A Rooted Map Invariant, Non-Orientability and Jack Symmetric Functions

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Abstract

The existence of a non-negative integer-valued invariant, called the Map-Jack invariant, for rooted maps has been conjectured by Goulden and Jackson [4]. It has the defining property that, if it is marked by an indeterminate b, then the generating series for rooted maps in locally orientable surfaces is a particular series, involving the Jack symmetric function $J_\lambda^{(b+1)}$, that specialises to the generating series for rooted maps in orientable surfaces for $b = 0$ (and clearly to all surfaces for $b = 1$). We propose a candidate Map-Jack invariant η that is determined by depth first search and by the local topological effect of deleting edges from maps, and give results that support its candidacy. In particular, we prove its correctness up to face partition. We also show how the algorithm for determining η may be associated with the Jack symmetric functions. This is achieved by means of the Laplace-Beltrami operator.

1 Introduction and a Conjecture

The invariant of rooted maps considered in this paper is derived from a particular presentation of the generating series for rooted maps in terms of Jack symmetric functions. We therefore begin with a brief review of the salient aspects of both maps and Jack symmetric functions.

1.1 Background to Map-Jack Invariants

A *(topological) map* $\mathfrak{m}$ is an embedding of a graph $G(\mathfrak{m})$, loops and multiple edges allowed, with vertices sent to points and edges sent to continuous paths (joining the images of the vertices to which the edge is incident), into a closed surface $\mathbb{S}(\mathfrak{m})$ without boundary (compact 2-manifold), such that the remainder of the surface has components each of which is homeomorphic to an open disk, called a *face*. The *degree* of a vertex or a face is the number of edges with which it is incident, counting with respect to multiplicities. A *partition* λ is a non-increasing sequence of non-negative integers $(\lambda_i)_{i\geqslant 1}$ with a finite number of terms that are non-zero. The *face partition* $\phi(\mathfrak{m})$ and *vertex partition* $\nu(\mathfrak{m})$ of a map $\mathfrak{m}$ are the partitions obtained from the degrees of the faces and vertices, respectively, of $\mathfrak{m}$. The number of edges of $\mathfrak{m}$ is denoted by $n(\mathfrak{m})$. A map is *orientable* if its surface is orientable, and is *non-orientable* if its surface is non-orientable. We use Tutte's convention [15] that *locally orientable* means either orientable or non-orientable. A map is *rooted* by selecting a side and an end of an edge. The sets of orientable and locally orientable rooted maps are denoted by $\mathfrak{O}$ and $\mathfrak{L}$, respectively. The subsets of $\mathfrak{O}$ and $\mathfrak{L}$ of maps $\mathfrak{m}$ with face partition ϕ and vertex partition ν are denoted by $\mathfrak{O}_{\phi,\nu}$ and $\mathfrak{L}_{\phi,\nu}$, and their cardinalites by $o_{\phi,\nu}$ and $l_{\phi,\nu}$. It will also be convenient to work with a purely combinatorial presentation of a map. We shall call the combinatorial presentation a *combinatorial map*, and this is defined in Section 2.

It is occasionally convenient to write λ as $(\cdots 2^{m_2(\lambda)}1^{m_1(\lambda)})$, where $m_j(\lambda)$ is the *multiplicity* of j in λ. For λ, its *length* $\ell(\lambda)$ is the number of positive parts it has and its *weight* $|\lambda|$ is the sum of its parts. If the weight of λ is n we say that λ is a partition of n, and write $\lambda \vdash n$. Lexicographic order is denoted by $\leqslant$ (so that $(1^2) \leqslant (2)$). The *monomial symmetric function* m_λ at $\mathbf{x} = (x_1, x_2, \dots)$ is $m_\lambda(\mathbf{x}) = \sum_\pi \prod_{i \geqslant 1} x_i^{\pi_i}$ where the sum is over all distinct permutations π of the partition λ and the *power sum symmetric function* $p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots$ is determined by $p_j(\mathbf{x}) = \sum_{i \geqslant 1} x_i^j$. Monomial and power sum symmetric functions each provide a basis for the ring Λ of symmetric functions. Thus the power sums p_j are algebraically independent, so any function from $\{p_j\}$ to a ring extends to a homomorphism from Λ to the ring. An inner product $\langle \cdot, \cdot \rangle_\alpha$ is defined on Λ by $\langle p_\lambda, p_\lambda \rangle_\alpha = \alpha^{\ell(\lambda)} \prod_i i^{m_i(\lambda)} m_i(\lambda)!$ and $\langle p_\lambda, p_\mu \rangle_\alpha = 0$ if $\lambda \neq \mu$. *Jack symmetric functions* $J_\lambda^{(\alpha)}$ are characterised by the following three properties:

1. $\left\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} \right\rangle_\alpha = 0$ if $\lambda \neq \mu$,
2. $[m_\lambda]\, J_\mu^{(\alpha)} = 0$ if $\lambda \not\leqslant \mu$, and
3. $\left[m_{(1^n)}\right] J_\mu^{(\alpha)} = n!$ if $\mu \vdash n$ (when a basis containing y is clear from context, the notation $[y]\, f$ indicates the coefficient of y in f).

Jack symmetric functions of degree 4 or less are given in Appendix A.

1.2 A Conjectured Map-Jack Invariant η and the Main Result

Jackson and Goulden [4] introduced the *Map-Jack coefficient* $\psi_{\phi,\nu}^{(\alpha)} \in \mathbb{Q}(\alpha)$, which is defined by $\Psi^{(\alpha)} = \sum_{n \geqslant 0} \sum_{\phi,\nu \vdash 2n} \psi_{\phi,\nu}^{(\alpha)} p_\phi(\mathbf{x}) p_\nu(\mathbf{y}) z^n$ where

$$\Psi^{(\alpha)} = 2\alpha z \frac{\partial}{\partial z} \log \sum_{n \geqslant 0} z^n \left[p_{(2^n)}(\mathbf{z})\right] \sum_\lambda \frac{J_\lambda^{(\alpha)}(\mathbf{x}) J_\lambda^{(\alpha)}(\mathbf{y}) J_\lambda^{(\alpha)}(\mathbf{z})}{\left\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)} \right\rangle_\alpha} . \tag{1.1}$$

They showed that $o_{\phi,\nu} = \psi_{\phi,\nu}^{(1)}$ and $l_{\phi,\nu} = \psi_{\phi,\nu}^{(2)}$ and then asked whether α has combinatorial significance in the generic case. Positive evidence that it may is provided by computing the explicit expressions for $\psi_{\phi,\nu}^{(b+1)}$ for ϕ, ν of small weight (see Appendix B or [2, 4]) directly from $\Psi^{(b+1)}$, for these are seen to be polynomials in b with non-negative integer coefficients. These observations prompted the following.

Conjecture 1.1 (Map-Jack Conjecture [4]). *There exists an invariant ϑ of rooted locally orientable maps such that (i) $\vartheta(\mathfrak{m}) \in \mathbb{N}$, (ii) $\vartheta(\mathfrak{m}) = 0$ if and only if $\mathfrak{m}$ is orientable, (iii) $\psi_{\phi,\nu}^{(b+1)} = \sum_{\mathfrak{m} \in \mathfrak{L}_{\phi,\nu}} b^{\vartheta(\mathfrak{m})}$, and (iv) ϑ has a natural combinatorial definition.*

A *Map-Jack invariant* is a function ϑ that satisfies Conjecture 1.1.

The main result of this paper is the formulation of an algorithm for computing an invariant η of rooted maps that we conjecture to be a Map-Jack invariant. This has been verified for all maps with at most four edges.

Map-Jack invariants also arise in geometry. The virtual Euler characteristics for the moduli spaces of real and complex algebraic curves can be expressed as a function of b, where $b = 1$ for the real case and $b = 0$ for the complex case [3]. This function is obtained from cell decompositions of the moduli spaces, and involves an alternating weighted sum of the number of monopoles (maps with exactly one vertex), with a prescribed number of edges and faces in locally orientable surfaces, in the real case, and in orientable surfaces, in the complex case. The lifting from rooted monopoles to the moduli spaces is well understood, so the conjectured existence of Map-Jack invariants suggests that b also has a natural interpretation. This leads to a natural (and potential geometrically important) question: what is the geometrical meaning of the coefficients of the powers of b? For example, we suspect that a knowledge of η may induce, through lifting, a geometrically meaningful construction in the moduli space of real curves. For further discussion, the reader is directed to [3].

1.3 Organisation of the paper

Section 2 provides the background on the combinatorialisation of maps and gives the background needed for the definition of η. Section 3 defines the conjectured Map-Jack invariant η and shows that it is dependent upon rooting. Evidence supporting η as a Map-Jack invariant is also given. Section 4 shows how the definition of η is associated with Jack symmetric functions, thereby providing further evidence that η is a Map-Jack invariant. The results and their impact on η are summarised in Section 5. The appendices contain expansions of Jack symmetric functions computed using the SF package of Stembridge [13].

2 Combinatorial Definitions for Maps

We begin by recalling a combinatorial definition of a rooted map and some aspects of the topological definition of a map. For further information, the reader is directed to [5] and [15]. We also introduce the definitions of a *submap* and a *directed map*, which are used in the definition of the invariant η. The combinatorial definition of a map is used throughout this paper as a convenient means for describing certain invariants for maps and algorithms for computing them. It will be useful to use $\mathfrak{m}$ to indicate both a (topological) map and the corresponding combinatorial map. The distinction between these will be resolved in each instance by the context.

2.1 Rooted Maps, Combinatorial Maps and Orientability

Let X be a finite, totally ordered set of even cardinality. The *cycle type* of a permutation $\pi : X \to X$ is the partition formed by the lengths of the disjoint cycles (orbits) of π. Let $M(X)$ be the set of *matchings* on X, these are fixed-point free involutions. The following definition is an adaptation of a definition by Tutte [15].

Definition 2.1 (Premap, Combinatorial Map, Rooted Map, [4]). *An ordered triple* $(\mu_1, \mu_2, \mu_3) \in M(X)^3$ *is a* premap *on* X *if* $\mu_1\mu_3 \in M(X)$. *The premap* $\mathfrak{m} = (\mu_1, \mu_2, \mu_3)$ *is a* (combinatorial) map *if the group* $\langle \mu_1, \mu_2, \mu_3 \rangle$ *acts transitively on* X. *The pair* $(\mathfrak{m}, r)$ *is a* rooted (combinatorial) map *if* $\mathfrak{m}$ *is a (combinatorial) map and* $r \in X$, *and* r *is called the* root.

