The virtual Euler characteristic of the moduli spaces of real and complex algebraic curves

I.P.Goulden[*], J.L.Harer[†] and D.M.Jackson[‡]

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Abstract

We determine the virtual Euler characteristics of the moduli spaces of s-pointed real and complex algebraic curves. In particular, for the space of real curves of genus g with a fixed point free involution, we find that the Euler characteristic is $(-2)^{s-1}(1-2^{g-1})(g+s-2)!B_g/g!$ where B_g is the gth Bernoulli number. This complements the result of Harer and Zagier that the Euler characteristic of the moduli space of complex algebraic curves is $(-1)^s(g+s-2)!B_{g+1}/(g+1)(g-1)!$

The proof uses Strebel differentials to triangulate the moduli spaces and some recent enumerative techniques to count cells. The approach involves a parameter that permits specialization of the formula to the real and complex cases. This suggests that the general expression itself may describe the Euler characteristics of some related moduli spaces, although we do not yet know what these spaces might be.

Contents

[*]Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1

[†]Department of Mathematics, Duke University, Durham, NC 27708-0320, U.S.A.

[‡]Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1

1 Introduction

The moduli space $\mathcal{M}_g^s$ of s-pointed complex curves of genus g has been an important object of study in many branches of mathematics for a long time. The global structure of these spaces is only partially understood; in particular, the cohomology of $\mathcal{M}_g^s$ is known in just a few cases. The primary tool for studying this cohomology has been a triangulation of $\mathcal{M}_g^s$ which is derived from work of Strebel on quadratic differentials by Harer [8]. In particular, the Euler characteristic of $\mathcal{M}_g^s$ was computed by Harer and Zagier [9] using this triangulation and an associated matrix model to count cells. Since that time, much work has been done in applying other matrix models and variants of the Strebel triangulation to the study of moduli spaces. This includes the alternative formulation by Penner [21] of the work in [9], the character-theoretic approach to the same work by Jackson [14], the exposition by Itzykson and Zuber [12], and most recently the beautiful work of Kontsevich [17] in establishing a conjecture of Witten [27] on intersection numbers of cohomology classes of $\overline{\mathcal{M}}_g^s$.

The moduli spaces of real algebraic curves have only come into the foreground more recently. Classically, a real algebraic curve C is the set of simultaneous zeros in $\mathbb{R}P^n$ of a collection of homogeneous polynomials. By taking the solution in $\mathbb{C}P^n$ of the same set of polynomials, we can associate to the curve a Riemann surface and an antiholomorphic involution on it (induced by complex conjugation) whose fixed point set is the real curve C. This suggest *defining* a real curve as a pair consisting of a complex curve X and an antiholomorphic involution τ and it is this point of view that we shall use here. For a fixed topological type of orientation reversing involution τ, there is then a moduli space $\mathcal{M}_g^s(\tau)$ whose points are isomorphism classes of pairs of this form. Seppälä, Silhol, Buser and others [2, 23, 24] have made some progress in understanding these spaces, but up to now little is known about the topology of $\mathcal{M}_g^s(\tau)$. In the present work we will apply the technique of the Strebel differential to obtain a triangulation of $\mathcal{M}_g^s(\tau)$. We then count cells using enumerative techniques similar to those that have been used to study $\mathcal{M}_g^s$.

Fortunately, work by Goulden and Jackson already exists in just the direction needed and provides the foundation for the current paper. The point of departure is to consider the generating series, M^O and M, for graphs embedded in orientable surfaces and in general surfaces, respectively. We call these combinatorial objects *maps.* M^O and M have been given in integral form as evaluations of expectation operators over Hermitian complex matrices and real symmetric matrices. For orientable surfaces these are familiar from the work of 't Hooft [11] and Bessis, Itzykson and Zuber [1], while the corresponding result for surfaces is given by Goulden and Jackson [4]. In the particular case of specialisation to the virtual Euler characteristics of the moduli spaces, the diagonalized forms of the associated expectations have a common generalisation as a Selberg integral whose evaluation is known [22]. The virtual Euler characteristics for $\mathcal{M}_g^s$ and $\mathcal{M}_g^s(\tau)$ are computed simultaneously by specialising the parameter, γ, that appears in the Selberg integral to $\gamma = 1$ and $\gamma = \frac{1}{2}$, respectively.

