INEQUIVALENT TRANSITIVE FACTORIZATIONS INTO TRANSPOSITIONS

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ABSTRACT. The question of counting minimal factorizations of permutations into transpositions that act transitively on a set has been studied extensively in the geometrical setting of ramified coverings of the sphere and in the algebraic setting of symmetric functions.

It is natural, however, from a combinatorial point of view to ask how such results are affected by counting up to equivalence of factorizations, where two factorizations are equivalent if they differ only by the interchange of adjacent factors that commute. We obtain an explicit and elegant result for the number of such factorizations of permutations with precisely two factors. The approach used is a combinatorial one that rests on two constructions.

We believe that this approach, and the combinatorial primitives that have been developed for the "cut and join" analysis, will also assist with the general case.

1. INTRODUCTION

1.1. **Minimal, transitive ordered factorizations.** Let $\mathcal{S}$ be a set with n elements, and let $\tau_1, \ldots, \tau_k$ be transpositions acting on $\mathcal{S}$. Then $f = (\tau_k, \ldots, \tau_1)$ is called an *ordered factorization* (where the context allows, we may sometimes refer to these simply as factorizations), of a permutation π in the symmetric group acting on $\mathcal{S}$, if $\pi = \tau_k \cdots \tau_1$ (with the convention that permutations are to act on the left of elements of $\mathcal{S}$). If, in addition, $\tau_1, \ldots, \tau_k$ act transitively on $\mathcal{S}$, then f is called *transitive*. If there are no such factorizations of π with fewer than k factors, then the factorization is called *minimal*. Since

$$k \geq n + l(\pi) - 2, \tag{1.1}$$

from [GJ2], where $l(\pi)$ denotes the number of cycles in the disjoint cycle representation of π, then

$$k = n + l(\pi) - 2 \tag{1.2}$$

for minimal, transitive ordered factorizations of π.

The number of minimal, transitive ordered factorizations of a permutation π into transpositions is clearly constant on the conjugacy class of π in the symmetric group. Suppose π is in the class indexed by the partition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of n with m parts, which we will denote by writing $\alpha \vdash n$ and $l(\alpha) = m$, respectively

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(and we call α the *cycle-type* of π). Then this number is given by

$$c_\alpha = (n+m-2)!\, n^{m-3} \prod_{j=1}^{m} \frac{\alpha_j^{\alpha_j}}{(\alpha_j - 1)!}, \tag{1.3}$$

from [H] and [GJ2]. Such factorizations of permutations occur in the study of the number of ramified covers of the sphere by curves of given genus, with branching above infinity, simple branching above other specified points and no other branching (see, for example, [A], [GJV] and [H]). In this case, the expression for k is a consequence of the Riemann-Hurwitz formula (1.2). These numbers, the Hurwitz numbers, are, up to a multiplicative factor, the number of minimal transitive ordered factorizations of a permutation in the class indexed by $\alpha \vdash n$, which specifies the ramification over infinity. They are studied extensively in algebraic geometry (see, for example, [ELSV] and [FP]).

1.2. **Inequivalent, minimal ordered factorizations.** Two factorizations are said to be *equivalent* if one can be transformed into the other by a sequence of exchanges of pairs of adjacent factors, provided the two commute (and we refer to this transformation as *commutation*). This defines an equivalence relation on the set of all minimal, transitive ordered factorizations, and if two factorizations f and g are equivalent, we write $f \sim g$. We adopt the convention that a pair of transpositions commutes when, together, they consist of four distinct elements. Thus, in particular, we do not allow a transposition to commute with itself. This convention simplifies the combinatorial analysis given in various parts of the paper, since special treatment is not required when two adjacent factors are identical.

In this paper we consider the problem of counting equivalence classes of minimal, transitive ordered factorizations with respect to this relation. Let $\widetilde{c}_\alpha$ be the number of inequivalent, minimal, transitive ordered factorizations of a permutation with cycle-type α, where α is a partition. For partitions with a single part, Eidswick [E] and Longyear [L] (see also [GJ1]) have proved that the $c_{(n)} = n^{n-2}$ ordered factorizations (from (1.3) with $l(\alpha) = m = 1$ and $\alpha_1 = n$) fall into

$$\widetilde{c}_{(n)} = \frac{1}{2n-1}\binom{3n-3}{n-1}$$

equivalence classes, where $n \geq 1$. Note that, since the product is a full cycle in this case, these factorizations are necessarily transitive. Consider the generating series for $\widetilde{c}_{(n)}$ given by

$$h(z) = \sum_{n \geq 1} \widetilde{c}_{(n)} z^{n-1}. \tag{1.4}$$

Then, from Lagrange's Implicit Function Theorem (see, for example, [GJ3]), $h(z)$ is the unique solution of the functional equation

$$h(z) = 1 + zh(z)^3. \tag{1.5}$$

Now, for partitions with two parts, consider the generating series for $\widetilde{c}_{(k,l)}$ given by

$$\Gamma(x,y) = \sum_{k,l \geq 1} \widetilde{c}_{(k,l)} \frac{x^k}{k} \frac{y^l}{l}, \tag{1.6}$$

where we define $\widetilde{c}_{(k,l)} = \widetilde{c}_{(l,k)}$ for $k < l$. Note that the coefficients are scaled by a factor of kl, which will be explained in the development. The main result of this paper is that this generating series can be expressed succinctly in terms of the generating series $h(z)$ given in (1.4) above.

Theorem 1.1 (Main result)**.**

$$\Gamma(x,y) = \log\left(1 + xyh(x)h(y)\frac{h(x)-h(y)}{x-y}\right).$$

There is a substantial history of investigations of ordered factorizations in the combinatorial literature (see, for example, [S]). They are also of interest in other areas; for example, the connection coefficients of the (conjugacy) class algebra of the symmetric group are a special case of enumerating ordered factorizations. The more recent investigations into transitive ordered factorizations (see, for example, [BIZ], or the description at the end of Section 1.1), have revealed that they have a rich structure. One aspect of this structure is the important role played by the series w, described as follows: For partitions with one part, from (1.3) we obtain $c_{(n)} = n^{n-2}$, which is the number of labelled trees on n vertices, and let

$$w(z) = \sum_{n\geq 1} n\, c_{(n)} \frac{z^n}{n!} = \sum_{n\geq 1} n^{n-1} \frac{z^n}{n!},$$

the exponential generating series for labelled, rooted trees on n vertices. Then, from Lagrange's Implicit Function Theorem [GJ3], $w(z)$ is the unique solution of the functional equation

$$w(z) = z\, e^{w(z)}.$$

Now, one of the unifying aspects for transitive ordered factorizations is how ubiquitously the series w appears in the generating series for such factorizations. For example, the generating series for the c_α, given in (1.3), when α has any fixed number of parts, can be expressed compactly in terms of $w(z_1), \dots, w(z_m)$, where $z_1, \dots, z_m$ are independent indeterminates, and α has m parts (see [GJ2]).

For the present investigation into inequivalent, transitive ordered factorizations, note that Theorem 1.1 expresses the generating series for the $\widetilde{c}_\alpha$, when α has two parts, compactly in terms of the series h. Moreover, h, defined in (1.4), is the ordinary generating series for the $\widetilde{c}_\alpha$ when α has one part. Thus h is an analogous series to w, and Theorem 1.1 gives some evidence that we might expect h to appear throughout various generating series for inequivalent, transitive ordered factorizations. We have been unable to determine an explicit expression for the generating series for the $\widetilde{c}_\alpha$, when α has three parts or more, but we conjecture that these series can be expressed compactly in terms of h, and that they too have a rich structure. The treatment of the $\widetilde{c}_{(n)}$ given in [GJ1] involved a special class of symmetric functions introduced by Macdonald [M], in which commutation was factored out by means of the Cartier-Foata monoid [CF]. We hope that Theorem 1.1 will shed light on the role that these algebraic methods might play in the general case.

1.3. **The cut and join operations.** The argument that we shall use to prove Theorem 1.1 rests on two combinatorial constructions obtained by a "cut-and-join" analysis of the effect on a permutation of multiplication by a transposition. The operations of *cut* and *join* are defined below. Let $\widehat{\mathcal{S}}$ be the set of all strings in

$\mathcal{S}$ in which no symbol occurs more than once. If $i_1 \dots i_r \in \widehat{\mathcal{S}}$, we denote the corresponding circular sequence (in which i_1 and i_r are adjacent) by $(i_1 \dots i_r)$. This we may also regard as a cycle on the elements of $\mathcal{S}$, so that $i_1 \mapsto i_2, i_2 \mapsto i_3, \dots, i_r \mapsto i_1$ under the action of this cycle. The string notation will be useful in the "cut-and-join" analysis of the effect of multiplying a permutation σ on $\mathcal{S}$ by a transposition $\tau = (ab)$. There are two cases, in which τ is referred to as a *cut* and a *join*, respectively, in the following result.