The elements of X are *positions* associated with a premap on X. We may think of them as points attached to each side of each end of an edge, so $|X| = 4n(\mathfrak{m})$. The matchings μ_1, μ_2 and μ_3 are the *face-move*, *edge-move*, and *vertex-move* matchings, respectively. An orbit of $\langle \mu_2, \mu_3 \rangle$, $\langle \mu_3, \mu_1 \rangle$, and $\langle \mu_1, \mu_2 \rangle$ is a *(combinatorial) face*, *(combinatorial) edge*, and *(combinatorial) vertex*, respectively. Each combinatorial edge is a four element subset of X. The *degree* of a combinatorial face or vertex is half its cardinality as an orbit of X. Invariants $\phi(\mathfrak{m})$, $\nu(\mathfrak{m})$ are defined by analogy with topological maps. If Z is an orbit of $\langle \mu_1, \mu_2, \mu_3 \rangle$, then the premap $(\mu_1|_Z, \mu_2|_Z, \mu_3|_Z)$, where $\mu_i|_Z$ is the restriction of the function μ_i to the domain Z, is a *connected component* of the premap $\mathfrak{m}$, and is a map on Z.

$(\mathfrak{m}^\partial, r)$. Associated with each premap $\mathfrak{m}$ is a graph $\Gamma(\mathfrak{m})$, the *matchings graph* [4], whose vertex set is X and edge set is a multi-set $\{\{x, \mu_k(x)\} : x < \mu_k(x), k = 1, \ldots, 3, x \in X\}$ (there may be multiple edges). The graph $\Gamma(\mathfrak{m})$ is vertex-3-regular and its edges have a *natural colouring* specified by the indices of μ_k.

An *isomorphism* between a premap (μ_1, μ_2, μ_3) on X and a premap (μ'_1, μ'_2, μ'_3) on X' is a bijection $\xi : X \to X'$ such that $\xi \circ \mu_k = \mu'_k \circ \xi$ for $1 \leqslant k \leqslant 3$, and an isomorphism between rooted maps has the further condition that $\xi(r) = r'$.

A *locally orientable surface* [15] is a compact 2-manifold. Associated with any combinatorial map $\mathfrak{m}$ is a topological map $\mathfrak{M}(\mathfrak{m})$ which is a *2-cell embedding* of the *underlying graph* $G(\mathfrak{m})$ in the *underlying surface* $\mathbb{S}(\mathfrak{m})$ of $\mathfrak{m}$. The surface $\mathbb{S}(\mathfrak{m})$ may be obtained by sewing discs to cycles of the matchings graph $\Gamma(\mathfrak{m})$ that are bicoloured in the natural colouring. A bijection, up to isomorphism, between topological maps and combinatorial maps, is given by a result sometimes known as the *Embedding Theorem* [5, 15]. Since combinatorial faces, edges and vertices correspond to topological faces, edges and vertices, the modifiers *topological* and *combinatorial* for the terms *map*, *face*, *edge* and *vertex* can henceforth be omitted when implied by the context.

Up to homeomorphism, two parameters, the Euler characteristic $\chi \in \mathbb{Z}$ and orientability, determine locally orientable surfaces. Every orientable surface is homeomorphic to a sphere with g handles for some integer $g \geqslant 0$, the *genus*, and the Euler characteristic for this surface is $\chi = 2 - 2g$. Every non-orientable surface is homeomorphic to a real projective plane with g handles ($\chi = 1 - 2g$) or Klein bottle with g handles ($\chi = -2g$). The Euler characteristic and orientability of a map $\mathfrak{m}$ are defined to be that of $\mathbb{S}(\mathfrak{m})$. Thus, by the Euler-Poincaré formula, $\chi(\mathfrak{m}) = \ell(\phi(\mathfrak{m})) - n(\mathfrak{m}) + \ell(\nu(\mathfrak{m}))$. It is easily seen that the underlying surface $\mathbb{S}(\mathfrak{m})$ is orientable if and only if the matchings graph $\Gamma(\mathfrak{m})$ is bipartite.

2.2 Submaps

The notion of a *submap* is required in the definition of the invariants η and β. A *ribbon graph* is a topological map with a hole punctured in each face. The map can be reconstructed from the ribbon graph by sewing disks onto the boundaries of the holes. Any connected subgraph G' of the underlying graph $G = G(\mathfrak{m})$ induces a ribbon graph, which is obtained by taking the ribbon graph of $\mathfrak{m}$ and deleting the disks and ribbons corresponding to vertices and edges, respectively, that do not belong to G'. The map reconstructed from the ribbon graph of G' by sewing disks into the holes is the *submap* $\mathfrak{s}$ of $\mathfrak{m}$ induced by the subgraph G' of $G(\mathfrak{m})$. A map $\mathfrak{m}$ and its submap $\mathfrak{s}$ need not have homeomorphic surfaces.

Although all submaps of an orientable map are orientable, a non-orientable map can have an orientable submap. Indeed, as we shall see, the invariants η and β depend on the orientability of certain submaps of a non-orientable map.

If e is an edge of $\mathfrak{m}$ and the graph $G(\mathfrak{m}) - e$ is connected, let $\mathfrak{m} - e$ be the submap of $\mathfrak{m}$ containing all the edges of $\mathfrak{m}$ but the edge e. If $G(\mathfrak{m}) - e$ is not connected, then it has two components. In this case, let $\mathfrak{m} - e$ be the pair of submaps of $\mathfrak{m}$ corresponding to the two components of $G(\mathfrak{m}) - e$.

2.3 Directed Maps

A *directed map* $\mathfrak{d}$ is a map, together with the following additional information: each edge is assigned a direction, each edge is assigned a label from $\mathbb{N}_n = \{1, 2, \ldots, n\}$, and each vertex is assigned a local (cyclic) orientation. The edge with label i is denoted by e_i. The set of all directed maps is denoted by $\mathfrak{D}$.

An edge *preserves* or *reverses* local orientations according to whether the local orientation at one end of the edge can be extended along the length the edge to the other end and be consistent with the local orientation there.

A directed map is *oriented* if all its edges preserve local orientations. If two directed maps differ only by their k-values (the third coordinate), then one is a *re-orientation* of the other. A directed map $\mathfrak{d}$ is *orientable* if it has an oriented re-orientation, and such a re-orientation is a *global orientation*. A directed map is orientable if and only if it is orientable as combinatorial map. An orientable directed map has exactly two global orientations.

When we encounter a map $\mathfrak{m}$ whose n edges are labelled with elements from a set S of positive integers different from $\mathbb{N}_n$, but the map $\mathfrak{m}$ otherwise meets the conditions for being a directed map, it will be useful to tranform $\mathfrak{m}$ into a directed map in a canonical way. The typical example is a submap $\mathfrak{s}$ of a directed map $\mathfrak{d}$ where $\mathfrak{s}$ is not strictly a directed map. The canonical transformation for these maps is to relabel the edges with labels from $\mathbb{N}_n$ in the same order as the edge labels in $S = \{i_1, \dots, i_n\}$ so that $i_j \mapsto j$.

3 The Conjectured Map-Jack Invariant η

The invariant η is defined by

$$\eta = \beta \circ \delta, \tag{3.1}$$

where the functions β and δ are defined by algorithms in Section 3.1 and Section 3.2, respectively, that act on maps. An additivity property of η is given in Section 3.4 and evidence supporting the conjecture that η is a Map-Jack invariant is given in Section 3.5.

3.1 The Invariant β of Directed Maps

Let $\mathfrak{d} \in \mathfrak{D}$ be a directed map, with $n \geqslant 2$ edges $e_1, e_2, \dots, e_n$. If $\mathfrak{d} - e_n$ is a directed map, let $\mathfrak{d}' = \mathfrak{d} - e_n$. Otherwise, $\mathfrak{d} - e_n$ has two components which, with re-indexing of edge labels, become two directed maps $\mathfrak{d}_1$ and $\mathfrak{d}_2$ where $\mathfrak{d}_1$ is the relabelling of the submap of $\mathfrak{d}$ containing edge e_1. Let $\mathfrak{d}' = (\mathfrak{d}_1, \mathfrak{d}_2)$ in this case. If $n = 1$, let $\mathfrak{d}' = \varnothing$. We classify directed maps $\mathfrak{d}$ into *deletion types*, defined below, according to the relationship between $\mathfrak{d}$ and $\mathfrak{d}'$.

Definition 3.1 (Deletion Type). *Let $\mathfrak{d} \in \mathfrak{D}$ be a directed map with n edges. The* deletion type *of the directed map $\mathfrak{d}$ is determined by the following exhaustive case analysis:*

1. *If $n = 1$, and:*
   (a) *If $\mathfrak{d}$ has one vertex, and:*
      i. *is orientable, then $\mathfrak{d}$ has* loop deletion type.
      ii. *is non-orientable, then $\mathfrak{d}$ has* cross-loop deletion type.
   (b) *If $\mathfrak{d}$ has two vertices, then $\mathfrak{d}$ has* link deletion type.
2. *If $n > 1$, and:*
   (a) *If there is a vertex of degree 1 incident to the edge e_n, then $\mathfrak{d}$ has* leaf deletion type.
   (b) *If e_n is not incident to a vertex of degree 1, and:*
      i. *If the ends of e_n are incident to two different faces in $\mathfrak{d}'$, and:*
         A. *If $\mathfrak{d}'$ has two components, then $\mathfrak{d}$ has* bridge deletion type.
         B. *If $\mathfrak{d}'$ has one component, and:*
            - *If $\mathfrak{d}'$ is orientable, and:*
               – *If $\mathfrak{d}$ is orientable, then $\mathfrak{d}$ has* handle deletion type.
               – *If $\mathfrak{d}$ is non-orientable, then $\mathfrak{d}$ has* cross-handle deletion type.
            - *If $\mathfrak{d}'$ is non-orientable, and:*

   - *If local orientation in $\mathfrak{d}$ is preserved by e_n, then $\mathfrak{d}$ has* handle deletion type.
   - *If local orientation in $\mathfrak{d}$ is reversed by e_n, then $\mathfrak{d}$ has* cross-handle deletion type.

ii. If both ends of e_n are incident to the same face in $\mathfrak{d}'$, and:

  A. If e_n is incident with two faces in $\mathfrak{d}$, then $\mathfrak{d}$ has border deletion type.

  B. If e_n is incident with one face in $\mathfrak{d}$, then the $\mathfrak{d}$ has cross-border deletion type.

Directed maps $\mathfrak{d}$ of each of these nine deletion types are given in Table 3.1. (The last column in the table relates to the partial differential equation (4.3) and is to be ignored for the moment.) Note that, in all but the loop type and the border type, e_n is an isthmus, that is, both sides of e_n are incident to a single face (see Table 3.1). Let the *deletion type* of e_n be that of $\mathfrak{d}$. Let the *deletion type* of a general edge e_i of $\mathfrak{d}$ be defined inductively as the deletion type of e_i as an edge of $\mathfrak{d}' = \mathfrak{d} - e_n$. (If $\mathfrak{d}'$ has two components $\mathfrak{d}_1, \mathfrak{d}_2$, then the deletion type of e_i in $\mathfrak{d}$ is the deletion type of the re-labelling of e_i in $\mathfrak{d}_j$.)