We conjecture that the parameter γ itself has geometric significance, perhaps that there is an appropriately defined moduli space involving γ that specializes to $\mathcal{M}_g^s$ and $\mathcal{M}_g^s(\tau)$ at $\gamma = 1$ and $\gamma = \frac{1}{2}$, respectively. This is related to a recent combinatorial conjecture for maps in Goulden and Jackson [3], in which γ is a parameter of nonorientability for maps. Moreover, $1/\gamma$ has a separate existence as the parameter indexing the Jack symmetric functions, which have been studied quite intensively in the combinatorial community in the last decade.

The paper is organized as follows. Section 2 gives the reduction of the calculation of the virtual Euler characteristic $\chi(\mathcal{M}_g^s(\tau_0))$ to the determination of the generating series for maps at particular settings of the variables. Integral expressions for the genus series are given in Section 3. In Section 4 we carry out this task in a general setting that specializes to both the complex case and the real case through the choice of the parameter γ. In Section 5 we discuss the combinatorial significance of γ and conjecture its role in connection with moduli spaces of curves.

2 The moduli spaces of real curves

2.1 The Teichmüller spaces and mapping class groups of nonorientable surfaces

By a real algebraic curve we will mean a pair (X, σ) where X is a Riemann surface and σ is an antiholomorphic involution. Two such curves (X_1, σ_1) and (X_2, σ_2) are isomorphic if there is a biholomorphism $\phi : X \to Y$ such that $\phi\sigma_1 = \sigma_2\phi$.

Fix a closed oriented surface F of genus g and let τ be an orientation reversing involution of F. The fixed point set $\mathsf{Fix}(\tau)$ of τ is a collection (possibly empty) of m disjoint simple closed curves which separates F into $\epsilon = 1$ or 2 connected components. If τ_1 is another such involution, τ is conjugate to τ_1 if and only if $m = m_1$ and $\epsilon = \epsilon_1$ (compare [23]). There are therefore $[(3g+4)/2]$ conjugacy classes of orientation reversing involutions on F. $\mathsf{Fix}(\tau)$ is non-separating in $g+1$ cases, while in the remaining cases $\mathsf{Fix}(\tau)$ separates F into two components, each of genus $(g-m+1)/2$.

Choose $2s$ distinct, ordered points $\{q_i\}$ on F and let τ be an orientation reversing involution of F which acts without fixed points on the $\{q_i\}$. We define the Teichmüller space $\mathcal{T}_g^s(\tau)$ to be the space of all isomorphism classes of quadruples $(X, \sigma, \{p_i\}, f)$ where (X, σ) is a real algebraic curve, $\{p_i\}$ is an ordered collection of $2s$ distinct points of X on which σ acts freely and

$$f : (X, \{p_i\}) \to (F, \{q_i\})$$

is a homeomorphism preserving the ordering of the points such that $\tau f = f\sigma$. (F is called a *marking* of $(X, \sigma, \{p_i\})$.) The quadruples $(X, \sigma, \{p_i\}, f)$ and $(Y, \mu, \{r_i\}, g)$ are isomorphic if there exists an isomorphism ϕ from (X, σ) to (Y, μ) with $\phi(p_i) = r_i$ for each i such that $g\phi$ is homotopic to f rel $\{p_i\}$.

