x}\right)^2f_2\right)\left(x\frac{\partial}{\partial x}f_0\right)\\ &+\frac{21}{17}\left(\left(x\frac{\partial}{\partial x}\right)^2f_0\right)\left(x\frac{\partial}{\partial x}f_2\right)+\frac{3899}{17}\left(\left(x\frac{\partial}{\partial x}\right)^2f_1\right)^2\\ &-\frac{3899}{34}\left(x\frac{\partial}{\partial x}f_1\right)\left(\left(x\frac{\partial}{\partial x}\right)^3f_1\right).\end{aligned} \tag{23}$$

Under the above specialization of the p_i's we have $s=w$ where w is the unique solution of the functional equation (21) and $\psi_i=w$ for all i. Thus from (4) and (5) we have

$$\begin{aligned}\left(x\frac{\partial}{\partial x}\right)^2f_0 &= w,\\ f_1 &= \frac{1}{24}\left(\log(1-w)^{-1}-w\right),\end{aligned}$$

and from Theorem 1.1

$$f_2=\frac{1}{5760}\left(\frac{4w^2}{(1-w)^4}+\frac{28w^3}{(1-w)^5}\right).$$

It is now a routine matter to evaluate each of the constituents of (23) as rational functions of w, using $xdw/dx=w/(1-w)$. Each side is a polynomial of degree 9 in $(1-w)^{-1}$ and they are readily shown to be equal with the aid of Maple.

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A Ramified coverings of the sphere by the triple torus

$$
\begin{aligned}
F_3(x,\mathbf{p}) \;=\; & \frac{1}{2^3 9!}\,\frac{35\,\psi_7 - 147\,\psi_6 + 205\,\psi_5 - 93\,\psi_4}{(1-\psi_1)^5} \\
& + \frac{1}{2^3 9!}\Big(-930\,\psi_2\,\psi_3 + 607\,{\psi_4}^2 + 1501\,{\psi_3}^2 + 2329\,\psi_2\,\psi_4 + 539\,\psi_2\,\psi_6 + 1006\,\psi_3\,\psi_5 \\
& - 3078\,\psi_3\,\psi_4 - 1938\,\psi_2\,\psi_5\Big)\Big/(1-\psi_1)^6 \\
& + \frac{1}{2^3 9!}\Big(13452\,\psi_2\,\psi_3\,\psi_4 + 2915\,{\psi_3}^3 - 16821\,\psi_2\,{\psi_3}^2 - 12984\,{\psi_2}^2\,\psi_4 \\
& + 12885\,{\psi_2}^2\,\psi_3 + 4284\,{\psi_2}^2\,\psi_5 - 1395\,{\psi_2}^3\Big)\Big/(1-\psi_1)^7 \\
& + \frac{1}{2^3 9!}\,\frac{22260\,{\psi_2}^3\,\psi_4 + 43050\,{\psi_2}^2\,{\psi_3}^2 - 55300\,{\psi_2}^3\,\psi_3 + 10710\,{\psi_2}^4}{(1-\psi_1)^8} \\
& + \frac{1}{2^3 9!}\,\frac{81060\,{\psi_2}^4\,\psi_3 - 31220\,{\psi_2}^5}{(1-\psi_1)^9} + \frac{245}{20736}\,\frac{{\psi_2}^6}{(1-\psi_1)^{10}}
\end{aligned}
$$

B The expression for $5760(T_0G_2 - T_1)(1 - \psi_1)^7$

Listed below for $i = 1, \ldots, 6$ is C_i, the contribution to $5760(T_0G_2 - T_1)(1 - \psi_1)^7$ from terms of total degree i in the a_i's. The $M_{l,m}$, N_k and ψ_i are series in the q_j's.