Definition 3.2. *The invariant $\beta : \mathfrak{D} \to \mathbb{N}$ is defined such that $\beta(\mathfrak{d})$ is the number of edges whose deletion type is cross-loop, cross-border, or cross-handle.*

Clearly, $\mathfrak{d}$ is orientable if and only if $\beta(\mathfrak{d}) = 0$. Contrapositively, $\beta(\mathfrak{d}) \geqslant 1$ if and only if $\mathfrak{d}$ is non-orientable. An upper bound on β is given by the following lemma.

Lemma 3.3. *Let $\mathfrak{d}$ be a directed map. Then $\beta(\mathfrak{d}) \leqslant 2 - \chi(\mathfrak{d})$ where $\chi(\mathfrak{d})$ is the Euler characteristic of $\mathfrak{d}$.*

Proof. Assume that $\mathfrak{d}$ has f faces, n edges, and v vertices. Each of the n edges of $\mathfrak{d}$ falls into one of the nine deletion types. The deletion of the edges from $\mathfrak{d}$ may disconnect $\mathfrak{d}$. Let $\mathfrak{C}_i$ be the union of the set of directed maps obtained by canonically re-labelling the component maps of the premap $\mathfrak{d}\downarrow_{\{e_1,\ldots,e_i\}}$ and a set of edgeless directed maps, one such edgeless map for each vertex of $\mathfrak{d}$ not incident to any edge in $\{e_1, \ldots, e_i\}$. Let f_i and c_i be the total number of faces and components, respectively, in the collection $\mathfrak{C}_i$. Each edgeless directed map contributes one face and one component. Let $k_i = f_i - c_i$. Then $k_0 = f_0 - c_0 = n - n = 0$ and $k_n = f_n - c_n = f - 1$. Let $g_i = k_i - k_{i-1}$. Then $f - 1 = g_1 + \cdots + g_n$. The effect on number of faces and components implies

$$g_i = \begin{cases} 1 & \text{if } e_i \text{ is a loop or border,} \\ 0 & \text{if } e_i \text{ is a leaf, link, bridge, cross-loop, or cross-border,} \\ -1 & \text{if } e_i \text{ is a handle, cross-handle.} \end{cases}$$

Thus the number of loops and borders is at least $f - 1$. Also, as there are v vertices and the tree edges (links, leaves and bridges) of $\mathfrak{d}$ form a spanning subgraph, there are $v - 1$ tree edges. Thus the number of links, leaves, loops, borders, and bridges is at least $(v-1)+(f-1)$, leaving at most $n-(v-1)-(f-1) = 2-\chi$ edges for cross-loops, cross-borders and cross-handles. $\square$

3.2 The Function δ

We define the function $\delta : \mathfrak{L} \to \mathfrak{D}$ from rooted to directed maps in two stages, by first defining a *canonical rooted map* by means of a *canonical isomorphism (labelling)* from any rooted map to a canonical rooted map, and then defining a *canonical directed map* associated with a canonical rooted map. Walsh and Lehman [9, 10, 16] use a similar method to canonically encode rooted orientable maps.

In terms of the topological map, the algorithm acts as follows. Once the algorithm visits a position of a vertex for the first time, it alternately steps by μ_1 and μ_2, cycling around the vertex. Then it backtracks, if needed, until it can step by μ_3 to an unvisited position, which must necessarily belong to an unvisited vertex. During backtracking, the algorithm might need to move to different vertices. A *forward step* by

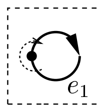
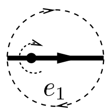
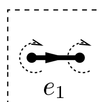
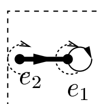
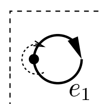
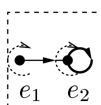
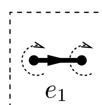
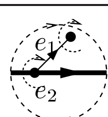
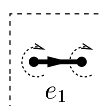
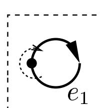
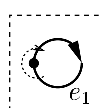
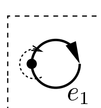
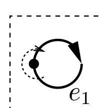
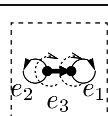
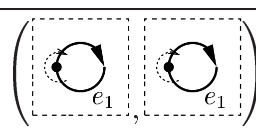
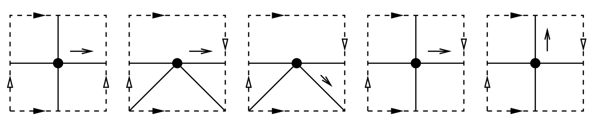
Deletion Type	Example $\mathfrak{d}$ (e_n bold)	Corresponding $\mathfrak{d} - e_n$	Contribution to (4.3)
Loop		$\varnothing$	$p_1^2 v$
Cross-loop		$\varnothing$	$bp_2 v$
Link		$\varnothing$	$2p_2 v^2$
Leaf			$\displaystyle\sum_{i\geqslant 1} 4ip_{i+2}v\frac{\partial D}{\partial p_i}$
Border			$\displaystyle\sum_{i,j\geqslant 1} (i+j-2)p_i p_j \frac{\partial D}{\partial p_{i+j-2}}$
Cross-border			$\displaystyle\sum_{i,j\geqslant 1} b(i+j-2)p_{i+j} \frac{\partial D}{\partial p_{i+j-2}}$
Handle			$\displaystyle\sum_{i,j\geqslant 1} ijp_{i+j+2} \frac{\partial^2 D}{\partial p_i \partial p_j}$
Cross-handle			$\displaystyle\sum_{i,j\geqslant 1} bijp_{i+j+2} \frac{\partial^2 D}{\partial p_i \partial p_j}$
Bridge			$\displaystyle\sum_{i,j\geqslant 1} 2ijp_{i+j+2} \frac{\partial D}{\partial p_i}\frac{\partial D}{\partial p_j}$

Table 3.1: Examples of the nine deletion types and their contribution to $\frac{\partial D}{\partial z}$ in (4.3)

μ_3 from x to y is a move along the side of a face. A new vertex is encountered at y. Then the algorithm cycles around the new vertex, ending at $z = \mu_2(y)$. Note that z belongs to the same face containing both y and x. If $w = \mu_3(z)$ is unvisited, the algorithm moves to w along the side of this face. Thus the algorithm visits the vertices incident to this face in cyclic order until an already visited vertex is encountered. The first position visited is the root vertex, which is assigned label 1. Each new position visited gets assigned the next available label. This algorithm is depth first search performed on the matchings graph of the map, using the matching indices as priorities for searching.

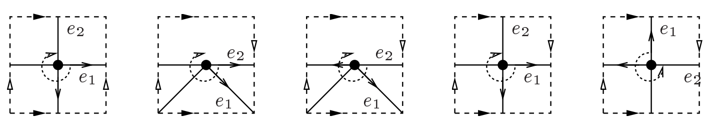
We say that an edge of a map is *exhausted* by the depth first algorithm once all four positions in the edge have been visited. In the *canonical directed map*, the labels of the edges are determined in the order in which they are exhausted by the canonical labelling algorithm. The local orientation of a vertex is determined by the order that algorithm visits the positions at the vertex. An edge is directed from the first end visited to the last end visited.

The five maps in $\mathfrak{L}_{(4),(4)}$ have just one vertex and are given in Figure 3.1. The step μ_3 is not used and thus the actions of the canonical labelling algorithm is very simple. The canonical directed maps associated with the five rooted maps of $\mathfrak{L}_{(4),(4)}$ are given in Figure 3.2.



(a) $\mathfrak{a}$ (b) $\mathfrak{b}$ (c) $\mathfrak{c}$ (d) $\mathfrak{d}$ (e) $\mathfrak{e}$

Figure 3.1: The set $\mathfrak{L}_{(4),(4)}$



(a) $\delta(\mathfrak{a})$ (b) $\delta(\mathfrak{b})$ (c) $\delta(\mathfrak{c})$ (d) $\delta(\mathfrak{d})$ (e) $\delta(\mathfrak{e})$

Figure 3.2: The five canonical directed maps in $\delta(\mathfrak{L}_{(4),(4)})$

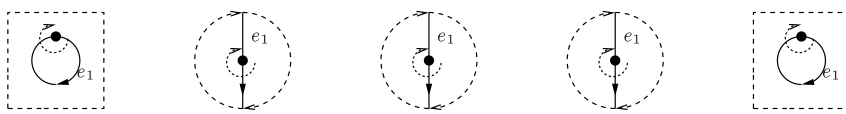
3.3 The Dependence of η on Rooting

We show by a small example that ϑ (and therefore η) necessarily depends on rooting. Direct calculation of the generating series $\psi^{(b+1)}_{(4),(4)}$ for $\mathfrak{L}_{(4),(4)}$ with (1.1) and Jack symmetric functions in Appendix A gives $\psi^{(b+1)}_{(4),(4)} = 3b^2 + b + 1$. Details of the calculation are given in Appendix A. Thus the values that a Map-Jack invariant ϑ takes on the set $\mathfrak{L}_{(4),(4)}$ of five rooted maps $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ given in Figure 3.1 are 2, 1 and 0, occurring with multiplicities 3,1 and 1, respectively. The underlying surface of $\mathfrak{a}$ is the torus, which is orientable, so $\vartheta(\mathfrak{a}) = 0$. The underlying surface of each of $\mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ is the Klein bottle, which is non-orientable, so ϑ takes the values 2 on three of the four maps and 1 on the remaining map. Suppose that ϑ does not depend on the rooting of a map. The rooted maps $\mathfrak{b}$ and $\mathfrak{c}$ differ only by their rooting, as do $\mathfrak{d}$ and $\mathfrak{e}$. Thus the supposition implies $\vartheta(\mathfrak{b}) = \vartheta(\mathfrak{c})$ and $\vartheta(\mathfrak{d}) = \vartheta(\mathfrak{e})$. Hence the values $\vartheta(\mathfrak{b})$ and $\vartheta(\mathfrak{d})$ both have multiplicity at least 2. There is only value with multiplicity at least 2, so the values $\vartheta(\mathfrak{b})$ and $\vartheta(\mathfrak{d})$ are the same. But then $\vartheta(\mathfrak{b})$ has multiplicity at least 4, which is not possible since the highest multiplicity is 3. Thus a Map-Jack invariant ϑ depends on the rooting of a map.

3.4 Examples and an Additivity Property of η

In this section, we complete the calculations of η on $\mathfrak{L}_{(4),(4)}$, and prove an additive property of η. We also provide evidence supporting the conjecture that η is a Map-Jack invariant.

We begin with an example of the calculation of η on the five maps in $\mathfrak{L}_{(4),(4)}$. The application of δ is given in Figure 3.2. The five maps resulting from deletion of e_2 are given in Figure 3.3. The subsequent application of β in two steps and the final results are given in Table 3.2.