Proposition 1.2. *Let $\tau = (ab)$ where $a, b \in \mathcal{S}$, and let σ be a permutation on $\mathcal{S}$. For the product $\tau\sigma$, there are two cases:*

1. *If a, b are on the same cycle of σ then this cycle is $(a\alpha b\beta)$ where $\alpha, \beta \in \widehat{\mathcal{S}}$. The cycles of $\tau\sigma$ are obtained from those of σ by deleting $(a\alpha b\beta)$ and adjoining $(a\alpha)$ and $(b\beta)$. In this case, we call τ a* cut *for σ.*
2. *If a, b are on different cycles of σ then the two cycles are $(a\alpha)$ and $(b\beta)$ where $\alpha, \beta \in \widehat{\mathcal{S}}$. The cycles of $\tau\sigma$ are obtained from those of σ by deleting $(a\alpha)$ and $(b\beta)$ and adjoining $(a\alpha b\beta)$. In this case, we call τ a* join *for σ.*

The transpositions in an ordered factorization can now be identified as cuts or joins, as follows. Let $f = (\tau_k, \dots, \tau_1)$, where $\tau_1, \dots, \tau_k$ are transpositions, and let $\sigma_j = \tau_j \cdots \tau_1$ for $j = 1, \dots, k$. Then, for $1 \le i \le k$,

1. τ_i is called a *cut* of f if it is a cut for σ_{i-1},
2. τ_i is called a *join* of f if it is a join for σ_{i-1}.

Thus, although throughout we apparently apply the term "cut" or "join" to a transposition τ_i in an absolute sense, it is always to be regarded as relative to the product, σ_{i-1}, of the transpositions that precede it (to the right) in the factorization f. We call σ_j the *j-subproduct* of f.

Now, in a minimal, transitive ordered factorization $f = (\tau_k, \dots, \tau_1)$ of permutation π, we have $k = n + l(\pi) - 2$ from (1.2). From [GJ2], as j runs from 1 to k, exactly $n - 1$ of the τ_j are joins, and $l(\pi) - 1$ are cuts. But, in the main result, we must consider minimal, transitive ordered factorizations of permutations π with two cycles, so $l(\pi) = 2$, and thus such factorizations have a single cut. To avoid a possible source of confusion, we will refer to this cut as the *unique cut.*

1.4. **Organization of the paper.** The proof of the main result is developed in this paper through a combinatorial analysis of the unique cut in equivalence classes of minimal, transitive ordered factorizations of a permutation with two cycles. There are a number of stages in this development. In Section 2 we provide a combinatorial model so that inclusion-exclusion can be applied. Then, in Sections 3 and 4, we give two constructions that, respectively, allow us to evaluate the generating series that arise as summands in the inclusion-exclusion model. The first of these, in Section 3, allows us to decompose each of these summands by means of a "switching" construction, thereby reducing the analysis to a single, simpler problem. The second of these, in Section 4, allows us to solve this simpler problem by means of a "commutation" construction, following a detailed analysis of the equivalence classes containing factorizations having a particular pair of elements as the unique cut.

The combinatorial primitives developed in this paper can also be used to obtain another proof of the functional equation (1.5) for factorizations of a single cycle. It is included for completeness in Appendix A.

2. Factorizations with distinguished possible cuts

2.1. **Possible cuts.** In the remainder of the paper, we restrict our attention to minimal, transitive ordered factorizations of permutations with exactly two cycles. Much of the analysis is detailed, and a number of examples are given, so we find it convenient to specify the elements on these two cycles by using positive integers with superscripts "1" and "2", to distinguish between them. Thus let $\mathcal{S}_m^i = \{1^i, \dots, m^i\}$, for $i = 1, 2$, and let the permutations act on $\mathcal{S} = \mathcal{S}_{n_1}^1 \cup \mathcal{S}_{n_2}^2$, a set of size $n_1 + n_2$, where n_1 and n_2 are positive integers. Now consider the set of all minimal, transitive ordered factorizations of any permutation on $\mathcal{S}_{n_1}^1 \cup \mathcal{S}_{n_2}^2$ consisting of an n_1-cycle on $\mathcal{S}_{n_1}^1$ and an n_2-cycle on $\mathcal{S}_{n_2}^2$. Let $\mathcal{F}$ denote the union of the sets of all such factorizations over $n_1, n_2 \geq 1$. Note that the number of transposition factors for factorizations on $\mathcal{S}_{n_1}^1 \cup \mathcal{S}_{n_2}^2$ in $\mathcal{F}$ is $n_1 + n_2$, from (1.2).

The next result gives a convenient characterization of the unique cut of a factorization in $\mathcal{F}$. Some terminology is needed to state the result compactly. We say that a permutation on $\mathcal{S}_{n_1}^1 \cup \mathcal{S}_{n_2}^2$ is *pure* if each of its cycles has either all of its elements in $\mathcal{S}_{n_1}^1$ or all of its elements in $\mathcal{S}_{n_2}^2$; otherwise, it will be called *mixed*. Thus, for example, a transposition is pure if both elements belong to $\mathcal{S}_{n_1}^1$ or to $\mathcal{S}_{n_2}^2$, or is mixed if one element belongs to $\mathcal{S}_{n_1}^1$ and the other belongs to $\mathcal{S}_{n_2}^2$ (since in a transposition all of the other cycles have one element only, which are the fixed points in the permutation). Such a transposition that appears as a factor in a factorization is then called a *pure factor* or *mixed factor*, respectively.

Proposition 2.1. *The unique cut of a factorization $f \in \mathcal{F}$ is the left-most mixed factor of f.*

Proof. Each element f of $\mathcal{F}$ is a factorization of a pure permutation $\sigma_{n_1+n_2}$, since $\sigma_{n_1+n_2}$ has two cycles, one consisting of the n_1 elements of $\mathcal{S}_{n_1}^1$, and the other consisting of the n_2 elements of $\mathcal{S}_{n_2}^2$. Now, since f is a transitive factorization, it must contain at least one mixed factor. But, if the last (left-most) mixed factor is a join, then $\sigma_{n_1+n_2}$ must also be a mixed permutation, which is a contradiction, so the left-most mixed factor is a cut. The result follows, since there is only one cut. $\square$

Before we give an example of this characterization of the unique cut for elements of $\mathcal{F}$, we define one further term. A factor in a factorization f that can become the unique cut of f by commutation of the factors of f is called a *possible cut* of f. In particular, the unique cut is always a possible cut.

Example 2.2. *For $n_1 = 10$ and $n_2 = 6$,*

$$\begin{gathered}((3^1 4^1), (1^2 6^2), (1^1 8^1), (6^1 3^2), (2^1 4^1), (1^1 1^2), (8^1 10^1), (1^2 2^2),\\ (5^1 4^2), (8^1 9^1), (4^1 5^2), (1^1 5^2), (4^1 4^2), (5^1 3^2), (6^1 7^1), (7^1 2^2))\end{gathered}$$

is a factorization in $\mathcal{F}$. It is a factorization of $(1^1 \dots 10^1)(1^2 \dots 6^2)$, with unique cut $(6^1 3^2)$. *There are four possible cuts, namely $(6^1 3^2), (1^1 1^2), (5^1 4^2)$, and $(4^1 5^2)$. It is a straightforward matter to verify that each of these can become the unique cut,* *by commuting the appropriate factors, and applying Proposition 2.1. Note, that for* $(4^1 5^2)$ *to become the unique cut,* $(2^1 4^1)$ *has to be commuted to the left of* $(6^1 3^2)$*.*

2.2. **Distinguished possible cuts and inclusion-exclusion.** As preparation for the inclusion-exclusion argument, let $\mathcal{F}_k^{\dagger}$, $k \geq 1$, consist of elements of $\mathcal{F}$ in which a subset consisting of k of the possible cuts are distinguished (we mark them with a "†" as a superscript). We call such possible cuts *distinguished possible cuts.* Thus each element of $\mathcal{F}$ with j possible cuts appears $\binom{j}{k}$ times in $\mathcal{F}_k^{\dagger}$, for $k \leq j$, once for each of the distinguished subsets. If k of the possible cuts in a factorization $f \in \mathcal{F}$ of π are distinguished, to create an element of $\mathcal{F}_k^{\dagger}$, for some $k \geq 1$, then we say that this element is also a factorization of π. Thus, whether factors are distinguished or not will make no difference in how they are multiplied together.

Let $\widetilde{f}^{\dagger}_{k,n_1,n_2}$ be the number of equivalence classes of elements of $\mathcal{F}_k^{\dagger}$ on $\mathcal{S}^1_{n_1} \cup \mathcal{S}^2_{n_2}$, for $k \geq 1$. Then the generating series for the $\widetilde{f}^{\dagger}_{k,n_1,n_2}$ that is exponential in the indeterminates marking both n_1 and n_2, is

$$\widetilde{F}^{\dagger}_k = \sum_{n_1,n_2\geq 1} \widetilde{f}^{\dagger}_{k,n_1,n_2} \frac{x^{n_1}}{n_1!}\frac{y^{n_2}}{n_2!}. \tag{2.1}$$

We now apply inclusion-exclusion to express the generating series $\Gamma(x,y)$, defined in (1.6), in terms of this series, for $k \geq 1$.

Lemma 2.3.

$$\Gamma(x,y) = \sum_{k\geq 1} (-1)^{k-1}\, \widetilde{F}^{\dagger}_k.$$

Proof. Let the number of equivalence classes of elements of $\mathcal{F}$ on $\mathcal{S}^1_{n_1} \cup \mathcal{S}^2_{n_2}$ be denoted by $\widetilde{f}_{n_1,n_2}$. Now, we apply the Principle of Inclusion and Exclusion, using the terminology of "properties". A property that an equivalence class may possess is to have a given mixed pair as a factor among its possible cuts. Thus $\widetilde{f}^{\dagger}_{k,n_1,n_2}$ is the number of classes with "at least" k properties, for $k \geq 1$, while $\widetilde{f}_{n_1,n_2}$ is the number of classes with "at least" 0 properties. Thus, from the Principle of Inclusion and Exclusion, we have

$$\widetilde{f}_{n_1,n_2} + \sum_{k\geq 1} (-1)^k \widetilde{f}^{\dagger}_{k,n_1,n_2} = 0,$$

for $n_1, n_2 \geq 1$, since there are no equivalence classes with exactly 0 properties (every factorization in $\mathcal{F}$ has a single cut, and thus at least one possible cut). But, by relabelling,

$$\widetilde{f}_{n_1,n_2} = (n_1-1)!(n_2-1)!\, \widetilde{c}_{(n_1,n_2)},$$

since there are $(n_i - 1)!$ choices for the n_i-cycle on $\mathcal{S}^i_{n_i}$, for $i = 1, 2$. The result follows from (1.6) and (2.1). □

3. The switching construction

3.1. **The switch.** Let $\overline{\mathcal{F}}$ be the set of all minimal, transitive ordered factorizations of each permutation with a cycle on a nonempty subset of the elements of S^1_∞ and a cycle on a nonempty subset of the elements of S^2_∞.