$$\begin{aligned}
C_1 = {} & -240\, M_{2,0} - 125\, M_{1,1} + 250\, M_{1,0} - 5\, M_{0,0} - 7\, M_{-2,2} + 3\,\psi_4 - 29\,\psi_3 + 21\,\psi_2 + 12\, M_{-2,3} \\
& - 5\, M_{-2,4} + 5\,\psi_5
\end{aligned}$$

$$\begin{aligned}
C_2 = {} & 20\,\psi_1\, M_{-2,4} + 29\,{\psi_3}^2 - 9\,\psi_1\,\psi_4 + 87\,\psi_1\,\psi_3 - 63\,\psi_1\,\psi_2 - 48\,\psi_1\, M_{-2,3} - 490\,\psi_2\, M_{1,0} \\
& + 130\,\psi_2\, M_{0,0} + 28\,\psi_1\, M_{-2,2} + 7\,\psi_3\,\psi_0 - 7\,\psi_3\, M_{-2,0} - 29\,\psi_2\, M_{-2,3} + 5\,\psi_5\,\psi_0 - 5\,\psi_5\, M_{-2,0} \\
& - 12\,\psi_4\,\psi_0 + 12\,\psi_4\, M_{-2,0} - 15\,\psi_1\,\psi_5 - 29\, M_{-2,2}\,\psi_3 + 50\, M_{-2,2}\,\psi_2 - 15\, M_{-2,1}\,\psi_4 \\
& + 36\, M_{-2,1}\,\psi_3 - 21\, M_{-2,1}\,\psi_2 + 625\,\psi_1\, M_{1,1} - 1250\,\psi_1\, M_{1,0} + 25\,\psi_1\, M_{0,0} - 5\, N_4 + 12\, N_3 \\
& - 7\, N_2 - 120\,\psi_3\, M_{0,0} + 30\,\psi_2\,\psi_3 - 79\,{\psi_2}^2 + 44\,\psi_4\,\psi_2 + 1200\,\psi_1\, M_{2,0}
\end{aligned}$$

$$\begin{aligned}
C_3 = {} & 15\,{\psi_1}^2\,\psi_5 + 36\,\psi_1\,\psi_4\,\psi_0 - 36\,\psi_1\,\psi_4\, M_{-2,0} - 15\,\psi_1\,\psi_5\,\psi_0 + 15\,\psi_1\,\psi_5\, M_{-2,0} \\
& + 45\,\psi_1\, M_{-2,1}\,\psi_4 - 108\,\psi_1\, M_{-2,1}\,\psi_3 + 63\,\psi_1\, M_{-2,1}\,\psi_2 + 87\,\psi_1\, M_{-2,2}\,\psi_3 + 20\,\psi_1\, N_4 \\
& - 42\,{\psi_1}^2\, M_{-2,2} + 9\,{\psi_1}^2\,\psi_4 - 87\,{\psi_1}^2\,\psi_3 + 63\,{\psi_1}^2\,\psi_2 + 72\,{\psi_1}^2\, M_{-2,3} - 30\,{\psi_1}^2\, M_{-2,4} \\
& - 29\, N_2\,\psi_3 + 50\, N_2\,\psi_2 - 15\, N_1\,\psi_4 + 36\, N_1\,\psi_3 - 21\, N_1\,\psi_2 + 1960\,\psi_1\,\psi_2\, M_{1,0} - 48\,\psi_1\, N_3 \\
& + 28\,\psi_1\, N_2 - 520\,\psi_1\,\psi_2\, M_{0,0} + 480\,\psi_1\,\psi_3\, M_{0,0} - 88\,\psi_1\,\psi_4\,\psi_2 - 29\,{\psi_3}^2\, M_{-2,0} - 58\,\psi_1\,{\psi_3}^2 \\
& - 116\, M_{-2,1}\,\psi_2\,\psi_3 + 86\,\psi_2\,\psi_3\, M_{-2,0} - 60\,\psi_1\,\psi_2\,\psi_3 + 44\,\psi_2\,\psi_4\,\psi_0 - 86\,\psi_2\,\psi_3\,\psi_0 \\
& - 44\,\psi_2\,\psi_4\, M_{-2,0} + 21\,\psi_0\,{\psi_2}^2 - 21\, M_{-2,0}\,{\psi_2}^2 + 158\,\psi_1\,{\psi_2}^2 + 100\, M_{-2,1}\,{\psi_2}^2 - 84\,{\psi_2}^2\, M_{-2,2} \\
& + 29\,{\psi_3}^2\,\psi_0 - 1250\,{\psi_1}^2\, M_{1,1} + 2500\,{\psi_1}^2\, M_{1,0} - 50\,{\psi_1}^2\, M_{0,0} - 2400\,{\psi_1}^2\, M_{2,0} \\
& - 21\,\psi_1\,\psi_3\,\psi_0 - 150\,\psi_1\, M_{-2,2}\,\psi_2 + 21\,\psi_1\,\psi_3\, M_{-2,0} + 87\,\psi_1\,\psi_2\, M_{-2,3} + 40\,{\psi_2}^3 - 29\, N_3\,\psi_2 \\
& - 245\,{\psi_2}^2\, M_{0,0} - 5\,\psi_5\, N_0 - 7\,\psi_3\, N_0 + 12\,\psi_4\, N_0 + 200\,\psi_3\,{\psi_2}^2
\end{aligned}$$