(a) $\delta(\mathfrak{a})-e_2$ (b) $\delta(\mathfrak{b})-e_2$ (c) $\delta(\mathfrak{c})-e_2$ (d) $\delta(\mathfrak{d})-e_2$ (e) $\delta(\mathfrak{e})-e_2$

Figure 3.3: Deletion of e_2

$\mathfrak{m}$	e_1	$\beta(\delta(\mathfrak{m}) - e_1)$	e_2	$\eta(\mathfrak{m})$
$\mathfrak{a}$	loop	0	handle	0
$\mathfrak{b}$	cross-loop	1	cross-border	2
$\mathfrak{c}$	cross-loop	1	cross-border	2
$\mathfrak{d}$	cross-loop	1	cross-border	2
$\mathfrak{e}$	loop	0	cross-handle	1

Table 3.2: The deletion types of e_1 and e_2

The unrooted map $\mathfrak{u}_*$ in Figure 3.4 is a counterexample to the hypothesis that, for each unrooted map $\mathfrak{u}$, the generating series for directings of $\mathfrak{u}$ with b marking β is an integer multiple of the generating series for rootings of $\mathfrak{u}$ with b marking η. The unrooted map $\mathfrak{u}_*$ has two rootings, and for each rooting η is three. But some directings have β equal to three and others have two.

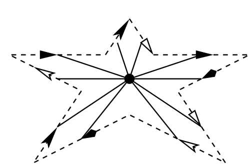


Figure 3.4: An unrooted map $\mathfrak{u}_*$ for which β and η act differently.

The following is an additivity property of η. A (topological) map $\mathfrak{m}$ is *separable* if a closed curve in the surface passes once through a single vertex and face, and also separates the surface into two components each containing edges of $\mathfrak{m}$. A *(non-separable) block* of $\mathfrak{m}$ is a non-separable submap maximal with respect to the subset of edges of $\mathfrak{m}$. Each edge of $\mathfrak{m}$ belongs to a unique block of a map $\mathfrak{m}$. The blocks of a map are not necessarily the submaps induced by the block of the underlying graph: the map $\mathfrak{a}$ in Figure 3.1(a) has one block while its underlying graph has two blocks. A *block* $\mathfrak{b}$ of a rooted map $\mathfrak{m}$ is a rooted map that is a block of $\mathfrak{m}$ as a combinatorial map with a root at the position whose canonical label in $\mathfrak{m}$ is minimal in $\mathfrak{b}$.

Theorem 3.4 ([2]). *Let $\mathfrak{m}$ be a rooted map and let its blocks be $\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_c$. Then $\eta(\mathfrak{m}) = \eta(\mathfrak{b}_1) + \eta(\mathfrak{b}_2) + \cdots + \eta(\mathfrak{b}_c)$.*

Proof. The canonical position labels of $\mathfrak{m}$ are consistent in order with the canonical position labels of each block. Thus the block decompsition of the directed map $\delta(\mathfrak{m})$ is into directed blocks $\delta(\mathfrak{b}_1), \ldots, \delta(\mathfrak{b}_c)$ (with appropriate re-labelling). Furthermore, the deletion types of each edge, regarded as an edge of $\mathfrak{m}$ and as an edge of a block $\mathfrak{b}_i$, are essentially the same with respect to being marked by b. $\square$

3.5 Evidence that η is a Map-Jack Invariant

The main evidence we have supporting that η is a Map-Jack invariant is provided by the following two theorems.

Theorem 3.5 ([4]). *Let $n \geqslant 1$ and $\nu \vdash 2n$. Then*

$$\sum_{\phi \vdash 2n} \psi_{\phi,\nu}^{(b+1)} = (b+1)^{n-\ell(\nu)+1} \sum_{\phi \vdash 2n} \psi_{\phi,\nu}^{(1)}. \tag{3.2}$$

Theorem 3.5 can be proved [4] using properties of Jack symmetric function (see also [2] for an adapted proof). Both this theorem and the following are independent of face partition and the number of faces, but they support the conjecture that η is a Map-Jack invariant in the following way. Since the power sum symmetric functions are algebraically independent, the mapping $\Upsilon : p_j(\mathbf{x}) \mapsto 1$ can be extended as a homomorphism, and it has the effect of suppressing the face partition. Let

$$H(\mathbf{x}, \mathbf{y}, z; b) = \sum_{\phi,\nu} h_{\phi,\nu}^{(b+1)} p_\phi(\mathbf{x}) p_\nu(\mathbf{y}) z^{|\phi|/2}. \tag{3.3}$$

where $h_{\phi,\nu}^{(b+1)} = \sum_{\mathfrak{m} \in \mathfrak{L}_{\phi,\nu}} b^{\eta(\mathfrak{m})}$ for $\phi, \nu \vdash 2n$. Since $h_{\phi,\nu}^{(1)} = o_{\phi,\nu} = \psi_{\phi,\nu}^{(1)}$, the theorem above and the theorem below together imply that $\Upsilon H = \Upsilon \Psi^{(b+1)}$, confirming a special case of the conjecture that η is a Map-Jack invariant which, in this notation, can be written as $H = \Psi^{(b+1)}$.

Theorem 3.6 ([2]). *Let $n \geqslant 1$ and $\nu \vdash 2n$. Then*

$$\sum_{\phi \vdash 2n} h_{\phi,\nu}^{(b+1)} = (b+1)^{n-\ell(\nu)+1} \sum_{\phi \vdash 2n} h_{\phi,\nu}^{(1)}. \tag{3.4}$$

Proof. Let an *edge diagram* be a quadruple (n, μ, V, T) where n is a positive integer, $\mu \in M(\mathbb{N}_{2n})$ is a matching, and $V, T \subseteq \mu^+ = \{i \in \mathbb{N}_{2n} : \mu(i) < i\}$ are disjoint sets such that if $v \in V$, $i \in \mathbb{N}_{2n}$ and $\mu(v) < i < v$, then $\mu(i) < v$. If $\mathfrak{m}$ is a canonical rooted map on $\mathbb{N}_{4n}$, then its associated *canonical edge diagram*, is denoted by $\varepsilon(\mathfrak{m})$, and is the edge diagram $\mathfrak{e} = (n, \mu, V, T)$ defined as follows. Let $\mu(i) = \lceil \mu_3(2i)/2 \rceil$. Let $V = \{i \in \mu^+ : \mu_2(2i-1) \neq 2i-2\}$ and $T = \{i \in \mu^+ : 2 \mid \mu_3(2i)\}$. This gives a one-to-one correspondence between edge diagrams and isomorphism classes of rooted maps. A drawing of an edge diagram consists of the following. The base line is a horizontal line segment with $2n$ selected internal points equally spaced.

The matching μ is represented by a set of semi-circular arcs drawn above the base line. If $\mu(i) = j$, then one of these arcs has its endpoints at the selected points i and j counting from left to right. A vertex is placed at the left endpoint of the base line and at each selected point $i \in V$. At each point $j \in T$, a small $\times$ is drawn.

Let $\mathfrak{E}_\nu$ be the set of edge diagrams whose corresponding rooted maps have n edges and vertex partition $\nu \vdash 2n$. Let $\mathfrak{E}^o_\nu$ be the subset of $\mathfrak{E}_\nu$ containing edge diagrams such that $T = \varnothing$, which is precisely the subset corresponding to orientable maps. The sum in the left hand side of (3.4) is the generating series for $\mathfrak{E}_\nu$ with b marking η. (Let $\eta(\mathfrak{e}) = \eta(\varepsilon^{-1}(\mathfrak{e}))$ for any edge diagram $\mathfrak{e}$.) Let $o : \mathfrak{E}_\nu \to \mathfrak{E}^o_\nu : (n, \epsilon, V, T) \mapsto (n, \epsilon, V, \varnothing)$. The function o removes twists from edge diagrams by replacing twisted arcs by untwisted arcs, and therefore gives edge diagrams of orientable maps. We claim that for each twist-free (orientable) edge diagram $\mathfrak{e}_o = (n, \epsilon, V, \varnothing) \in \mathfrak{E}^o_\nu$, the generating series for its pre-image set $o^{-1}(\mathfrak{e}_o)$, where b marks η, is

$$(b+1)^{n-\ell(\nu)+1}. \tag{3.5}$$

To prove Theorem (3.6) it suffices to prove the claim (3.5), because the right hand side of (3.4) is (3.5) multiplied by the number of edge diagrams in $\mathfrak{E}^o_\nu$. The proof is by induction on the number of arcs n. Suppose $n > 1$. Let $\mathfrak{e} \in o^{-1}(\mathfrak{e}_o)$. Let a and a_o be the rightmost arcs of the edge diagrams $\mathfrak{e}$ and $\mathfrak{e}_o$ respectively. Let $\mathfrak{e}' = \mathfrak{e} - a$ and $\mathfrak{e}'_o = \mathfrak{e}_o - a_o$. Then $o(\mathfrak{e}') = \mathfrak{e}'_o$. There are two cases to consider: either both a and a_o possess vertices, or equivalently that $n \in V$, or they both do not.

Case (a): The arc a possesses a vertex ($n \in V$). It follows that the edge associated with a is classified into the leaf type by the invariant β. Therefore, $\eta(\mathfrak{e}) = \eta(\mathfrak{e}')$. This holds for all $\mathfrak{e} \in o^{-1}(\mathfrak{e}_o)$. Therefore the generating series for $o^{-1}(\mathfrak{e}_o)$ equals the generating series for $o^{-1}(\mathfrak{e}'_o)$, which equals $(b+1)^{(n-1)-(\ell(\nu)-1)+1}$ by induction. This proves (3.5) for this case.

Case(b): The arc a does not possess a vertex ($n \notin V$). Now either a does or does not carry a twist ($n \in T$ or $n \notin T$). Then, depending on the classification by β of the edge associated with a,

$$\eta(\mathfrak{e}) = \begin{cases} \eta(\mathfrak{e}') & \text{if } a \text{ has border or handle deletion type,} \\ \eta(\mathfrak{e}') + 1 & \text{if } a \text{ has cross-border or cross-handle deletion type.} \end{cases}$$

(Note that bridge deletion type of a is ruled out by the canonical labelling algorithm.) Let $\tilde{\mathfrak{e}}$ be the edge diagram identical to $\mathfrak{e}$ except that its rightmost arc $\tilde{a}$ which is twisted if a is not, and is not twisted if a is. (Equivalently, $\tilde{T} = T \cup \{n\} \setminus (T \cap \{n\}$.) Clearly this operation is a pairing, $\tilde{\tilde{\mathfrak{e}}} = \mathfrak{e}$ and the effects of deletion of a and $\tilde{a}$ are the same, $\tilde{\mathfrak{e}}' = \mathfrak{e}'$. In fact, $\mathfrak{e}$ and $\tilde{\mathfrak{e}}$ are the only two edge diagram solutions $\mathfrak{f}$ determined by $\mathfrak{f}' = \mathfrak{e}'$ and $o(\mathfrak{f}) = \mathfrak{e}_o$.