Suppose that a factorization $f \in \overline{\mathcal{F}}$ has factors $\tau_c = (w^1x^2)$ and $\tau_{c-1} = (y^1z^2)$, for some $c \geq 2$, where (w^1x^2) is the (unique) cut of f, and (y^1z^2) is a possible cut of f. Then we say that $((w^1x^2), (y^1z^2))$ are *at the cut* of f.

Proposition 3.1. *Suppose that $(w^1x^2), (y^1z^2)$ are possible cuts of a factorization $f \in \overline{\mathcal{F}}$. Then the factors of f can be commuted to obtain a factorization $g \in \overline{\mathcal{F}}$, where $g \sim f$, so that $((w^1x^2), (y^1z^2))$ are at the cut of g.*

Proof. Since (w^1x^2) is a possible cut of f, the factors of f can be commuted to obtain an equivalent factorization $e \in \overline{\mathcal{F}}$ so that (w^1x^2) is the unique cut of e. Then in e there are factors $\tau_c = (w^1x^2)$ and $\tau_d = (y^1z^2)$, and since (y^1z^2) is also a possible cut of e, we have $d < c$.

Now we describe an algorithm that will enable us to commute factors of e, so that (w^1x^2) is still the unique cut, but (y^1z^2) is immediately to its right: At each stage we have a set of factors $\mathcal{M}$; initially $\mathcal{M} = \{\tau_d\}$. Then, for $i = d+1, \ldots, c-1$, if τ_i does not commute with some factor in $\mathcal{M}$, we update $\mathcal{M}$ by inserting τ_i into $\mathcal{M}$.

Now, we establish some properties of the final $\mathcal{M}$ that is constructed by this algorithm. For each factor $\tau_j \in \mathcal{M}$, it is easy to prove by induction that, for some $m \geq 0$, there exists a sequence $j = i_0 > i_1 > \cdots > i_m = d$, where $\tau_{i_0}, \ldots, \tau_{i_m} \in \mathcal{M}$, such that τ_{i_r} does not commute with $\tau_{i_{r+1}}$, for $r = 0, \ldots, m-1$. Thus, τ_c must commute with all elements of $\mathcal{M}$, since if τ_c does not commute with some $\tau_j \in \mathcal{M}$, then we have a sequence $\tau_c, \tau_j = \tau_{i_0}, \ldots, \tau_{i_m} = \tau_d$, of pairwise noncommuting factors that appears in left-to-right order in e; this would make it impossible to commute τ_d to the left of τ_c, which contradicts the fact that τ_d is a possible cut, by Proposition 2.1. Also, by construction in the above algorithm, if τ_i is not in $\mathcal{M}$ for some $c > i > d$, then τ_i commutes with all τ_j in $\mathcal{M}$, such that $i > j$.

These observations allow us to commute factors of e as follows: Move the elements of $\mathcal{M}$ except τ_d, maintaining their left-to-right order, immediately to the left of τ_c; move τ_d immediately to the right of τ_c. All required commutations are legitimate, from the above observations, and the resulting factorization is therefore a suitable choice of g. □

Let $\overline{\mathcal{F}}^{\dagger}_k$, $k \geq 1$, consist of elements of $\overline{\mathcal{F}}$ in which a subset consisting of k of the possible cuts are distinguished by marking them with a "†" as a superscript. Again, whether factors are distinguished or not will make no difference in how they are multiplied together.

Suppose that $(a_1^1b_1^2)^{\dagger}, \ldots, (a_k^1b_k^2)^{\dagger}$ are the distinguished possible cuts of $f \in \overline{\mathcal{F}}^{\dagger}_k$ where $k \geq 1$, and $a_1 < \cdots < a_k$ (note that possible cuts are all pairwise commuting, so $a_i \neq a_j$ and $b_i \neq b_j$, for $i \neq j$). Then we define the *index* of f as $\mathcal{I}(f) = \{a_1, \ldots, a_k\}$. For $k \geq 2$, we define $\vartheta(f)$, the *switch* of f, as follows: Commute the factors of f to obtain f', so that $((a_1^1b_1^2)^{\dagger}, (a_2^1b_2^2)^{\dagger})$ are at the cut of f' (this is always possible, from Proposition 3.1). Then $\vartheta(f)$ is the ordered factorization obtained from f' by replacing the factor $(a_1^1b_1^2)^{\dagger}$ by $(a_1^1b_2^2)^{\dagger}$, and replacing the factor $(a_2^1b_2^2)^{\dagger}$ by $(a_2^1b_1^2)^{\dagger}$ (note that this does not uniquely define $\vartheta(f)$, but all $\vartheta(f)$ constructed in this way are equivalent).

In the next result, we show that the switch of a factorization has a remarkable decomposition property. Some notation is needed to state the result. If f, g are ordered lists of transpositions, let $f \cdot g$ denote the ordered list obtained by appending the elements of f to those of g. Let $\mathrm{supp}(f)$ denote the union of the sets on which the transpositions listed in f act. We write $\{\mathcal{A}_1, \ldots, \mathcal{A}_p\} \Vdash \mathcal{A}$ to indicate that $\{\mathcal{A}_1, \ldots, \mathcal{A}_p\}$ is a (set) partition of $\mathcal{A}$.

Lemma 3.2. *Suppose that $(a_1^1 b_1^2)^\dagger, \ldots, (a_k^1 b_k^2)^\dagger$ are the distinguished possible cuts of $f \in \overline{\mathcal{F}}_k^\dagger$ where $k \geq 2$, and $a_1 < \cdots < a_k$. Then*

$$\vartheta(f) \sim \vartheta_1(f) \cdot \vartheta_2(f),$$

where $\vartheta_1(f), \vartheta_2(f)$ satisfy the following

1. *$\vartheta_1(f) \in \overline{\mathcal{F}}_j^\dagger$, $\vartheta_2(f) \in \overline{\mathcal{F}}_{k-j}^\dagger$, for some $j = 1, \ldots, k-1$,*
2. *$\{\mathrm{supp}(\vartheta_1(f)), \mathrm{supp}(\vartheta_2(f))\} \Vdash \mathrm{supp}(f)$,*
3. *$\{\mathcal{I}(\vartheta_1(f)), \mathcal{I}(\vartheta_1(f))\} \Vdash \mathcal{I}(f)$,*
4. *$(a_1^1 b_2^2)^\dagger$ is the cut of $\vartheta_1(f)$, and $(a_2^1 b_1^2)^\dagger$ is the cut of $\vartheta_2(f)$.*

Proof. Suppose that f is a factorization of permutation π_1, and thus f' (as described in the construction of $\vartheta(f)$ above) is also a factorization of π_1. Suppose that $\vartheta(f)$ is a factorization of π_2. Let the product of the factors to the left and right of $(a_1^1 b_1^2)^\dagger, (a_2^1 b_2^2)^\dagger$ in f' be γ and δ, respectively. Now, let $(a_1^1 b_1^2)^\dagger (a_2^1 b_2^2)^\dagger = (a_1^1 b_1^2)(a_2^1 b_2^2) = \alpha_1$ (we ignore the fact that these transpositions are distinguished in determining their product), and $(a_1^1 b_2^2)(a_2^1 b_1^2) = \alpha_2$, so

$$\gamma \alpha_1 \delta = \pi_1, \quad \gamma \alpha_2 \delta = \pi_2.$$

But $\alpha_2 = \alpha_3 \alpha_1$, where $\alpha_3 = (a_1^1 a_2^1)(b_1^2 b_2^2)$, and we have

$$\pi_2 = \gamma \alpha_2 \delta = \gamma \alpha_3 \alpha_1 \delta = \gamma \alpha_3 \gamma^{-1} \gamma \alpha_1 \delta = \gamma \alpha_3 \gamma^{-1} \pi_1. \tag{3.1}$$

If $\gamma(x)$ is the image of element x under γ, then, by conjugation,

$$\gamma \alpha_3 \gamma^{-1} = (\gamma(a_1^1)\ \gamma(a_2^1))\ (\gamma(b_1^2)\ \gamma(b_2^2)). \tag{3.2}$$

But all factors of γ are pure, from Proposition 2.1 (since $(a_1^1 b_1^2)^\dagger$ is the unique cut of f'), so γ is pure, and thus $\gamma(a_i^1) \in \mathcal{S}_\infty^1$ and $\gamma(b_i^2) \in \mathcal{S}_\infty^2$, for $i = 1, 2$. Now, since $f \in \overline{\mathcal{F}}$, then π_1 consists of exactly two cycles, one consisting entirely of elements of $\mathcal{S}_\infty^1$, and the other consisting entirely of elements of $\mathcal{S}_\infty^2$. Therefore, $(\gamma(a_1^1)\ \gamma(a_2^1))$ and $(\gamma(b_1^2)\ \gamma(b_2^2))$ are cuts for π_1, and it follows from Proposition 1.2, and (3.1), (3.2), that π_2 consists of exactly four cycles, say A_1, A_2, B_1, B_2, where A_1, A_2 consist entirely of elements of $\mathcal{S}_\infty^1$, and B_1, B_2 consist entirely of elements of $\mathcal{S}_\infty^2$. Let A_1 contain $\gamma(a_1^1)$, so A_2 contains $\gamma(a_2^1)$, and let B_1 contain $\gamma(b_1^2)$, so B_2 contains $\gamma(b_2^2)$.