$$
\begin{aligned}
C_4 \;=\;& 200\,{\psi_2}^2\,\psi_3\,\psi_0 - 200\,{\psi_2}^2\,\psi_3\,M_{-2,\,0} - 100\,\psi_0\,{\psi_2}^3 - 3\,{\psi_1}^3\,\psi_4 + 29\,{\psi_1}^3\,\psi_3 - 21\,{\psi_1}^3\,\psi_2 \\
& - 48\,{\psi_1}^3\,M_{-2,\,3} + 20\,{\psi_1}^3\,M_{-2,\,4} - 140\,{\psi_2}^3\,M_{-2,\,1} + 100\,M_{-2,\,0}\,{\psi_2}^3 + 44\,{\psi_1}^2\,\psi_4\,\psi_2 \\
& - 2940\,{\psi_1}^2\,\psi_2\,M_{1,\,0} + 780\,{\psi_1}^2\,\psi_2\,M_{0,\,0} - 720\,{\psi_1}^2\,\psi_3\,M_{0,\,0} + 28\,{\psi_1}^3\,M_{-2,\,2} \\
& + 172\,\psi_1\,\psi_2\,\psi_3\,\psi_0 + 88\,\psi_1\,\psi_2\,\psi_4\,M_{-2,\,0} - 172\,\psi_1\,\psi_2\,\psi_3\,M_{-2,\,0} + 232\,\psi_1\,M_{-2,\,1}\,\psi_2\,\psi_3 \\
& - 200\,\psi_1\,M_{-2,\,1}\,{\psi_2}^2 + 29\,{\psi_1}^2\,{\psi_3}^2 + 72\,{\psi_1}^2\,N_3 - 42\,{\psi_1}^2\,N_2 - 30\,{\psi_1}^2\,N_4 - 88\,\psi_1\,\psi_2\,\psi_4\,\psi_0 \\
& - 58\,\psi_1\,{\psi_3}^2\,\psi_0 + 58\,\psi_1\,{\psi_3}^2\,M_{-2,\,0} + 30\,{\psi_1}^2\,\psi_2\,\psi_3 - 42\,\psi_1\,\psi_0\,{\psi_2}^2 + 42\,\psi_1\,M_{-2,\,0}\,{\psi_2}^2 \\
& + 86\,\psi_3\,N_0\,\psi_2 - 79\,{\psi_1}^2\,{\psi_2}^2 + 100\,N_1\,{\psi_2}^2 - 29\,{\psi_3}^2\,N_0 - 21\,{\psi_2}^2\,N_0 - 40\,\psi_1\,{\psi_2}^3 \\
& - 116\,N_1\,\psi_2\,\psi_3 - 108\,\psi_1\,N_1\,\psi_3 + 63\,\psi_1\,N_1\,\psi_2 + 735\,\psi_1\,{\psi_2}^2\,M_{0,\,0} - 36\,\psi_1\,\psi_4\,N_0 \\
& + 21\,\psi_1\,\psi_3\,N_0 + 87\,\psi_1\,N_3\,\psi_2 + 15\,\psi_1\,\psi_5\,N_0 - 45\,{\psi_1}^2\,M_{-2,\,1}\,\psi_4 + 87\,\psi_1\,N_2\,\psi_3 \\
& - 150\,\psi_1\,N_2\,\psi_2 + 45\,\psi_1\,N_1\,\psi_4 - 200\,\psi_1\,\psi_3\,{\psi_2}^2 - 21\,{\psi_1}^2\,\psi_3\,M_{-2,\,0} - 87\,{\psi_1}^2\,\psi_2\,M_{-2,\,3} \\
& + 15\,{\psi_1}^2\,\psi_5\,\psi_0 - 15\,{\psi_1}^2\,\psi_5\,M_{-2,\,0} - 36\,{\psi_1}^2\,\psi_4\,\psi_0 + 36\,{\psi_1}^2\,\psi_4\,M_{-2,\,0} + 108\,{\psi_1}^2\,M_{-2,\,1}\,\psi_3 \\
& - 87\,{\psi_1}^2\,M_{-2,\,2}\,\psi_3 + 150\,{\psi_1}^2\,M_{-2,\,2}\,\psi_2 + 21\,{\psi_1}^2\,\psi_3\,\psi_0 - 63\,{\psi_1}^2\,M_{-2,\,1}\,\psi_2 - 5\,{\psi_1}^3\,\psi_5 \\
& + 168\,\psi_1\,{\psi_2}^2\,M_{-2,\,2} + 2400\,{\psi_1}^3\,M_{2,\,0} + 1250\,{\psi_1}^3\,M_{1,\,1} - 2500\,{\psi_1}^3\,M_{1,\,0} + 50\,{\psi_1}^3\,M_{0,\,0} \\
& - 84\,N_2\,{\psi_2}^2 - 44\,\psi_4\,N_0\,\psi_2 + 140\,{\psi_2}^4
\end{aligned}
$$