Now note that a is a border if and only if $\tilde{a}$ has cross-border type, and a is a handle if and only if $\tilde{a}$ has cross-handle type. Thus for all $\mathfrak{e} \in o^{-1}(\mathfrak{e}_o)$: $b^{\eta(\mathfrak{e})} + b^{\eta(\tilde{\mathfrak{e}})} = (b+1)b^{\eta(\mathfrak{e}')}$. Therefore the generating series for $o^{-1}(\mathfrak{e}_o)$ is

$$\sum_{\mathfrak{e} \in o^{-1}(\mathfrak{e}_o)} b^{\eta(\mathfrak{e})} = (b+1) \sum_{\mathfrak{e}' \in o^{-1}(\mathfrak{e}'_o)} b^{\eta(\mathfrak{e}')}.$$

By the inductive hypothesis, this equals $(b+1)^{(n-1)-\ell(\nu')+2}$. $\square$

Theorem 3.6 implies that, if face partition is ignored, that η is a Map-Jack invariant.

4 A Further Connexion between η and Jack Symmetric Functions

We now show how the algorithm for determining η is associated with Jack symmetric functions, which provides further supporting evidence that η is a Map-Jack invariant.

4.1 A Partial Differential Equation Involving β

Although η is marked by a generating series H for rooted maps (see (3.3)), it is also fruitful to consider the following closely related generating series D for directed maps defined by

$$D = \sum_{\mathfrak{d}\in\mathfrak{D}} b^{\beta(\mathfrak{d})} p_{\phi(\mathfrak{d})} v^{\ell(\nu(\mathfrak{d}))} \frac{z^{n(\mathfrak{d})}}{n(\mathfrak{d})!} \tag{4.1}$$

where $p_\phi = p_{\phi_1} p_{\phi_2} \ldots$ for a partition ϕ. Vertices are marked by v; edges, which are labelled, are marked by z, exponentially; the invariant β is marked by b; and faces of degree i are marked by the power sum symmetric functions p_i (which are algebraically independent and, in this section, can be regarded as indeterminates since none of their other properties are used). Note that $\mathfrak{D}$ includes all directed maps, including non-canonical maps. So D enumerates a set different from the set that H enumerates.

The series D is close to H in the sense that a simple relationship,

$$D = \frac{1}{4}\Theta H, \tag{4.2}$$

appears to hold, where Θ is the homomorphism defined by $\Theta : p_j(\mathbf{y}) \mapsto 2v$ and $\Theta : z \mapsto 2z$ and $\Theta : p_j(\mathbf{x}) \mapsto p_j$. The interpretation for $\frac{1}{4}\Theta$ as a transformation of enumerative series is to switch from counting rooted maps to directed maps, by factoring through by appropriate multiplicities. The substitution $p_j(\mathbf{y}) \mapsto 2v$ suppresses the vertex degree but introduces a local orientation of vertices. The substitution $z \mapsto 2z$ introduces edge directions. The factor $\frac{1}{4}$ accounts for the root edge and root vertex not needing directing and local orientation, respectively. Edge labels for directed maps are accounted for by the fact that the generating series D is exponential in z whereas as H is ordinary. These interpretations of the action of $\frac{1}{4}\Theta$ on enumerative series immediately prove that $D(b) = \frac{1}{4}\Theta H(b)$ in the special cases of $b \in \{0, 1\}$, and direct calculations prove the relationship for terms with low degree in z.

Although D is less refined than H (in not marking vertex partition), it has the advantage that the coefficients satisfy a recursion. This recursion extends recursions of Tutte [14] and Walsh [16] for certain subsets of $\mathfrak{D}$: planar maps and orientable maps, respectively. This recursion is expressed in a differential form, whose impact on Conjecture 1.1 is discussed further in Section 4.2, in the theorem below.

Theorem 4.1. *D satisfies the partial differential equation*

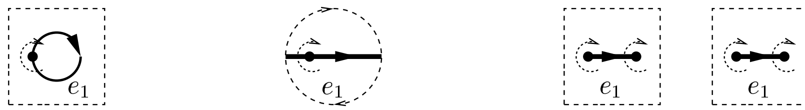
$$\begin{aligned}\frac{\partial}{\partial z} D &= (p_1^2 v + b p_2 v + 2 p_2 v^2) + \sum_{i \geqslant 1} 4 i p_{i+2} v \frac{\partial D}{\partial p_i} \\ &\quad + \sum_{i,j \geqslant 1} \left((i+j-2)(p_i p_j + b p_{i+j}) \frac{\partial D}{\partial p_{i+j-2}} + ij p_{i+j+2} \left(2 \left(\frac{\partial D}{\partial p_i} \right) \left(\frac{\partial D}{\partial p_j} \right) + (b+1) \frac{\partial^2 D}{\partial p_i \partial p_j} \right) \right). \end{aligned} \tag{4.3}$$

Proof. The proof involves the enumerative consequences of the case analysis of edge deletion. The left hand side of (4.3) is

$$\frac{\partial}{\partial z} D = \sum_{\mathfrak{d}\in\mathfrak{D}} b^{\beta(\mathfrak{d})} p_{\phi(\mathfrak{d})} v^{\ell(\nu(\mathfrak{d}))} \frac{z^{n(\mathfrak{d})-1}}{(n(\mathfrak{d})-1)!}.$$

The right hand side of (4.3) is obtained by summing over all deletion types (there are nine), and then for each of these deletion types, all instances of the directed map $\mathfrak{d}' = \mathfrak{d} - e_n$, (or pairs $(\mathfrak{d}_1, \mathfrak{d}_2)$ in the bridge deletion type), and then for each possible $\mathfrak{d}'$, all ways that e_n can be added to $\mathfrak{d}'$ to form $\mathfrak{d}$. To complete the proof, it suffices to verify that the contributions to $\frac{\partial D}{\partial z}$ of each of the nine deletion types of e_n are exactly as given in Table 3.1.

Case (1): $\mathfrak{d}$ has loop deletion type. The unique directed map $\mathfrak{d}$ having loop deletion type has one vertex and face partition $\phi(\mathfrak{d}) = (1^2)$ and is given in Figure 4.1(a). By definition $\beta(\mathfrak{d}) = 0$. Therefore the contribution of this case to $\frac{\partial D}{\partial z}$ is $p_1^2 v$.



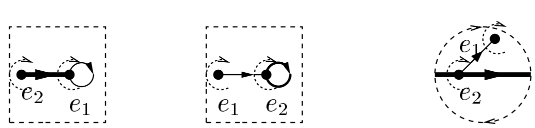
(a) Loop (b) Cross-loop (c) Link

Figure 4.1: Directed maps with $n = 1$

Case (2): $\mathfrak{d}$ has cross-loop deletion type. The unique directed map $\mathfrak{d}$ having cross-loop deletion type has one vertex and face partition $\phi(\mathfrak{d}) = (2)$ and is given in Figure 4.1(b). By definition $\beta(\mathfrak{d}) = 1$. Therefore the contribution of this case to $\frac{\partial D}{\partial z}$ is $bp_2 v$.

Case (3): $\mathfrak{d}$ has link deletion type. Each of the two directed maps $\mathfrak{d}$ having link deletion type has two vertices and face partition $\phi(\mathfrak{d}) = (2)$. The two directed maps are given in Figure 4.1(c) and only differ in that e_1 has a different sign. By definition $\beta(\mathfrak{d}) = 0$. Therefore the contribution of this case to $\frac{\partial D}{\partial z}$ is $2p_2 v^2$.

Case (4): $\mathfrak{d}$ has leaf deletion type. An example of leaf deletion type with $n = 2$ is given in Figure 4.2(a). When e_n has leaf deletion type, e_n must be attached to a corner of a face of the directed map $\mathfrak{d}' = \mathfrak{d} - e_n$. Let i be the degree of this face. This face of degree i in $\mathfrak{d}'$, once edge e_n is added, is transformed into a face of degree $i+2$ in $\mathfrak{d}$. To account for this action enumeratively, $p_{i+2}\frac{\partial D}{\partial p_i}$ represents the replacement of a face marked with p_i by a face marked with a face marked with p_{i+2}. Each face of degree i has i corners to which e_n may be attached. This introduces a factor i. Also, at the end of e_n opposite to $\mathfrak{d}'$ is a new vertex, which must be accounted for with a factor v. The edge e_n in $\mathfrak{d}$ may be directed towards $\mathfrak{d}'$ or away from $\mathfrak{d}'$, introducing a factor of 2. Since $\mathfrak{d}$ is a directed map, the new vertex of the leaf e_n may be given a local orientation consistent (along e_n) with the local orientation in $\mathfrak{d}'$ of the other vertex of e_n, or inconsistent with it, introducing another factor of 2. Each value of $i \geqslant 1$ corresponds to a distinct set of directed maps $\mathfrak{d}$ of link deletion type. The total contribution to $\frac{\partial D}{\partial z}$ is $\sum_{i\geqslant 1} 4ip_{i+2}v\frac{\partial D}{\partial p_i}$.



(a) Leaf (b) Border (c) Cross-Border

Figure 4.2: Some directed maps with $n = 2$

Case (5): $\mathfrak{d}$ has border deletion type. An example of border deletion type with $n = 2$ is given in Figure 4.2(b). Let the faces on either side of e_n be f_1 and f_2. Let f_1 be the face which is earlier with respect to the local orientation at the tail of e_n. Let f_1 and f_2 have degrees i and j, respectively. Then in $\mathfrak{d}' = \mathfrak{d} - e_n$, the removal of e_n merges f_1 and f_2 into a single face f of degree $i+j-2$. For each face of f of each $\mathfrak{d}'$, having degree $i+j-2$, there are $i+j-2$ corners where the tail of e_n may be attached. Therefore, using D to enumerate the choices for $\mathfrak{d}'$, differentiating by p_{i+j-2} selects and removes some face f in $\mathfrak{d}'$, yielding $\frac{\partial D}{\partial p_{i+j-2}}$; then, multiplying by $(i+j-2)p_i p_j$ accounts for the faces in f_1 and f_2, and the number of choices for the position of the tail of e_n in the face of f. Over all values of the degrees i and j, the contribution to $\frac{\partial D}{\partial z}$ is $\sum_{i,j\geqslant 1}(i+j-2)p_j p_j \frac{\partial D}{\partial p_{(i+j-2)}}$.