Since the automorphism group of any Riemann surface X is finite, there are only finitely many antiholomorphic involutions σ of X (any two differ by an automorphism). The natural map $\mathcal{T}_g^s(\tau) \to \mathcal{T}_g^{2s}$ which forgets the involution is therefore finite-to-one. Topologize $\mathcal{T}_g^s(\tau)$ via this map. By choosing an appropriate pair of pants decomposition of F, it is straightforward to use Fenchel-Nielsen coordinates to show that $\mathcal{T}_g^s(\tau)$ is homeomorphic to Euclidean space of dimension $3g-3+2s$. (Here $3g-3$ dimensions come from the isomorphism class of the real curve while $2s$ more dimensions come from the locations of s of the points and these determine that of the others.)

It can be shown that when $(X, \sigma, \{p_i\}, f)$ is isomorphic to $(Y, \mu, \{r_i\}, g)$ using the definition above, the homotopy from $g\phi$ to f can be replaced by an isotopy H_t such that at each time t, $H_t(p_i) = q_i$ and $H_t\sigma = \tau H_t$. Therefore, if we let

$$\pi_X : X \to X/\sigma$$

and

$$\pi_F : F \to F/\tau$$

be the quotient maps, $\overline{p}_i = \pi(p_i)$ and $\overline{q}_i = \pi(q_i)$, the isomorphism class of $(X, \sigma, \{p_i\}, f)$ is determined uniquely by $(X/\sigma, \{\overline{p}_i\})$ and the homotopy class $\mathsf{rel}\ \{\overline{p}_i\}$ of the homeomorphism of pairs $(X/\sigma, \{\overline{p}_i\}) \to (F/\tau, \{\overline{q}_i\})$ induced by f. Thus $\mathcal{T}_g^s(\tau)$ may be thought of as the Teichmüller space of non-orientable surfaces of genus g (Euler characteristic $1-g$) with s ordered points and a local orientation of the surface at each point.

Next, recall that the mapping class group Γ_g^{2s} is the group of isotopy classes $\mathsf{rel}\ \{q_i\}$ of self-homeomorphisms of F that fix each q_i. We define the mapping class group $\Gamma_g^s(\tau)$ to be the subgroup of Γ_g^{2s} consisting of all mapping classes which admit a representative which commutes with τ. It can be shown that two such representatives are homotopic $\mathsf{rel}\ \{q_i\}$ if and only if they are isotopic via an isotopy which commutes with τ at each time. Since every self-homeomorphism of F/τ has a unique orientation preserving lift to F which commutes with τ (the two lifts differ by τ which is orientation reversing), $\Gamma_g^s(\tau)$ may be identified with the group of isotopy classes ($\mathsf{rel}\ \{\overline{q}_i\}$) of self-homeomorphisms of F/τ which preserve each point $\overline{q}_i$ *and* preserve a local orientation of F/τ at each $\pi(q_i)$. (Notice that unlike the usual mapping class group of a surface with boundary, we allow boundary components of F/τ to be permuted and we do *not* require that homeomorphisms or isotopies of homeomorphisms fix boundary curves pointwise.) The group $\Gamma_g^s(\tau)$ acts properly discontinuously on $\mathcal{T}_g^s(\tau)$ via the usual action of Γ_g^{2s} on $\mathcal{T}_g^{2s}$. The quotient will be denoted $\mathcal{M}_g^s(\tau)$ and called the moduli space of s-pointed real algebraic curves. It is an orbifold (V-manifold) of real dimension $3g-3+2s$ and, like the moduli space of complex curves, it has a finite cover which is an ordinary manifold. When two involutions of F are conjugate, there is a natural identification of one moduli space with the other, so there are $[(3g+4)/2]$ such moduli spaces for each g and s.