$$
\begin{aligned}
C_5 \;=\;& 100\,{\psi_2}^3\,N_0 + 200\,\psi_1\,{\psi_2}^2\,\psi_3\,M_{-2,\,0} + 1960\,{\psi_1}^3\,\psi_2\,M_{1,\,0} + 140\,\psi_1\,{\psi_2}^3\,M_{-2,\,1} \\
& - 200\,\psi_1\,{\psi_2}^2\,\psi_3\,\psi_0 + 140\,{\psi_2}^4\,\psi_0 - 140\,{\psi_2}^4\,M_{-2,\,0} - 48\,{\psi_1}^3\,N_3 + 28\,{\psi_1}^3\,N_2 + 20\,{\psi_1}^3\,N_4 \\
& - 7\,{\psi_1}^4\,M_{-2,\,2} + 12\,{\psi_1}^4\,M_{-2,\,3} - 5\,{\psi_1}^4\,M_{-2,\,4} - 12\,{\psi_1}^3\,\psi_4\,M_{-2,\,0} + 100\,\psi_1\,\psi_0\,{\psi_2}^3 \\
& - 100\,\psi_1\,M_{-2,\,0}\,{\psi_2}^3 - 520\,{\psi_1}^3\,\psi_2\,M_{0,\,0} + 480\,{\psi_1}^3\,\psi_3\,M_{0,\,0} - 36\,{\psi_1}^3\,M_{-2,\,1}\,\psi_3 \\
& + 21\,{\psi_1}^3\,M_{-2,\,1}\,\psi_2 + 29\,{\psi_1}^3\,M_{-2,\,2}\,\psi_3 - 50\,{\psi_1}^3\,M_{-2,\,2}\,\psi_2 - 7\,{\psi_1}^3\,\psi_3\,\psi_0 + 7\,{\psi_1}^3\,\psi_3\,M_{-2,\,0} \\
& + 29\,{\psi_1}^3\,\psi_2\,M_{-2,\,3} - 5\,{\psi_1}^3\,\psi_5\,\psi_0 + 5\,{\psi_1}^3\,\psi_5\,M_{-2,\,0} + 12\,{\psi_1}^3\,\psi_4\,\psi_0 - 735\,{\psi_1}^2\,{\psi_2}^2\,M_{0,\,0} \\
& - 21\,{\psi_1}^2\,\psi_3\,N_0 - 87\,{\psi_1}^2\,N_3\,\psi_2 - 15\,{\psi_1}^2\,\psi_5\,N_0 + 15\,{\psi_1}^3\,M_{-2,\,1}\,\psi_4 + 36\,{\psi_1}^2\,\psi_4\,N_0 \\
& - 21\,{\psi_1}^2\,M_{-2,\,0}\,{\psi_2}^2 + 100\,{\psi_1}^2\,M_{-2,\,1}\,{\psi_2}^2 - 84\,{\psi_1}^2\,{\psi_2}^2\,M_{-2,\,2} + 29\,{\psi_1}^2\,{\psi_3}^2\,\psi_0 \\
& - 29\,{\psi_1}^2\,{\psi_3}^2\,M_{-2,\,0} + 86\,{\psi_1}^2\,\psi_2\,\psi_3\,M_{-2,\,0} - 116\,{\psi_1}^2\,M_{-2,\,1}\,\psi_2\,\psi_3 - 87\,{\psi_1}^2\,N_2\,\psi_3 \\
& + 150\,{\psi_1}^2\,N_2\,\psi_2 - 45\,{\psi_1}^2\,N_1\,\psi_4 + 108\,{\psi_1}^2\,N_1\,\psi_3 - 63\,{\psi_1}^2\,N_1\,\psi_2 - 86\,{\psi_1}^2\,\psi_2\,\psi_3\,\psi_0 \\
& - 44\,{\psi_1}^2\,\psi_2\,\psi_4\,M_{-2,\,0} - 172\,\psi_1\,\psi_3\,N_0\,\psi_2 + 88\,\psi_1\,\psi_4\,N_0\,\psi_2 + 44\,{\psi_1}^2\,\psi_2\,\psi_4\,\psi_0 \\
& + 232\,\psi_1\,N_1\,\psi_2\,\psi_3 - 200\,\psi_1\,N_1\,{\psi_2}^2 + 58\,\psi_1\,{\psi_3}^2\,N_0 + 42\,\psi_1\,{\psi_2}^2\,N_0 + 168\,\psi_1\,N_2\,{\psi_2}^2 \\
& + 21\,{\psi_1}^2\,\psi_0\,{\psi_2}^2 - 1200\,{\psi_1}^4\,M_{2,\,0} - 625\,{\psi_1}^4\,M_{1,\,1} + 1250\,{\psi_1}^4\,M_{1,\,0} - 25\,{\psi_1}^4\,M_{0,\,0} \\
& - 140\,N_1\,{\psi_2}^3 - 200\,\psi_3\,N_0\,{\psi_2}^2
\end{aligned}
$$