Case (6): $\mathfrak{d}$ has cross-border deletion type. An example of cross-border deletion type with $n=2$ is given in

Figure 4.2(c). In $\mathfrak{d}' = \mathfrak{d} - e_n$, the corners to which e_n attachs will belong again to a single face, say f. To construct $\mathfrak{d}$ from $\mathfrak{d}'$, topologically, a circular hole is cut in the interior of f and a *cross-cap* is inserted into this hole by identifying its boundary hole with the circular boundary of a Möbius band. Then e_n is drawn in the resulting surface by travelling through the cross-cap that has resulted from gluing the Möbius band to the hole. Equivalently, attach a narrow ribbon from one segment to another segment of the boundary of the circular hole, and add a twist to the ribbon, and draw e_n along this ribbon. As the resulting surface still has a single hole with one circular topological boundary, a disk is attached. Let $i-1$ and $j-1$ be the number of sides of f between the two corners where e_n is attached. In the cross-border deletion type, e_n does not divide the face f, but makes a new face of degree $i+j$. Each pair of positive values of i and j represents a distinct possibility for $\mathfrak{d}$, so we sum over all $i,j \geqslant 1$. As in the border deletion type, there are $i+j-2$, choices of a corner in f for the tail of e_n. By definition $\beta(\mathfrak{d}) = \beta(\mathfrak{d}')+1$, so a factor of b is introduced to account for the increase in this invariant. Therefore the contribution to $\frac{\partial D}{\partial z}$ directed maps $\mathfrak{d}$ having the cross-border deletion type is $\sum_{i,j\geqslant 1}(i+j-2)bp_{i+j}\frac{\partial D}{\partial p_{i+j-2}}$.

Case (7): $\mathfrak{d}$ has handle deletion type. In $\mathfrak{d}' = \mathfrak{d} - e_n$, the face f, see Figure 4.3(b), decomposes into two faces, see Figure 4.3(a). The corners of $\mathfrak{d}'$ to which e_n's head and tail are attached belong to distinct faces f^+ and f^-, of degrees say i and j, respectively. Topologically, one may add a handle to the surface of $\mathfrak{d}'$. The ends of the handle are inserted into the faces f^+ and f^-. There are two topologically distinct ways to do this, depending on the choice of circular direction the ends of the handle are glued to the surface. We shall shortly determine how to decide which one applies in this deletion type. As in Figure 4.3, the edge e_n is drawn from the given corner of the former face f^-, along the length of the handle, to the given corner of the face f^+. The handle therefore joins the faces f^- and f^+, into the single face f of $\mathfrak{d}$. The degree of f is $i+j+2$, the sum of the degrees of f^- and f^+, plus two, for the two sides of e_n. This accounts for

Figure 4.3: Two faces joined by a handle

a contribution of $p_{i+j+2}\frac{\partial^2 D}{\partial p_i \partial p_j}$. But in $\mathfrak{d}'$, the face f^+ has i corners, to any of which e_n can be attached. Similarly, there are j choices of a corner for f^-. Thus we must multiply by ij, to account for all possible $\mathfrak{d}$ that can be obtained from each $\mathfrak{d}'$. The faces f^+ and f^- can have any positive integer degrees, so we sum over $i,j \geqslant 1$. The contribution to $\frac{\partial D}{\partial z}$ is $\sum_{i,j\geqslant 1} ijp_{i+j+2}\frac{\partial^2 D}{\partial p_i \partial p_j}$. The rule for the topological glueing of the handle, or topological cylinder, to the surface of $\mathfrak{d}'$ is as follows. First cut circular holes out of the surface of $\mathfrak{d}'$. The ends of the topological cylinder are identified with the circular boundaries of these two holes. Once the first end is attached, the second end may be attached in two ways according to the cyclical direction used to identify the circular boundaries. There is a unique direction to identify these circular boundaries. If $\mathfrak{d}$ is orientable, it is the direction that causes the path along e_n in $\mathfrak{d}$ to preserve the global orientation. If $\mathfrak{d}$ is non-orientable, it is the direction that causes the path along e_n to preserve the local orientations of the vertices at both ends of e_n.

Case (8): $\mathfrak{d}$ has cross-handle deletion type. This deletion type is identical to the handle deletion type, except as follows. The edge e_n must reverse the orientations that would be preserved for e_n of handle deletion type. The contribution is multiplied by b, because $\beta(\mathfrak{d}) = \beta(\mathfrak{d}') + 1$. Hence the contribution to $\frac{\partial D}{\partial z}$ is b times the contribution of the handle deletion type, and is therefore $\sum_{i,j\geqslant 1} bijp_{i+j+2}\frac{\partial^2 D}{\partial p_i \partial p_j}$.

Case (9): $\mathfrak{d}$ has bridge deletion type. Two directed maps, say $\mathfrak{d}^-$ and $\mathfrak{d}^+$, are the components of $\mathfrak{d} - e_n$, where $\mathfrak{d}^-$ and $\mathfrak{d}^+$ contain the corner to which is attached e_n's tail and head, respectively. As in the handle and cross-handle deletion types, two faces of degrees i and j in $\mathfrak{d} - e_n$, are joined to form a face of degree $i+j+2$. There is the same number ij of choices for the corners of attachment. The joining cylinder is from surface containing $\mathfrak{d}^-$ to the surface containing $\mathfrak{d}^+$. The cylinder may preserve the local orientations of the end-vertices of e_n or may reverse them, introducing a factor of 2. The edges in $\mathfrak{d}^-$ and $\mathfrak{d}^+$ are be re-labelled so that the the union of the labels is $\{1, 2, \ldots, n-1\}$. Enumeratively, this relabelling is accounted for by the product lemma (for exponential generating series) since edges in D are marked exponentially by z. Thus the contribution to $\frac{\partial D}{\partial z}$ is $\sum_{i,j\geqslant 1} 2ijp_{i+j+2}\frac{\partial D}{\partial p_i}\frac{\partial D}{\partial p_j}$. □

4.2 Evidence that D is Expressible with Jack Symmetric Functions

The theorem below gives evidence that strongly suggests that D, the generating series for directed maps with respect to β (which is essentially η) marked by b, may be expressible in the "same form" as $\Psi^{(b+1)}$ in (1.1), thereby supporting Conjecture 1.1. This is because the Jack symmetric functions are eigenfunctions of the Laplace-Beltrami operator $\square^\alpha$, which is defined below and shown to take a form relevant to the Map-Jack conjecture.

Lemma 4.2. *Let $\square_n^\alpha$ be an operator whose action on homogeneous symmetric functions of degree r is given by*

$$\square_n^\alpha = -(n-1)r + \sum_{1\leqslant i<j\leqslant n} \frac{\frac{x_i^2\partial}{\partial x_i} - \frac{x_j^2\partial}{\partial x_j}}{x_i - x_j} + \sum_{1\leqslant i\leqslant n} \frac{\alpha}{2}\frac{x_i^2\partial^2}{\partial x_i^2}. \tag{4.4}$$

The limit $\square^\alpha = \lim_{n\to\infty}\square_n^\alpha$ exists [11], and is the Laplace-Beltrami operator. *The value of the limit is may also be expressed as the following operator:*

$$\square^\alpha = \frac{1}{2}\left(\sum_{k\geqslant 1}(\alpha-1)(k-1)p_k k\frac{\partial}{\partial p_k}\right) + \frac{1}{2}\left(\sum_{i,j\geqslant 1} p_i p_j (i+j)\frac{\partial}{\partial p_{i+j}} + \alpha p_{i+j} ij\frac{\partial^2}{\partial p_i \partial p_j}\right). \tag{4.5}$$

Proof. In the following discussion it is convenient, when clear from context, to write p_λ for $p_\lambda(x_1, \ldots, x_n)$. To study the action of $\square_n^\alpha$ on $p_\lambda = p_{\lambda_1}p_{\lambda_2}\ldots$, write $\square_n^\alpha = \delta_n^0 + \delta_n^1 + \delta_n^2$ where $\delta_n^0 = -(n-1)r$, $\delta_n^1 = \sum_{1\leqslant i<j\leqslant n}(\frac{x_i^2\partial}{\partial x_i} - \frac{x_j^2\partial}{\partial x_j})/(x_i - x_j)$ and $\delta_n^2 = \sum_{1\leqslant i\leqslant n}\frac{\alpha}{2}\frac{x_i^2\partial^2}{\partial x_i^2}$ are differential operators of orders zero, one and two, respectively.

Because δ_n^1 is a first order differential operator, its action on the product p_λ is determined by its action on the factors p_k. Clearly $\frac{x_i^2\partial}{\partial x_i}p_k = kx_i^{k+1}$, thus $\delta_n^1 p_k = \sum_{1\leqslant i<j\leqslant n} k\frac{x_i^{k+1} - x_j^{k+1}}{x_i - x_j} = k\sum_{1\leqslant i<j\leqslant n}\sum_{0\leqslant l\leqslant k} x_i^l x_j^{k-l}$,

which is symmetric in the x_i and equal to $k(n-1)p_k+\frac{1}{2}\sum_{1\leqslant l\leqslant k-1}k(p_lp_{k-l}-p_k)$. Now treating the p_j as indeterminates, this can be rewritten in form $\left(\sum_{k\geqslant 1}\frac{k(n-1)p_k\partial}{\partial p_k}-\sum_{k\geqslant 1}\frac{1}{2}\frac{k(k-1)p_k\partial}{\partial p_k}+\frac{1}{2}\sum_{i,j\geqslant 1}\frac{(i+j)p_ip_j\partial}{\partial p_{i+j}}\right)p_k$. Since $\delta_n^0 p_\lambda=-(n-1)|\lambda|p_\lambda=-(n-1)\sum_k\lambda_kp_\lambda$, the operator δ_n^0 equals the first order differential operator $-(n-1)\sum_k kp_k\frac{\partial}{\partial p_k}$, so that the term r is eliminated. Therefore $\delta_n^0+\delta_n^1=-\sum_{k\geqslant 1}\frac{1}{2}\frac{k(k-1)p_k\partial}{\partial p_k}+\frac{1}{2}\sum_{i,j\geqslant 1}(i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}$, so that dependence on n is removed. Direct expansion shows

$$\delta_n^2 p_\lambda=\frac{\alpha}{2}\left(\sum_{k,l}klp_{k+l}\frac{\partial^2}{\partial p_k\partial p_l}+\sum_k k(k-1)p_k\frac{\partial}{\partial p_k}\right)p_\lambda.$$

In this form, $\square_n^\alpha$ does not depend on n, so the limit as $n\to\infty$ is well-defined, with the form the given in (4.5). $\square$

We now derive a differential equation for the generating series E for disconnected maps.