Now, let β_1 be the set consisting of the elements of A_1 together with the elements of B_2, and β_2 consist of the elements of A_2 together with the elements of B_1. Denote the size of β_1 by m_1, and the size of β_2 by m_2. The number of factors in f (and thus also in f', $\vartheta(f)$), is $m_1 + m_2$, from (1.2), and the number of factors in a transitive ordered factorization of π_2 is at least $m_1 + m_2 + 2$, from (1.1), so $\vartheta(f)$ is not a *transitive* factorization of π_2. However, since α_2 and γ are products of factors in $\vartheta(f)$, we deduce that the factors of $\vartheta(f)$ act transitively on β_1, and act transitively on β_2. Thus there can be no factor in $\vartheta(f)$ containing both an element of β_1 and an element of β_2. This means that the factors in $\vartheta(f)$ fall into two classes, those consisting of a pair from β_1, and those consisting of a pair from β_2; let the ordered list of factors of the first kind be denoted by $\vartheta_1(f)$, and the ordered list of factors of the second kind be denoted by $\vartheta_2(f)$. Now, all factors of $\vartheta_1(f)$ commute with all factors of $\vartheta_2(f)$, so, in particular, we may write

$$\vartheta(f) \sim \vartheta_1(f) \cdot \vartheta_2(f).$$

But, $\vartheta_1(f)$ is a transitive ordered factorization of A_1B_2, so it must have at least m_1 factors, from (1.1). Similarly, $\vartheta_2(f)$ is a transitive ordered factorization of A_2B_1, and must have at least m_2 factors. Since $\vartheta(f)$ has exactly $m_1 + m_2$ factors, from above, then $\vartheta_1(f)$, $\vartheta_2(f)$ must have exactly m_1, m_2 factors, respectively, and so, from (1.2), they are minimal, transitive ordered factorizations. All of the stated properties of $\vartheta_1(f)$, $\vartheta_2(f)$ follow from the development above, noting that $\beta_i = \mathrm{supp}(\vartheta_i(f))$, for $i = 1, 2$. □

Example 3.3. *For the factorization given in Example 2.2, suppose we distinguish three of the possible cuts, to obtain the following factorization:*

$$\begin{aligned} f \;=\; &((3^14^1),(1^26^2),(1^18^1),(6^13^2)^\dagger,(2^14^1),(1^11^2)^\dagger,(8^110^1),(1^22^2),\\ &(5^14^2),(8^19^1),(4^15^2)^\dagger,(1^15^2),(4^14^2),(5^13^2),(6^17^1),(7^12^2)). \end{aligned} \tag{3.3}$$

Thus, $f \in \overline{\mathcal{F}}_3^\dagger$, and we have $\mathcal{I}(f) = \{1, 4, 6\}$. Now we can commute factors in f to obtain an equivalent f', with $((1^11^2)^\dagger, (4^15^2)^\dagger)$ at the cut:

$$\begin{aligned} f' \;=\; &((3^14^1),(1^26^2),(1^18^1),(2^14^1),(1^11^2)^\dagger,(4^15^2)^\dagger,(8^110^1),(1^22^2),\\ &(5^14^2),(8^19^1),(6^13^2)^\dagger,(1^15^2),(4^14^2),(5^13^2),(6^17^1),(7^12^2)), \end{aligned}$$

so we have

$$\begin{aligned} \vartheta(f) \;=\; &((3^14^1),(1^26^2),(1^18^1),(2^14^1),(1^15^2)^\dagger,(4^11^2)^\dagger,(8^110^1),(1^22^2),\\ &(5^14^2),(8^19^1),(6^13^2)^\dagger,(1^15^2),(4^14^2),(5^13^2),(6^17^1),(7^12^2)). \end{aligned}$$

Now, in the notation of the proof of Lemma 3.2, f is a factorization of $\pi_1 = (1^1 \cdots 10^1)(1^2 \cdots 6^2)$, and we have $\gamma\alpha_3\gamma^{-1} = (2^18^1)(5^26^2)$. Thus, we have $\pi_2 = (1^18^19^110^1)(2^1 \cdots 7^1)(1^2 \cdots 4^26^2)(5^2)$. Now, the unique cuts of $\vartheta_1(f)$ and $\vartheta_2(f)$ are $((1^15^2)^\dagger$ and $(4^11^2)^\dagger$, respectively, so $\vartheta_1(f)$ is a factorization of $(1^18^19^110^1)(5^2)$, and $\vartheta_2(f)$ is a factorization of $(2^1 \cdots 7^1)(1^2 \cdots 4^26^2)$. Thus, we have $\mathrm{supp}(\vartheta_1(f)) = \{1^1, 8^1, 9^1, 10^1, 5^2\}$, and $\mathrm{supp}(\vartheta_2(f)) = \{2^1, \dots, 7^1, 1^2, \dots, 4^2, 6^2\}$. These sets allow us to determine immediately to which factorization each factor of $\vartheta(f)$ belongs, and we obtain

$$\vartheta_1(f) \;=\; ((1^18^1),(1^15^2)^\dagger,(8^110^1),(8^19^1),(1^15^2)), \tag{3.4}$$

$$\begin{aligned} \vartheta_2(f) \;=\; &((3^14^1),(1^26^2),(2^14^1),(4^11^2)^\dagger,(1^22^2),(5^14^2),(6^13^2)^\dagger,\\ &(4^14^2),(5^13^2),(6^17^1),(7^12^2)). \end{aligned} \tag{3.5}$$

Note that $\mathcal{I}(\vartheta_1(f)) = \{1\}$ and $\mathcal{I}(\vartheta_2(f)) = \{4, 6\}$.

The switch is reversed as follows. Given two factorizations $f_1 \in \overline{\mathcal{F}}_i^\dagger$ and $f_2 \in \overline{\mathcal{F}}_j^\dagger$, for $i, j \geq 1$, where $\mathrm{supp}(f_1), \mathrm{supp}(f_2)$ are disjoint, suppose that the smallest element of $\mathcal{I}(f_i)$ is s_i, and that $s_1 < s_2$ (if $s_1 > s_2$, interchange f_1 and f_2). Then commute factors of f_i, if necessary, to obtain g_i, where $(s_i^1t_i^2)^\dagger$, for some t_i, is the unique cut of g_i, for $i = 1, 2$. Suppose that $g_i = u_i \cdot (s_i^1t_i^2)^\dagger \cdot v_i$, for $i = 1, 2$, and let $g = u_1 \cdot u_2 \cdot (s_1^1t_2^2)^\dagger(s_2^1t_1^2)^\dagger \cdot v_1 \cdot v_2$. Then we have recovered the factorization $f' = g$ in the switch, so that $\vartheta_1(f) = g_1$, $\vartheta_2(f) = g_2$; of course, all possible f from which this f' could arise in the construction are equivalent to f'.

3.2. **The Iterated Switching Algorithm .** We now apply the switch iteratively, to decompose a factorization in $\overline{\mathcal{F}}_k^{\dagger}$ into k factorizations.

Algorithm 3.4 (Iterated Switching Algorithm)**.** *The* input *is $f \in \overline{\mathcal{F}}_k^{\dagger}$, with $k \geq 2$. Suppose the index of f is $\mathcal{I}(f) = \{a_1, \ldots, a_k\}$.*

Step 1: *Set $f_1 = f$.*

Step i: *(For $i = 2, \ldots, k$) Determine the unique t_i so that $a_i \in \mathcal{I}(f_{t_i})$, where $1 \leq t_i \leq i-1$. Apply ϑ to f_{t_i}; set $f_i = \vartheta_2(f_{t_i})$ and replace f_{t_i} by $\vartheta_1(f_{t_i})$.*

The output *is $(f_1, \ldots, f_k)$ and $(t_2, \ldots, t_k)$.*

Example 3.5. *If the input is the factorization f given in Example 3.3, then $k = 3$ and $\mathcal{I}(f) = \{1, 4, 6\}$.*

Step 1: *Set $f_1 = f$, where f is given in (3.3).*

Step 2: *We have $a_2 = 4$, and $a_2 \in \mathcal{I}(f_1)$, so $t_2 = 1$. Then apply ϑ to f_1, setting $f_2 = \vartheta_2(f_1) = \vartheta_2(f)$, given in (3.5), and replacing f_1 by $\vartheta_1(f_1) = \vartheta_1(f)$, given in (3.4). For later use, we record the fact that*

$$f_1 = ((1^1 8^1), (1^1 5^2)^{\dagger}, (8^1 10^1), (8^1 9^1), (1^1 5^2)) \tag{3.6}$$

Step 3: *We have $a_3 = 6$, and $a_3 \in \mathcal{I}(f_2)$, so $t_3 = 2$. Then apply ϑ to f_2. First, commute factors of f_2 to obtain an equivalent factorization with $((4^1 1^2)^{\dagger}, (6^1 3^2)^{\dagger})$ at the cut:*

$$((3^1 4^1), (1^2 6^2), (2^1 4^1), (4^1 1^2)^{\dagger}, (6^1 3^2)^{\dagger}, (1^2 2^2), (5^1 4^2), (4^1 4^2), (5^1 3^2), (6^1 7^1), (7^1 2^2)).$$