$$
\begin{aligned}
C_6 \;=\;& 5\,{\psi_1}^3\,\psi_5\,N_0 - 12\,{\psi_1}^3\,\psi_4\,N_0 + 7\,{\psi_1}^3\,\psi_3\,N_0 + 29\,{\psi_1}^3\,N_3\,\psi_2 - 116\,{\psi_1}^2\,N_1\,\psi_2\,\psi_3 \\
& + 86\,{\psi_1}^2\,\psi_3\,N_0\,\psi_2 - 44\,{\psi_1}^2\,\psi_4\,N_0\,\psi_2 + 240\,{\psi_1}^5\,M_{2,0} + 125\,{\psi_1}^5\,M_{1,1} - 250\,{\psi_1}^5\,M_{1,0} \\
& + 5\,{\psi_1}^5\,M_{0,0} + 100\,{\psi_1}^2\,N_1\,{\psi_2}^2 - 29\,{\psi_1}^2\,{\psi_3}^2\,N_0 - 21\,{\psi_1}^2\,{\psi_2}^2\,N_0 - 84\,{\psi_1}^2\,N_2\,{\psi_2}^2 \\
& + 29\,{\psi_1}^3\,N_2\,\psi_3 - 50\,{\psi_1}^3\,N_2\,\psi_2 + 15\,{\psi_1}^3\,N_1\,\psi_4 - 36\,{\psi_1}^3\,N_1\,\psi_3 + 21\,{\psi_1}^3\,N_1\,\psi_2 \\
& + 245\,{\psi_1}^3\,{\psi_2}^2\,M_{0,0} + 12\,{\psi_1}^4\,N_3 - 7\,{\psi_1}^4\,N_2 - 5\,{\psi_1}^4\,N_4 + 200\,\psi_1\,\psi_3\,N_0\,{\psi_2}^2 \\
& - 100\,\psi_1\,{\psi_2}^3\,N_0 + 140\,\psi_1\,N_1\,{\psi_2}^3 - 490\,{\psi_1}^4\,\psi_2\,M_{1,0} + 130\,{\psi_1}^4\,\psi_2\,M_{0,0} - 120\,{\psi_1}^4\,\psi_3\,M_{0,0} \\
& - 140\,{\psi_2}^4\,N_0
\end{aligned}
$$