Theorem 4.3. *Let*

$$\nabla^\alpha=\sum_{k\geqslant 0}\left(2\alpha(v+1)-2+(\alpha-1)(k-1)\right)p_{k+2}k\frac{\partial}{\partial p_k}+\sum_{i,j\geqslant 1}p_ip_j(i+j-2)\frac{\partial}{\partial p_{i+j-2}}+\alpha p_{i+j+2}ij\frac{\partial^2}{\partial p_i\partial p_j} \tag{4.6}$$

and $E=e^{(2/\alpha)D}|_{v\mapsto v/2,\ z\mapsto z/2}$, *then*

$$\left(\frac{\partial}{\partial z}-\nabla^\alpha\right)E=\frac{1}{2\alpha}v\left(p_1^2+(\alpha-1)p_2+vp_2\right)E. \tag{4.7}$$

Proof. The proof of (4.7) follows from Theorem 4.1 as detailed below. First note that $\left[z^0\right]D=0$, which ensures that E is well-defined as a power series in z. The following notation will be convenient. Let H_v be the substitution operator for $v\mapsto v/2$. Similarly, let $H_z: z\mapsto z/2$. Let F be $2/\alpha$. Let $C_0=p_1^2v+(\alpha-1)p_2v+2p_2v^2$ and $C_1=4v$. Consider F, C_0, and C_1 to be operators acting by multiplication. Let $d_z=\frac{\partial}{\partial z}$. Let $p_i^\perp=i\frac{\partial}{\partial p_i}$ as in Macdonald [11, Ex. I.5.3(c)]. For two operators f and g let $fg: X\mapsto f(g(X))$ and $f*g: X\mapsto f(X)g(X)$. With this notation $E=e^{FH_vH_zD}$ and (4.3) is

$$\left(d_z-C_1\sum_{i\geqslant 1}\left(p_{i+2}p_i^\perp-\sum_{i,j\geqslant 1}(p_ip_j+(\alpha-1)p_{i+j})p_{i+j-2}^\perp+\alpha p_{i+j+2}p_i^\perp p_j^\perp+2p_{i+j+2}p_i^\perp*p_j^\perp\right)\right)D=C_0. \tag{4.8}$$

The important actions on E are $d_zE=EFH_vd_zH_zD=EFH_vH_z\frac{1}{2}d_zD$, $p_{i+1}p_i^\perp E=EFH_vH_zp_{i+1}p_i^\perp D$, $p_ip_jp_{i+j-2}^\perp E=EFH_vH_zp_ip_jp_{i+j-1}^\perp D$, and $p_{i+j+2}p_i^\perp p_j^\perp E=EFH_vH_zp_{i+j+2}(p_i^\perp p_j^\perp+Fp_i^\perp*p_j^\perp)D$. Move the terms $(\alpha-1)p_{i+j}p_{i+j-2}^\perp$ in the double sum of (4.8) to the single sum as terms $(\alpha-1)(i+1)p_{i+2}p_i^\perp$. Next, apply H_vH_z and multiply the result by EF, to get

$$\left(d_z-\sum_{i\geqslant 1}\left(((i+1)(\alpha-1)+H_vC_1)p_{i+2}p_i^\perp\right)-\sum_{i,j\geqslant 1}\left((p_ip_j)p_{i+j-2}^\perp+\alpha p_{i+j+2}p_i^\perp p_j^\perp\right)\right)E=EFH_vC_0. \tag{4.9}$$

$\square$

The crucial property of Jack symmetric functions that we invoke is that Jack symmetric functions are eigenfunctions of the Laplace-Beltrami operator $\square^\alpha$, which may be re-expressed as $\square^\alpha = \square_0^\alpha$ where

$$\square_c^\alpha = \frac{1}{2}\left(\sum_{k\geqslant 1}(\alpha-1)(k-1)p_{k+c}k\frac{\partial}{\partial p_k}\right) + \frac{1}{2}\left(\sum_{i,j\geqslant 1} p_i p_j (i+j-c)\frac{\partial}{\partial p_{i+j-c}} + \alpha p_{i+j-c} ij \frac{\partial^2}{\partial p_i \partial p_j}\right).$$

Further details on Jack symmetric functions are given by Macdonald [11], Stanley [12], Lapointe and Vinet [8], Knop and Sahi [7], and also [1].

Now

$$\nabla^\alpha = 2\square_2^\alpha + \sum_{k\geqslant 1}\left(\alpha\left(v+1\right)-1\right)p_{k+2}k\frac{\partial}{\partial p_k}.$$

The similarity between ∇^α and the Laplace-Beltrami operator $\square_0^\alpha$ (in the sense that ∇^α is $\square_0^\alpha$ with a shift in the lower argument and an additional term) is strong *combinatorially derived* evidence suggesting how the Jack symmetric functions appear in $\Psi^{(\alpha)}$. Moreover, it also suggests that the differential equation (4.7) for E is central to showing that η is a Map-Jack invariant.

5 Concluding Comments

We have described a candidate Map-Jack invariant η, and have provided evidence for its validity in two ways. First, we have established the special case of the conjecture that η is a Map-Jack invariant, namely $\Upsilon H = \Upsilon\Psi^{(b+1)}$ through Theorems 3.5 and 3.6. Second, we have shown that another special case of this conjecture, namely $\Theta H = \Theta\Psi^{(b+1)}$, can be broken down into two problems, one combinatorial and the other algebraic, for which we have evidence. The combinatorial problem is to prove that $D = \frac{1}{4}\Theta H$. The evidence for this is that β and η are very closely related, and that the relation holds for $b \in \{0,1\}$ and for all b for the low degree terms in z. The algebraic problem is to prove that

$$\left(\frac{\partial}{\partial z} - \nabla^\alpha - \Lambda\right) Z\frac{1}{4}\Theta\Psi^{(b+1)} = 0,$$

where Λ is the operator applied in the right hand side of (4.7). The evidence for this is the close relation between ∇^α and $\square^\alpha$, the correctness of the cases $b=0$ and $b=1$, and the direct calculations for low degrees in z. We believe that further investigation of these two problems will lead to a full understanding of the connexion between η and Jack symmetric functions.

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A Examples of Jack Symmetric Functions

The expansions of Jack symmetric functions $J_\lambda^{(\alpha)}$ into the power sum basis for $|\lambda| \leqslant 4$ are

$$\begin{aligned}
J_{\varnothing}^{(\alpha)} &= 1, & J_{(1^2)}^{(\alpha)} &= p_{(1^2)} - p_{(2)}, & J_{(1^3)}^{(\alpha)} &= p_{(1^3)} - 3p_{(21)} + 2p_{(3)},\\
J_{(1)}^{(\alpha)} &= p_{(1)}, & J_{(2)}^{(\alpha)} &= p_{(1^2)} + \alpha p_{(2)}, & J_{(21)}^{(\alpha)} &= p_{(1^3)} + (\alpha-1)p_{(21)} - \alpha p_{(3)},\\
& & & & J_{(3)}^{(\alpha)} &= p_{(1^3)} + 3\alpha p_{(21)} + 2\alpha^2 p_{(3)},
\end{aligned}$$

$$\begin{array}{lllllll}
J^{(\alpha)}_{(1^4)} & = p_{(1^4)} & -6p_{(21^2)} & +3p_{(2^2)} & +8p_{(31)} & -6p_{(4)}, \\
J^{(\alpha)}_{(21^2)} & = p_{(1^4)} & +(\alpha-3)p_{(21^2)} & -\alpha p_{(2^2)} & -2(\alpha-1)p_{(31)} & +2\alpha p_{(4)}, \\
J^{(\alpha)}_{(2^2)} & = p_{(1^4)} & +2(\alpha-1)p_{(21^2)} & +(\alpha^2+\alpha+1)p_{(2^2)} & -4\alpha p_{(31)} & -\alpha(\alpha-1)p_{(4)}, \\
J^{(\alpha)}_{(31)} & = p_{(1^4)} & +(3\alpha-1)p_{(21^2)} & -\alpha p_{(2^2)} & +2\alpha(\alpha-1)p_{(31)} & -2\alpha^2 p_{(4)}, \\
J^{(\alpha)}_{(4)} & = p_{(1^4)} & +6\alpha p_{(21^2)} & +3\alpha^2 p_{(2^2)} & +8\alpha^2 p_{(31)} & +6\alpha^3 p_{(4)}.
\end{array}$$

We now calculate $\psi^{(b+1)}_{(4),(4)}$. By the algebraic independence of power sum symmetric functions we substitute $p_j(\mathbf{x}) = p_j(\mathbf{y}) = 0$ for $j \neq 4$ and $p_j(\mathbf{z}) = 0$ for $j \neq 2$ in $\Psi^{(b+1)}$. This substitution does not affect the coefficient $\psi^{(b+1)}_{(4),(4)}$. Values of n other than 0 and 4 in (1.1) do not contribute to $\psi^{(b+1)}_{(4),(4)}$. The terms $\left\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda \right\rangle_\alpha$ can be calculated directly by means of a formula (see Macdonald [11]). Noting that $\left[z^2\right] 2\alpha \frac{z\partial}{\partial z} \log(1+z^2A) = 4\alpha A$, we have

$$\psi^{(b+1)}_{(4),(4)} = 4\alpha\left(\frac{9}{2(3+\alpha)(2+\alpha)(1+\alpha)\alpha} - \frac{\alpha}{(3+\alpha)(1+\alpha)^2} + \frac{(\alpha-1)^2(\alpha^2+\alpha+1)}{4(2+\alpha)(2\alpha+1)(1+\alpha)^2} - \frac{\alpha^2}{(1+\alpha)^2(3\alpha+1)} + \frac{9\alpha^4}{2(3\alpha+1)(2\alpha+1)(1+\alpha)}\right) \tag{A.1}$$

which simplifies to $3\alpha^2 - 5\alpha + 3 = 3b^2 + b + 1$.

B Further Examples of Interesting Map-Jack Coefficients

We define the *rooting partition* $\rho^n_{\phi,\nu}$ to have a part for unrooted map $\mathfrak{u}$ in the set $\mathfrak{N}_{\phi,\nu} = \mathfrak{L}_{\phi,\nu} \setminus \mathfrak{O}_{\phi,\nu}$ of size equal to the number of rootings of $\mathfrak{u}$. Thus $\rho^n_{\phi,\nu} \vdash \psi^{(2)}_{\phi,\nu} - \psi^{(1)}_{\phi,\nu}$. Comparison of $\rho^n_{\phi,\nu}$ with $\psi^{(b+1)}_{\phi,\nu}$ can be used to show that in many cases that a Map-Jack invariant cannot be an invariant of unrooted maps. If the partition formed by collecting the coefficients of nonzero degree terms of $\psi^{(b+1)}_{\phi,\nu}$ cannot be refined into the partition $\rho^n_{\phi,\nu}$ in the sense the latter cannot be obtained by breaking the parts of the former into smaller parts, then a Map-Jack invariant cannot be an invariant of unrooted maps for $\mathfrak{N}_{\phi,\nu}$. As shown earlier, the coefficient $\psi^{(b+1)}_{(4),(4)} = 3b^2 + b + 1$ gives partition (31) which cannot be refined into $\rho^n_{(4),(4)} = (2^2)$. Other indices (ϕ,ν) with $\rho^n_{\phi,\nu}$ and $\psi^{(b+1)}_{\phi,\nu}$ such that the former is not a refinement of the latter include: $((3^2),(6))$ with (3^2) and $1+b+5b^2$; $((42^2),(8))$ with (84^4) and $6+6b+18b^2$; $((4^2),(4^2))$ with (42^4) and $3+3b+9b^2$; $((6),(6))$ with $(12^1 6^3 3^3 2)$ and $13b+13b^2+15b^3$; $((4^2),(8))$ with $(8^6 4^3 2)$ and $19b+19b^2+24b^3$; and $((8),(8))$ with $(16^{19} 8^{19} 4^7 2^2)$ and $21(1+b)^2 + 160b^2(1+b) + 105b^4$. More rooting partitions are given in [2] and these partitions were useful in compiling the polygonal representations of some of the maps in [6].