Then, we immediately obtain

$$\begin{aligned} \vartheta(f_2) &= ((3^1 4^1), (1^2 6^2), (2^1 4^1), (4^1 3^2)^{\dagger}, (6^1 1^2)^{\dagger}, (1^2 2^2), (5^1 4^2), \\ &\quad (4^1 4^2), (5^1 3^2), (6^1 7^1), (7^1 2^2)). \end{aligned}$$

Now, in the notation of the proof of Lemma 3.2, f_2 is a factorization of $\pi_1 = (2^1 \cdots 7^1)(1^2 \cdots 4^2 6^2)$, and we have $\gamma \alpha_3 \gamma^{-1} = (2^1 6^1)(3^2 6^2)$. Thus, we have $\pi_2 = (2^1 \cdots 5^1)(6^1 7^1)(1^2 2^2 6^2)(3^2 4^2)$. Now, the unique cuts of $\vartheta_1(f_2)$ and $\vartheta_2(f_2)$ are $((4^1 3^2)^{\dagger}$ and $(6^1 1^2)^{\dagger}$, respectively, so $\vartheta_1(f)$ is a factorization of $(2^1 \cdots 5^1)(3^2 4^2)$, and $\vartheta_2(f)$ is a factorization of $(6^1 7^1)(1^2 2^2 6^2)$. Thus,

$$f_2 = ((3^1 4^1), (2^1 4^1), (4^1 3^2)^{\dagger}, (5^1 4^2), (4^1 4^2), (5^1 3^2)), \tag{3.7}$$

$$f_3 = ((1^2 6^2), (6^1 1^2)^{\dagger}, (1^2 2^2), (6^1 7^1), (7^1 2^2)). \tag{3.8}$$

The output is (f_1, f_2, f_3), displayed in (3.6), (3.7), (3.8), and $(t_2, t_3) = (1, 2)$.

The next result gives an enumeratively useful property of the Iterated Switching Algorithm.

Theorem 3.6. *For the output $(f_1, \ldots, f_k)$ and $(t_2, \ldots, t_k)$ of the Iterated Switching Algorithm, we have*

1. *For $i = 1, \ldots, k$, $f_i \in \overline{\mathcal{F}}_1^{\dagger}$ with $\mathcal{I}(f_i) = \{a_i\}$.*
2. $\{\mathrm{supp}(f_1), \ldots, \mathrm{supp}(f_k)\} \Vdash \mathrm{supp}(f)$.
3. *For $i = 2, \ldots, k$, $1 \leq t_i \leq i - 1$.*

Moreover, the Iterated Switching Algorithm is reversible (up to membership in an equivalence class).

Proof. By induction, it is easy to prove from Lemma 3.2 that, for $i = 1, \dots, k$, after Step i we have $(f_1, \dots, f_i)$, where

1. $f_j \in \overline{\mathcal{F}}^{\dagger}_{m_j}$, where $m_j \geq 1$ for $j = 1, \dots, i$, and $m_1 + \cdots + m_i = k$.
2. $\{\mathcal{I}(f_1), \dots, \mathcal{I}(f_i)\} \Vdash \{a_1, \dots, a_k\}$, where $a_j \in \mathcal{I}(f_j)$, for $j = 1, \dots, i$.
3. $\{\mathrm{supp}(f_1), \dots, \mathrm{supp}(f_i)\} \Vdash \mathrm{supp}(f)$.

This immediately establishes that $1 \leq t_{i+1} \leq i$, for $i = 1, \dots, k-1$. Knowing the t_j's, we can straightforwardly reverse each step of the algorithm (up to equivalence, as discussed following Example 3.3). The result follows. □

3.3. **Enumerative consequences of the Iterated Switching Algorithm.** Theorem 3.6 immediately gives an expression for the generating series $\Gamma(x, y)$, defined in (1.6).

Corollary 3.7.

$$\Gamma(x, y) = \log(1 + \widetilde{F}_1^{\dagger}).$$

Proof. Apply the Iterated Switching Algorithm (Algorithm 3.4) to representatives of the equivalence classes of $\mathcal{F}_k^{\dagger}$, for $k \geq 2$. The exponential generating series for the input f is $\widetilde{F}_k^{\dagger}$. We now determine the exponential generating series for the output $(f_1, \dots, f_k)$ and $(t_2, \dots, t_k)$ of the algorithm. There are two parts to this. First, the exponential generating series for $(f_1, \dots, f_k)$, where

1. $f_j \in \mathcal{F}_1^{\dagger}$, for $j = 1, \dots, k$,
2. $\{\mathrm{supp}(f_1), \dots, \mathrm{supp}(f_k)\} \Vdash \mathrm{supp}(f)$,

is $\left(\widetilde{F}_1^{\dagger}\right)^k$. But the condition on the a_j's in Theorem 3.6 for the output imposes a canonical order on the f_j's, so the exponential generating series for the output $(f_1, \dots, f_k)$ of the Iterated Switching Algorithm is

$$\frac{1}{k!}\left(\widetilde{F}_1^{\dagger}\right)^k.$$

Second, the number of choices for the output $(t_2, \dots, t_k)$, is $(k-1)!$, so multiplying this into the above generating series, we have

$$\widetilde{F}_k^{\dagger} = \frac{1}{k}\left(\widetilde{F}_1^{\dagger}\right)^k \tag{3.9}$$

since the Iterated Switching Algorithm is reversible (up to equivalence). The result follows immediately from (3.9) and Lemma 2.3. □

4. The Commutation Construction

4.1. **Factorizations with a fixed unique cut.** In Corollary 3.7, we have reduced the evaluation of the series $\Gamma(x, y)$ to the determination of $\widetilde{F}_1^{\dagger}$. The remainder of the paper is concerned with this determination, and thus we shall complete the proof of the main result.

In considering factorizations of a permutation with two cycles, it is convenient to choose these to be the canonical cycles $C^i = \left(1^i \dots n_i^i\right)$ on $\mathcal{S}_{n_i}^i$ for $i = 1, 2$. Let $\mathcal{D}_{n_1, n_2}$ denote the subset of $\mathcal{F}$ consisting of all minimal, transitive ordered factorizations of C^1C^2, in which the unique cut is $(1^1 1^2)$. From (1.2), members of $\mathcal{D}_{n_1,n_2}$ have n_1+n_2 factors. Let $\widetilde{d}_{n_1,n_2}$ be the number of equivalence classes of $\mathcal{D}_{n_1,n_2}$. Note that $\mathcal{D}_{n_1,n_2}$ is not closed under commutation, for factorizations with more than one possible cut, so equivalence classes of $\mathcal{D}_{n_1,n_2}$ are subsets of equivalence classes of $\mathcal{F}$, in general. Let

$$\widetilde{D}(x,y)=\sum_{n_1,n_2\geq 1}\widetilde{d}_{n_1,n_2}x^{n_1}y^{n_2}, \tag{4.1}$$

the ordinary generating series for the $\widetilde{d}_{n_1,n_2}$.

Proposition 4.1.

$$\widetilde{F}_1^{\dagger}=\widetilde{D}(x,y).$$

Proof. For the elements of $\mathcal{F}_1^{\dagger}$, there are $(n_i-1)!$ choices for the n_i-cycle on $\mathcal{S}^i_{n_i}$, for $i=1,2$, and $n_1\,n_2$ choices for the distinguished possible cut. Then, by relabelling, we obtain

$$\widetilde{f}^{\dagger}_{1,n_1,n_2}=n_1(n_1-1)!\,n_2(n_2-1)!\,\widetilde{d}_{n_1,n_2},$$

and the result follows from (4.1) and (2.1). ☐

This result shows that to determine the series $\widetilde{F}_1^{\dagger}$, and thus complete the proof of the main result, it is sufficient to determine the series $\widetilde{D}(x,y)$. We begin this, in the following proposition, with a number of detailed combinatorial results about the structure of elements of $\mathcal{D}_{n_1,n_2}$. One piece of notation will be convenient in stating these results. For $i=1$ or 2, a sequence $a^i_1\ldots a^i_m$ of elements in $\mathcal{S}^i_{n_i}$ is said to be *C^i-ordered* if the order of the elements of the sequence is consistent with their circular order in the cycle C^i (that is, there is a (unique) j such that $1\leq a_j<a_{j+1}<\ldots<a_m<a_1<\ldots<a_{j-1}\leq n_i$).

Proposition 4.2. *Let $f=(\tau_{n_1+n_2},\ldots,\tau_1)\in\mathcal{D}_{n_1,n_2}$ and let σ_j be the j-subproduct of f. Let τ_c denote the unique cut of f. Then:*

1. *For $i=1,2$, the elements from $\mathcal{S}^i_{n_i}$ appearing in each cycle of σ_j are C^i-ordered for $j=1,\ldots,n_1+n_2$.*
2. *No cycle of σ_j, for $j<c$, has a subsequence of the form $c^1d^2e^1f^2$ where $d^1,e^1\in\mathcal{S}^1_{n_1}$ and $d^2,f^2\in\mathcal{S}^2_{n_2}$.*
3. *For $i=1,2$, if $a^i,b^i\in\mathcal{S}^i_{n_i}$ are on the same cycle of σ_j for some j, with $1\leq j\leq n_1+n_2$, then a^i,b^i are on the same cycle of σ_m for all $m>j$.*
4. *σ_{c-1} has exactly one mixed cycle.*
5. *For $i=1,2$, $\sigma_c^{-1}\left(1^i\right)=p^i_i$ for some $1\leq p_i\leq n_i$,*
6. *Suppose $g=(\tau'_{n_1+n_2},\ldots,\tau'_1)\in\mathcal{D}_{n_1,n_2}$, where $g\sim f$. Let $\tau'_{c'}$ denote the unique cut of g, and ω_j the j-subproduct of g. For $i=1,2$, $\sigma_c^{-1}(1^i)=\omega_{c'}^{-1}(1^i)$.*

Proof. From Proposition 1.2, the effect of a sequence of subsequent joins on the elements of a cycle is to keep them together on cycles that are formed by the joins, and to maintain their relative circular order around such cycles. Moreover, f is a factorization of C^1C^2, which is a pure permutation with the two cycles C^1 and C^2, and so it is these two cycles that will ultimately be formed by the sequence of joins in f.

However, f contains one cut, so we must analyze the effect of this cut in the sequence of joins. The cut is $\tau_c=(1^1 1^2)$, so σ_{c-1} has a cycle containing 1^1 and 1^2, and this cycle is replaced, in σ_c, by two cycles, one containing 1^1, and the other 1^2, from Proposition 1.2. Thus (4) holds. But τ_j is a pure factor for all $j > c$, from Proposition 2.1, so $\sigma_k = \tau_{k+1}^{-1} \cdots \tau_c^{-1} C^1 C^2$ is a pure permutation for all $k \geq c$. In particular, σ_c is a pure permutation, so (5) holds. Also, the cycle in σ_{c-1} containing 1^1 and 1^2 must be of the form $(1^1 u_1 1^2 u_2)$, where u_i is a string consisting entirely of elements of $\mathcal{S}_{n_i}^i$, for $i = 1, 2$. This cycle is replaced by the two cycles $(1^1 u_1)$ and $(1^2 u_2)$ in σ_c, so the cut has no effect on the relative positions around cycles of the elements of $\mathcal{S}_{n_1}^1$ or $\mathcal{S}_{n_2}^2$. Thus, we have parts (1), (3) and (2).

Part (6) follows since we cannot commute a factor involving 1^i with $\left(1^1 1^2\right)$. □

4.2. **Classification of factors with respect to the fixed cut.** The following lemma enables us to classify the factors $\tau_v = (ab)$ in $f \in \mathcal{D}_{n_1,n_2}$ with respect to the unique cut $\left(1^1 1^2\right)$ by determining the subsets of $\mathcal{S}_{n_1}^1 \times \mathcal{S}_{n_1}^1, \mathcal{S}_{n_2}^2 \times \mathcal{S}_{n_2}^2$ (thus τ_v is pure) and $\mathcal{S}_{n_1}^1 \times \mathcal{S}_{n_2}^2$ (thus τ_v is mixed) to which (ab) can belong. It is necessary to introduce the following sets where p_i, for $i = 1, 2$, is given by Proposition 4.2 (5). For $i = 1, 2$, let

$$\mathcal{L}^i = \left\{ m^i : 1 < m < p_i \right\}, \quad \mathcal{G}^i = \left\{ m^i : p_i < m \leq n_i \right\}.$$

Then (for the pure pairs) let

$$\begin{aligned} \mathcal{K}_1 &= \left(\mathcal{L}^1 \times \mathcal{L}^1\right) \cup \left(\mathcal{L}^2 \times \mathcal{L}^2\right), & \mathcal{K}_2 &= \left(\mathcal{G}^1 \times \mathcal{G}^1\right) \cup \left(\mathcal{G}^2 \times \mathcal{G}^2\right), \\ \mathcal{K}_3 &= \left(\left\{p_1^1\right\} \times \mathcal{L}^1\right) \cup \left(\left\{p_2^2\right\} \times \mathcal{L}^2\right), & \mathcal{K}_4 &= \left(\left\{1^1\right\} \times \mathcal{G}^1\right) \cup \left(\left\{1^2\right\} \times \mathcal{G}^2\right), \end{aligned}$$

$$\mathcal{K}_5 = \left(\left\{1^1\right\} \times \left(\mathcal{L}^1 \cup \left\{p_1^1\right\}\right)\right) \cup \left(\left\{1^2\right\} \times \left(\mathcal{L}^2 \cup \left\{p_2^2\right\}\right)\right),$$

and (for the mixed pairs)

$$\mathcal{K}_6 = \left(\mathcal{L}^1 \cup \left\{1^1, p_1^1\right\}\right) \times \left(\mathcal{L}^2 \cup \left\{1^2, p_2^2\right\}\right).$$

Lemma 4.3. *Let* $(\tau_{n_1+n_2}, \ldots, \tau_1) \in \mathcal{D}_{n_1,n_2}$. *Then*

$$\tau_v \in \begin{cases} \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_4, & \text{if } \tau_v \text{ is on the left of the unique cut } (1^1 1^2), \\ \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_5 \cup \mathcal{K}_6, & \text{if } \tau_v \text{ is on the right of the unique cut } (1^1 1^2). \end{cases}$$

Proof. Let the unique cut be τ_c. A case analysis is required. In all cases, τ_v is a join when $v \neq c$.

For pure pairs: From Proposition 4.2 (5), for $i = 1$ and 2, there exists a p_i with $1 \leq p_i \leq n_i$, such that $\sigma_c(p_i^i) = 1^i$. Thus, in σ_c, there is a cycle C containing p_i^i and 1^i, of the form $(p_i^i 1^i \ldots)$. There are three cases.

Case 1: Suppose that $\tau_v = (a^i b^i)$ where $a^i, b^i \in \mathcal{S}_{n_i}^i \setminus \{1^i, p_i^i\}$ for $i = 1$ or 2. There are two subcases.

A: If $v > c$ then $v - 1 \geq c$. Thus, from Proposition 4.2 (3), since 1^i and p_i^i are on the cycle C of σ_c, then they are also on the same cycle C', say, of σ_{v-1}. Since τ_v is a join for σ_{v-1}, then at most one of a^i and b^i is on C'. There are two subcases of this case.

(i) If one of a^i and b^i is on C', suppose it to be a^i, without loss of generality. Then C' has the form

$$(a^i \ldots 1^i \ldots p_i^i \ldots) \text{ or } (a^i \ldots p_i^i \ldots 1^i \ldots).$$

But $\sigma_v = \tau_v \sigma_{v-1}$ so σ_v has a cycle of the form

$$(a^i \ldots 1^i \ldots p_i^i \ldots b^i \ldots) \text{ or } (a^i \ldots p_i^i \ldots 1^i \ldots b^i \ldots),$$

respectively. But, from Proposition 4.2 (1), this means that $1 < p_i < b < a$ or $1 < b < a < p_i$, respectively, so $\tau_v \in \mathcal{K}_1 \cup \mathcal{K}_2$.

(ii) If neither a^i nor b^i is on C', then, from Proposition 4.2 (3), 1^i and p_i^i are on one cycle of σ_v, and, since τ_v is a join, a^i and b^i lie on another cycle of σ_v. Then these cycles will be joined at some stage by the successive multiplication of $\tau_{v+1}, \ldots, \tau_{n_1+n_2}$, giving a cycle of the form

$$(1^i \ldots p_i^i \ldots a^i \ldots b^i \ldots) \text{ or } (1^i \ldots p_i^i \ldots b^i \ldots a^i \ldots),$$

(in which case, from Proposition 4.2 (1), $a^i, b^i \in \mathcal{K}_1$) or of the form

$$(1^i \ldots a^i \ldots b^i \ldots p_i^i \ldots) \text{ or } (1^i \ldots b^i \ldots a^i \ldots p_i^i \ldots),$$

(in which case, from Proposition 4.2 (1), $a^i, b^i \in \mathcal{K}_2$). Thus $\tau_v \in \mathcal{K}_1 \cup \mathcal{K}_2$.

B: If $v < c$, then a^i and b^i are on the same cycle of σ_v since τ_v is a join, so, from Proposition 4.2 (3), a^i and b^i are on the same cycle of σ_c. There are two subcases of this case.

(i) If a^i and b^i are on C, then C has the form

$$(p_i^i 1^i \ldots a^i \ldots b^i \ldots) \text{ or } (p_i^i 1^i \ldots b^i \ldots a^i \ldots).$$

Then from Proposition 4.2 (1) we have $\tau_v \in \mathcal{K}_1 \cup \mathcal{K}_2$.

(ii) If neither a^i nor b^i is on C, then the cycle of σ_c containing a^i and b^i and the cycle C will be joined at some stage by successive multiplication of $\tau_{c+1}, \ldots, \tau_{n_1+n_2}$, and, as in Case 1 A(ii), we conclude from Proposition 4.2 (1) that $\tau_v \in \mathcal{K}_1 \cup \mathcal{K}_2$.

Case 2: Suppose that $\tau_v = (p_i^i a^i)$ where $a^i \in \mathcal{S}_{n_i}^i \setminus \{1^i, p_i^i\}$ for $i = 1$ or 2. There are two subcases.

A: If $v > c$, then $v - 1 \geq c$. Thus, from Proposition 4.2 (3), since 1^i and p_i^i are on the cycle C of σ_c, then they are also on the same cycle C', say, of σ_{v-1}. Moreover, a^i is not on C' since τ_v is a join and therefore p_i^i, a^i are on different cycles of σ_{i-1}. Thus, C' has the form $(p_i^i \ldots 1^i \ldots)$ so, under the action of τ_v, σ_v has a cycle of the form

$$(p_i^i \ldots 1^i \ldots a^i \ldots).$$

Then from Proposition 4.2 (1), $1 < a < p_i$, so $\tau_v \in \mathcal{K}_3$.