C Illustration of Theorem 4.1

We now illustrate the use of Theorem 4.1 by first determining the initial terms of D and then checking that these correspond to the appropriate sets of maps. Let $D = \sum_{n\geqslant 1} D_n \frac{z^n}{n!}$. The partial differential equation (4.3) gives a recurrence equation for D_{n+1} in terms of $D_1, \ldots, D_n$. Since $\left[z^0\right] \frac{\partial}{\partial z} D = D_1$, it follows immediately that $D_1 = vp_{(1^2)} + \left(2v^2 + bv\right) p_{(2)}$. Similarly, $D_2 = [z] \frac{\partial}{\partial z} D$, so we have immediately that

$$D_2 = \underbrace{4p_3v\frac{\partial D_1}{\partial p_1} + 4\cdot 2p_4v\frac{\partial D_1}{\partial p_2}}_{\text{leaf deletion type: } i=1 \text{ and } i=2} + \underbrace{2(1+2-2)(p_1p_2+bp_3)\frac{\partial D_1}{\partial p_1}}_{\text{border deletion type: } \{i,j\}=\{1,2\}} + \underbrace{(2+2-2)(p_2^2+bp_4)\frac{\partial D_1}{\partial p_2}}_{\text{border deletion type: } i=j=2} +$$
$$\underbrace{2(1+3-2)(p_1p_3+bp_4)\frac{\partial D_1}{\partial p_2}}_{\text{border deletion type: } \{i,j\}=\{1,3\}} + \underbrace{p_4(b+1)\frac{\partial^2}{\partial p_1^2}D_1}_{\text{handle and cross-handle deletion types: } i=j=1} \tag{C.1}$$

and thus

$$D_2 = 4vp_{(21^2)} + \left(4v^2+2bv\right)p_{(2^2)} + \left(16v^2+8bv\right)p_{(31)} + \left(16v^3+20bv^2+2(1+b+3b^2)v\right)p_{(4)}. \tag{C.2}$$

The coefficients D_1 and D_2 may be confirmed independently by listing the directed maps that they enumerate. Four directed maps are enumerated by D_1. They are given in Figures 4.1(a)–(c). Three of these are in the sphere, one of loop deletion type (see Figure 4.1(a)) and two of link deletion type (see Figure 4.1(c)). The directed maps of link deletion type may be distinguished by whether the edge preserves or reverses local orientation. The value of β on these three maps is zero and the face partition for the loop is (1^2), while the face partition for the links are each (2). The fourth directed map is in the real projective plane and has cross-loop deletion type (see Figure 4.1(b)). The value of β is one and the face partition is (2). Noting the number of vertices immediately confirms the value of D_1 given above.

Eighty directed maps are enumerated by D_2. The eleven distinct unrooted maps with two edges are given in Table C.1. Each of these may be directed in a certain number of ways. Different ways of directing an unrooted map may give a different value for β. These eleven maps and the additional information about directing them and the resulting contributions to D_2 are given in Table C.1, together with their index number [6]. The contribution to D_2 from β from each unrooted map is determined as follows.

Unrooted Map	Index in [6]	ϕ	k	n	Contribution to $\left[p_\phi v^k \frac{z^n}{n!}\right] D$
$\mathfrak{u}_1$	$0\cdot 3$	(21^2)	1	2	4
$\mathfrak{u}_2$	$0\cdot 5$	(2^2)	2	2	4
$\mathfrak{u}_3$	$0\cdot 4$	(31)	2	2	16
$\mathfrak{u}_4$	$0\cdot 6$	(4)	3	2	16
$\mathfrak{u}_5$	$\tilde{1}\cdot 3$	(2^2)	1	2	$2b$
$\mathfrak{u}_6$	$\tilde{1}\cdot 2$	(31)	1	2	$8b$
$\mathfrak{u}_7$	$\tilde{1}\cdot 5$	(4)	2	2	$4b$
$\mathfrak{u}_8$	$\tilde{1}\cdot 4$	(4)	2	2	$16b$
$\mathfrak{u}_9$	$1\cdot 1$	(4)	1	2	2
$\mathfrak{u}_{10}$	$\tilde{2}\cdot 1$	(4)	1	2	$2b+2b^2$
$\mathfrak{u}_{11}$	$\tilde{2}\cdot 2$	(4)	1	2	$4b^2$

Table C.1: Unrooted maps with two edges

$\mathfrak{u}_1$–$\mathfrak{u}_4$, $\mathfrak{u}_9$: These unrooted maps are orientable, so $\beta(\mathfrak{d}) = 0$ for each directed map $\mathfrak{d}$ associated with one of these unrooted maps.

$\mathfrak{u}_5$–$\mathfrak{u}_8$: These non-orientable maps have $\chi = 1$, so $\beta(\mathfrak{d}) = 1$ for each directed map $\mathfrak{d}$ associated with one of these unrooted maps.

$\mathfrak{u}_{10}$: There are four directed maps $\mathfrak{d}$ associated with this unrooted map. For two of these, e_1 has link type and e_2 has cross-handle type, so $\beta(\mathfrak{d}) = 1$. For the other two e_1 has cross-loop type and e_2 has cross-border type, so $\beta(\mathfrak{d}) = 2$.

$\mathfrak{u}_{11}$: For any directed map $\mathfrak{d}$ associated with this unrooted map, e_1 has cross-loop and and e_2 has cross-border types, so $\beta(\mathfrak{d}) = 2$.

D Glossary of Notation

$\langle \cdot, \cdot \rangle_\alpha$	A scalar product defined on Λ.
α	A parameter of Jack symmetric functions, with $\alpha = b + 1$.
$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$	The five rooted maps in $\mathfrak{L}_{(4)(4)}$. A block is also indicated by $\mathfrak{b}$.
b	An indeterminate marking invariants η and β, with $b = \alpha - 1$.
β	An invariant of directed maps.
$\square_0^\alpha$	The Laplace-Beltrami operator.
$\Gamma(\mathfrak{m})$	The matchings graph of a map $\mathfrak{m}$.
D	A generating series for directed maps defined in (4.1) with b marking β.
$\mathfrak{D}$	The set of directed maps.
$\mathfrak{d}$	A directed map, or one of the rooted maps in $\mathfrak{L}_{(4),(4)}$.
δ	A mapping from rooted maps to directed maps.
e_i	An edge of a map, in particular the edge with label i in a directed map.
$\mathfrak{e}$	An edge diagram, or one of the rooted maps in $\mathfrak{L}_{(4),(4)}$.
η	The proposed Map-Jack invariant: $\eta = \beta \circ \delta$.
$\phi(\mathfrak{m})$	The face partition of the map $\mathfrak{m}$.
$G(\mathfrak{m})$	The underlying graph of a map.
H	A generating series $H = H(\mathbf{x}, \mathbf{y}, z; b)$ for rooted maps in locally orientable surfaces, with b marking η.
$J_\lambda^{(\alpha)}$	Jack symmetric function indexed by partition λ.
$\mathfrak{L}$	The set of locally orientable rooted maps.
$\mathfrak{L}_{\phi,\nu}$	The subset of $\mathfrak{L}$ with face partition ϕ and vertex partition ν.
$l_{\phi,\nu}$	The number of maps in $\mathfrak{L}_{\phi,\nu}$.
Λ	The ring of symmetric functions. Also an operator on symmetric functions.
λ, ϕ, ν	Partitions
ℓ, $\ell_\mathfrak{m}$	The canonical labelling function of a map $\mathfrak{m}$
$\ell(\lambda)$	The length $\sum_{j \geqslant 1} m_j(\lambda)$ of the partition λ.
$\lvert\lambda\rvert$	The weight $\sum_{i \geqslant 1} \lambda_i$ of the partition λ.
$\lambda \vdash n$	The partition λ has weight n.
$M(X)$	The set of matchings on a set X.
$\mathfrak{M}$	A topological map.
$\mathfrak{m}$	A premap, a map, or a rooted map.
$\mathfrak{m} \downarrow_Y$	A submap induced by a set Y.
$m_j(\lambda)$	The multiplicity of part j in a partition λ.

m_λ	Monomial symmetric function indexed by partition λ
μ_1, μ_2, μ_3	The face-move, edge-move and vertex-move matchings of a map or premap.
$\mathfrak{N}_{\phi,\nu}$	The set $\mathfrak{L}_{\phi,\nu} \setminus \mathfrak{O}_{\phi,\nu}$ of non-orientable maps with vertex and face partition ϕ and ν, respectively.
$n(\mathfrak{m})$	The number of edges of the map $\mathfrak{m}$.
$\mathfrak{O}$	The set of orientable rooted maps.
$\mathfrak{O}_{\phi,\nu}$	The subset of $\mathfrak{O}$ with face partition ϕ and vertex partition ν.
o	A function sending a locally orientable edge diagram to an orientable one.
$o_{\phi,\nu}$	The number of maps in $\mathfrak{O}_{\phi,\nu}$.
p_λ	The product $p_{\lambda_1} p_{\lambda_2} \ldots$ of indeterminates p_j, which can mark faces or vertices. Also, the power sum symmetric function indexed by partition λ.
$\Psi^{(b+1)}$	A power series involving Jack symmetric functions defined in (1.1).
$\psi_{\phi,\nu}^{(b+1)}$	A coefficient of the power series $\Psi^{(b+1)}$.
$\mathbb{Q}(\alpha)$	The field of rational functions in indeterminate α.
$\mathfrak{r}$	A canonical rooted map.
$\mathbb{S}(\mathfrak{m})$	The underlying surface of a map.
$\mathfrak{s}$	A submap.
ϑ	An arbitrary Map-Jack invariant.
$\frac{1}{4}\Theta$	A power series transformation for switching from rooted maps to directed maps
$\mathfrak{u}$	An unrooted map.
Υ	A power series transformation for suppressing face partition
v, y	Indeterminates used to mark vertices.
$\nu(\mathfrak{m})$	The vertex partition of the map $\mathfrak{m}$.
∇^α	A differential operator.
X	The set of positions of a map.
$\chi(\mathfrak{m})$	The Euler characteristic of a map $\mathfrak{m}$.
$\mathbf{x}$	A family of indeterminates $x_1, x_2, \ldots$ where $p_j(\mathbf{x})$ marks degree j faces.
$\mathbf{y}$	A family of indeterminates $y_1, y_2, \ldots$ where $p_j(\mathbf{y})$ marks degree j vertices.
z	An indeterminate used to mark edges.