B: If $v < c$, then a^i and p_i^i are on the same cycle of σ_v, since τ_v is a join, so from Proposition 4.2 (3), a^i and p_i^i are on the same cycle of σ_c. Thus a^i is also on C, so C has the form

$$(p_i^i 1^i \ldots a^i \ldots).$$

Then from Proposition 4.2 (1), $1 < a < p_i$, so $\tau_v \in \mathcal{K}_3$.

Case 3: Suppose that $\tau_v = (1^i a^i)$ where $a^i \in \mathcal{S}_{n_i}^i \setminus \{1^i\}$ for $i = 1$ or 2. There are two subcases.

A: If $v > c$, then $v - 1 \geq c$. Thus, from Proposition 4.2 (3), since 1^i and p_i^i are on the cycle C of σ_c, then they are also on the same cycle C', say, of σ_{v-1}. Moreover, a^i is not on C' since τ_v is a join and therefore $1^i, p_i^i$ are on different cycles of σ_{v-1}. Thus, C' has the form $(1^i \dots p_i^i \dots)$ so, under the action of τ_v, σ_v has a cycle of the form

$$(1^i \dots p_i^i \dots a^i \dots).$$

Then, by Proposition 4.2 (1), $p_i < a \le n_i$, so $\tau_v \in \mathcal{K}_4$.

B: If $v < c$, then 1^i and a^i are on the same cycle of σ_v since τ_v is a join, so 1^i and a^i are on the same cycle of σ_c, from Proposition 4.2 (3). Thus a^i is also on C. Then either $a^i = p_i^i$, or $a^i \ne p_i^i$, in which case C has the form $(p_i^i 1^i \dots a^i \dots)$, so, by Proposition 4.2 (1), $1 < a < p_i$. Combining these possibilities we have $1 < a \le p_i$, so $\tau_v \in \mathcal{K}_5$.

This completes the proof of the Lemma for the pure pairs.

For mixed pairs: Suppose that $\tau_v = (a^1 b^2)$ for $a^1 \in \mathcal{S}^1_{n_1}$ and $b^2 \in \mathcal{S}^2_{n_2}$. Then $v \le c$ from Proposition 2.1. If $v < c$, then τ_v is a join, from Proposition 2.1, so a^1 and b^2 lie on the same cycle of σ_v. Since, from Proposition 4.2 (4), there is precisely one cycle $\widehat{C}$, say, in σ_{c-1} containing elements of both $\mathcal{S}^1_{n_1}$ and $\mathcal{S}^2_{n_2}$, this cycle certainly contains a^1 and b^2. Now, $\sigma_{c-1} = (1^1 1^2)\sigma_c$, so $\sigma_{c-1}(p_1^1) = 1^2$ and $\sigma_{c-1}(p_2^2) = 1^1$, from Proposition 4.2 (5), so $\widehat{C}$ contains $a^1, b^2, p_1^1, 1^2, p_2^2, 1^1$. Moreover, because of the action of σ_{c-1}, p_1^1 and 1^2 appear consecutively on $\widehat{C}$, and so also do p_2^2 and 1^1. Combining these facts with Proposition 4.2 (2), we see that $\widehat{C}$ has the form

$$\left(p_1^1 1^2 \dots b^2 \dots p_2^2 1^1 \dots a^1 \dots\right).$$

Moreover, a^1 may assume the values 1^1 or p_1^1, and b^2 may assume the values 1^2 or p_1^2. Then, from Proposition 4.2 (1), $1 \le a \le p_1$ and $1 \le b \le p_2$, so $\tau_v \in \mathcal{K}_6$.

This completes the proof of the Lemma for the mixed pairs. □

In the next lemma, we commute factors, and prove that the selection of representatives from equivalence classes of $\mathcal{D}_{n_1,n_2}$ can be restricted to a special form.

Lemma 4.4. *Let $f \in \mathcal{D}_{n_1,n_2}$. Then there exists $f^\star \in \mathcal{D}_{n_1,n_2}$ such that $f^\star \sim f$, where for any factor τ in $f^\star$,*

$$\tau \in \begin{cases} \mathcal{K}_2 \cup \mathcal{K}_4, & \text{if } \tau \text{ is on the left of the unique cut } (1^1 1^2), \\ \mathcal{K}_1 \cup \mathcal{K}_3 \cup \mathcal{K}_5 \cup \mathcal{K}_6, & \text{if } \tau \text{ is on the right of the unique cut } (1^1 1^2). \end{cases}$$

Proof. If $\mathcal{A}$ and $\mathcal{B}$ are subsets of transpositions acting on $\mathcal{S}^1_{n_1} \cup \mathcal{S}^2_{n_2}$, we will write $\mathcal{A} \leftrightsquigarrow \mathcal{B}$ if $\alpha\beta = \beta\alpha$ for all $\alpha \in \mathcal{A}$ and all $\beta \in \mathcal{B}$.

Since $f \in \mathcal{D}_{n_1,n_2}$ then, from Lemma 4.3, the factors on the left of the unique cut $(1^1 1^2)$ in f are in $\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_4$, and the factors on the right of $(1^1 1^2)$ are in $\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_5 \cup \mathcal{K}_6$. Now

$$\mathcal{K}_2 \leftrightsquigarrow \left\{(1^1 1^2)\right\} \cup \mathcal{K}_1 \cup \mathcal{K}_3 \cup \mathcal{K}_5 \cup \mathcal{K}_6$$

since the set on which the transpositions in $\mathcal{K}_2$ act is disjoint from the set on which the transpositions in $\left\{(1^1 1^2)\right\} \cup \mathcal{K}_1 \cup \mathcal{K}_3 \cup \mathcal{K}_5 \cup \mathcal{K}_6$ act. Thus the left-most factor of f on the right of $(1^1 1^2)$ that is in $\mathcal{K}_2$ can be commuted to the position immediately to the left of $(1^1 1^2)$. Repeat this operation for each such factor of $\mathcal{K}_2$. Also,

$$\mathcal{K}_1 \cup \mathcal{K}_3 \leftrightsquigarrow \left\{(1^1 1^2)\right\} \cup \mathcal{K}_2 \cup \mathcal{K}_4,$$

by a similar argument, so the right-most factor of f on the left of $(1^1 1^2)$ that is in $\mathcal{K}_1 \cup \mathcal{K}_3$ can be commuted to the position immediately to the right of $(1^1 1^2)$. Repeat this operation for each such factor of $\mathcal{K}_1 \cup \mathcal{K}_3$.

Let $f^\star$ be the result of this construction. Then $f^\star \sim f$, and the factors of $f^\star$ to the left of $(1^1 1^2)$ are in $\mathcal{K}_2 \cup \mathcal{K}_4$, while the factors of $f^\star$ to the right of $(1^1 1^2)$ are in $\mathcal{K}_1 \cup \mathcal{K}_3 \cup \mathcal{K}_5 \cup \mathcal{K}_6$. Note that, of the $\mathcal{K}_i$'s, only $\mathcal{K}_6$ contains mixed factors, and no elements of $\mathcal{K}_6$ are moved to the left of $(1^1 1^2)$ in the construction of $f^\star$, so we conclude from Proposition 2.1 that $f^\star \in \mathcal{D}_{n_1,n_2}$, and the result follows. □

4.3. **Representatives of equivalence classes with the fixed cut.** We are now able to exploit the special form for factorizations in Lemma 4.4, to give a canonical representation for the equivalence classes of $\mathcal{D}_{n_1,n_2}$.

Lemma 4.5. *Let $f \in \mathcal{D}_{n_1,n_2}$. Then for unique p_1, p_2 with $1 \leq p_1 \leq n_1$ and $1 \leq p_2 \leq n_2$,*

$$f \sim l_1 \cdot l_2 \cdot ((1^1 1^2)) \cdot r,$$

where

1. *For $1 = 1,2$, l_i is a minimal, transitive ordered factorization of $(1^i (p_i + 1)^i \dots n_i^i)$,*
2. *r is a minimal, transitive ordered factorization of $(1^1 \dots p_1^1 1^2 \dots p_2^2)$.*

Proof. Let $f \in \mathcal{D}_{n_1,n_2}$, and let $f^\star$ be the factorization of $C^1 C^2$ obtained by means of the algorithm in Lemma 4.4. Let $f^\star = l \cdot ((1^1 1^2)) \cdot r$, and suppose that l and r are ordered factorizations of λ and ρ, respectively. Then $C^1 C^2 = \lambda (1^1 1^2) \rho$, where λ is a product of transpositions in $\mathcal{K}_2 \cup \mathcal{K}_4$ and ρ is a product of transpositions in $\mathcal{K}_1 \cup \mathcal{K}_3 \cup \mathcal{K}_5 \cup \mathcal{K}_6$.

Now the transpositions in $\mathcal{K}_2 \cup \mathcal{K}_4$ act on the set $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \{1^1, 1^2\}$, so the elements of the complement of this set with respect to $\mathcal{S}^1_{n_1} \cup \mathcal{S}^2_{n_2}$ are fixed points of λ. Thus $\lambda(k^1) = k^1$ for $1 < k \leq p_1$, and $\lambda(k^2) = k^2$ for $1 < k \leq p_2$.

Similarly, the transpositions in $\mathcal{K}_1 \cup \mathcal{K}_3 \cup \mathcal{K}_5 \cup \mathcal{K}_6$ act on the set $\mathcal{L}^1 \cup \mathcal{L}^2 \cup \{1^1, 1^2, p_1^1, p_2^2\}$, so the elements of the complement of this set with respect to $\mathcal{S}^1_{n_1} \cup \mathcal{S}^2_{n_2}$ are fixed points of ρ. Thus $\rho(k^1) = k^1$ for $p_1 < k \leq n_1$, and $\rho(k^2) = k^2$ for $p_2 < k \leq n_2$.

Also, if the unique cut of $f = (\tau_{n_1+n_2}, \dots, \tau_1)$ is τ_c, then from Proposition 4.2 (5), there exist unique p_1 and p_2, where $1 \leq p_1 \leq n_1$, $1 \leq p_2 \leq n_2$, such that $\sigma_c(p_i^i) = 1^i$, for $i = 1, 2$. Thus, from Proposition 4.2 (6), we have $(1^1 1^2)\rho(p_i^i) = 1^i$, for $i = 1, 2$, so $\rho(p_1^1) = 1^2$ and $\rho(p_2^2) = 1^1$.

But $\lambda = C^1 C^2 \rho^{-1} (1^1 1^2)$, so for $p_1 < k \leq n_1$, then $\lambda(k^1) = C^1 C^2 \rho^{-1} (1^1 1^2)(k^1) = C^1(k^1)$ and, similarly $\lambda(k^2) = C^2(k^2)$ for $p_2 < k \leq n_2$. In addition, $\lambda(1^1) = C^1 C^2 \rho^{-1} (1^1 1^2)(1^1) = (p_1 + 1)^1$ and, similarly, $\lambda(1^2) = (p_2 + 1)^2$. Thus $(1^i (p_i + 1)^i \dots n_i^i)$, for $i = 1, 2$, are cycles of λ, and the remaining cycles are fixed points, so

$$\lambda = \left(1^1 (p_1 + 1)^1 \dots n_1^1\right) \left(1^2 (p_2 + 1)^2 \dots n_2^2\right),$$

and, finally,

$$\rho = (1^1 1^2) \lambda^{-1} C^1 C^2 = (1^1 \dots p_1^1 1^2 \dots p_2^2).$$

Now we return to the factorizations l and r. Let m_1 and m_2 denote the number of factors in l and r, respectively. The factorization r must act transitively on the elements of the (p_1+p_2)-cycle in ρ, so from (1.1), we have $m_2 \geq p_1+p_2-1$. Similarly, the factorization l must act transitively on the elements of the (n_1-p_1+1)-cycle in λ, and on the elements of the (n_2-p_2+1)-cycle in λ, so $m_1 \geq (n_1-p_1)+(n_2-p_2)$. Combining these inequalities, we obtain

$$m_1 + m_2 \geq (p_1 + p_2 - 1) + (n_1 - p_1 + n_2 - p_2) = n_1 + n_2 - 1.$$

But $m_1 + m_2 = n_1 + n_2 - 1$, since f (and $f^\star$) has $n_1 + n_2$ factors, so each of the above inequalities is an equality. Thus, we conclude from (1.2) that r is a minimal, transitive ordered factorization of $(1^1 \dots p_1^1 1^2 \dots p_2^2)$. In addition, we conclude that l consists of two disjoint sets of factors, one giving a minimal, transitive ordered factorization l_1 of $(1^1(p_1+1)^1 \dots n_1^1)$, and the second giving a minimal, transitive ordered factorization l_2 of $(1^2(p_2+1)^2 \dots n_2^2)$. Moreover, since the factors in l_i act only on $\mathcal{S}_{n_i}^i$, for $i = 1, 2$, each factor of l_1 commutes with each factor of l_2, so we have $l \sim l_1 \cdot l_2$, and the result follows from Lemma 4.4. □

4.4. **An enumerative theorem for factorizations with a fixed cut.** In the following result, we use Lemma 4.5 to determine $\widetilde{D}(x,y)$, and thus complete the proof of the main result.

Theorem 4.6.

$$\widetilde{D}(x,y) = xy\, h(x)h(y)\frac{h(x)-h(y)}{x-y}.$$

Proof. The ordinary generating series for equivalence classes of $\widetilde{\mathcal{D}}_{n_1,n_2}$, for $n_1, n_2 \geq 1$, is $\widetilde{D}(x,y)$, from (4.1). In Lemma 4.5, the ordinary generating series for equivalence classes of l_1 is $h(x)$ and of l_2 is $h(y)$, from (1.4). The ordinary generating series for equivalence classes of r is

$$\sum_{p_1,p_2\geq 1} \widetilde{c}_{(p_1+p_2)}\, x^{p_1}y^{p_2} = xy\frac{h(x)-h(y)}{x-y},$$

and the result follows. □

The proof of the main result: The main result (Theorem 1.1) follows immediately, by combining Corollary 3.7, Proposition 4.1 and Theorem 4.6.

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Appendix A. The case of a single cycle

We conclude with a synoptic treatment of inequivalent, minimal ordered factorizations of a full cycle, which have been enumerated elsewhere in [E], [L] and [GJ1]. The functional equation for the generating series for these factorizations has been stated in (1.5), and is due to Longyear [L]. It can also be derived from a canonical representation of equivalence classes of these factorizations into three other factorizations of a full cycle, and we state such a representation without proof in this Appendix (as Lemma A.4), for completeness. The development is very similar to that given in Section 4 of this paper, and the canonical representation is comparable in purpose to Lemma 4.5. We simply state the results here, since the interested reader will be able to supply the details.

The permutations act on $\{1.\dots,n\}$, where n is a positive integer, and we consider factorizations of the canonical n-cycle $C_n=(1\cdots n)$. Let $\mathcal{B}_n$ be the set of minimal, transitive ordered factorizations of C_n. Then the elements of $\mathcal{B}_n$ have $n-1$ factors, with no cuts, from the discussion at the end of Section 1.3. The first result identifies two parameters, denoted by q and p, whose values are fixed for the equivalence classes of $\mathcal{B}_n$.

Proposition A.1. *Let $f=(\tau_{n-1},\dots,\tau_1)\in\mathcal{B}_n$ and let σ_j be the j-subproduct of f. Suppose $g=\left(\tau'_{n-1},\dots,\tau'_1\right)\in\mathcal{B}_n$, where $g\sim f$, and ω_j denotes the j-subproduct of g. If τ_c is the right-most factor of f that moves 1 and $\tau'_{c'}$ is the right-most factor of g that moves 1, then, for $n\geq 2$,*

1. *$\tau_c=\tau'_{c'}=(1q)$, for some $1<q\leq n$,*
2. *$\sigma_c^{-1}(1)=\omega_{c'}^{-1}(1)=p$, for some $q\leq p\leq n$.*

The next result classifies the factors of $f\in\mathcal{B}_n$ with respect to the canonical factor $(1q)$, given in Part 1 of the above result, using the parameter p, given in Part 2 of the above result. For this purpose, let $\mathcal{X} = \{x \colon 1 < x < q\}$, $\mathcal{Y} = \{x \colon q < x < p\}$, and $\mathcal{Z} = \{x \colon p < x \leq n\}$. Let

$$\begin{array}{lll} \mathcal{U}_1 = \mathcal{X} \times \mathcal{X}, & \mathcal{U}_2 = \mathcal{Y} \times \mathcal{Y}, & \mathcal{U}_3 = \mathcal{Z} \times \mathcal{Z}, \\ \mathcal{U}_4 = \{1\} \times \mathcal{Y}, & \mathcal{U}_5 = \{q\} \times \mathcal{X}, & \\ \mathcal{U}_6 = \{q\} \times (\mathcal{Y} \cup \{p\}), & \mathcal{U}_7 = \{p\} \times \mathcal{Y}. & \end{array}$$

Lemma A.2. *Let $f = (\tau_{n-1}, \dots, \tau_1) \in \mathcal{B}_n$. Then, for $n \geq 2$,*

$$\tau_v = \begin{cases} \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4 \cup \mathcal{U}_5 \cup \mathcal{U}_7, & \text{if } \tau_v \text{ is on the left of } (1q), \\ \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_6 \cup \mathcal{U}_7, & \text{if } \tau_v \text{ is on the right of } (1q). \end{cases}$$

Lemma A.3. *Let $f = (\tau_{n-1}, \dots, \tau_1) \in \mathcal{B}_n$. Then there exists $f^\star \in \mathcal{B}_n$ such that $f^\star \sim f$, where for any factor τ in $f^\star$, for $n \geq 2$,*

$$\tau_v = \begin{cases} \mathcal{U}_1 \cup \mathcal{U}_3 \cup \mathcal{U}_4 \cup \mathcal{U}_5, & \text{if } \tau_v \text{ is on the left of } (1q), \\ \mathcal{U}_2 \cup \mathcal{U}_6 \cup \mathcal{U}_7, & \text{if } \tau_v \text{ is on the right of } (1q). \end{cases}$$

With the aid of these, we may now obtain a canonical representation of the equivalence classes of $\mathcal{B}_n$. The decomposition is the analogue of Lemma 4.5 for $\mathcal{B}_n$.

Lemma A.4. *Let $f \in \mathcal{B}_n$. Then for unique p, q with $1 < q \leq p \leq n$, $n \geq 2$,*

$$f \sim l_1 \cdot l_2 \cdot (1q) \cdot r,$$

where

1. *l_1 is a minimal, transitive ordered factorization of $(1\,(p+1)\dots n)$,*
2. *l_2 is a minimal, transitive ordered factorization of $(2\,3\dots q)$,*
3. *r is a minimal, transitive ordered factorization of $(q\,(q+1)\dots p)$.*

This result leads immediately to the functional equation for $h(z)$ that is given in (1.5).

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