2.2 The orbifold Euler characteristics

The Euler characteristic of an orbifold $\mathcal{M}$ which has a finite manifold branched covering $\widetilde{\mathcal{M}}$ of degree d is defined to be

$$\chi(\mathcal{M}) = \frac{\chi(\widetilde{\mathcal{M}})}{d}$$

where $\chi(\widetilde{\mathcal{M}})$ is the ordinary Euler characteristic of $\widetilde{\mathcal{M}}$. This number is independent of the choice of $\widetilde{\mathcal{M}}$ because Euler characteristics multiply by degree for unbranched coverings. Recall that the Euler characteristic of a group G which is virtually torsion free is defined similarly,

$$\chi(G) = \frac{\chi(\tilde{G})}{d}$$

where $\tilde{G}$ is a torsion free subgroup of finite index d in G. Since $\mathcal{T}_g^s(\tau)$ is contractible and $\Gamma_g^s(\tau)$ acts properly discontinuously on $\mathcal{T}_g^s(\tau)$ with quotient $\mathcal{M}_g^s(\tau)$,

$$\chi(\mathcal{M}_g^s(\tau)) = \chi(\Gamma_g^s(\tau)).$$

2.3 Statement of the main result

We now state the main result of the present paper, and one of its corollaries. Let τ_0 be an orientation reversing involution of F without fixed points. Let B_g be the gth Bernoulli number, defined by the exponential generating series

$$B(t) = \frac{t}{e^t - 1} = \sum_{j\geq 0} B_j \frac{t^j}{j!}. \tag{1}$$

Theorem 2.1. *For $g \geq 1, s \geq 0, g+s > 1$,*

$$\chi(\mathcal{M}_g^s(\tau_0)) = (-2)^{s-1}(1-2^{g-1})\frac{(g+s-2)!}{g!}B_g.$$

From this theorem it is clear that $\chi(\mathcal{M}_g^s(\tau_0)) = 0$ when g is odd and $g+s>1$ since $B_{2n+1} = 0$ for $n \geq 1$. The group $\Gamma_1(\tau_0)$ is the subgroup of Γ_1 consisting of all maps which commute with τ_0. Now Γ_1 is isomorphic to $SL_2(\mathbb{Z})$ since every element of Γ_1 is determined by the map it induces on $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. The map τ_0 induces

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

on $\pi_1(T^2)$, so $\Gamma_1(\tau_0)$ is simply $\{\pm I\}$. Since Γ_0 and Γ_0^2 are trivial, so are $\Gamma_0(\tau_0)$ and $\Gamma_0^1(\tau_0)$. Also, π_0^2 is infinite cyclic, isomorphic to $\Gamma_0^2(\tau_0)$. Therefore the above theorem may be completed with the statements:

$$\chi(\mathcal{M}_1^0(\tau_0)) = \tfrac{1}{2}, \quad \chi(\mathcal{M}_0^s(\tau_0)) = 1 \quad \text{for } s = 0 \text{ or } 1, \quad \chi(\mathcal{M}_0^s(\tau_0)) = 0 \quad \text{for all } s \geq 2.$$

For all g and s with $g+s \geq 1$ there are exact sequences

$$1 \to \pi_g^s \to \Gamma_g^{s+1}(\tau_0) \to \Gamma_g^s(\tau_0) \to 1$$

where π_g^s is the fundamental group of the orientable surface of genus g with $2s$ points removed. The map

$$\Gamma_g^{s+1}(\tau_0) \to \Gamma_g^s(\tau_0)$$

is obtained by forgetting one of the pairs of points, say (q_{2s+1}, q_{2s+2}), which are interchanged by τ_0. Its kernel is π_g^s because the isotopy to the identity of an element of $\Gamma_g^{s+1}(\tau_0)$ which fixes $\{q_1, ..., q_{2s}\}$ creates and is created by moving the point q_{2s+1} along a loop representing an element of the fundamental group of $F - \{q_1, ..., q_{2s}\}$. (F occurs here instead of F/τ_0 because the points q_i are not permuted. Equivalently, the self-maps of F/τ which we consider are all locally orientable at the points $\pi(q_i)$, so only orientation preserving loops lie in $\Gamma_g^{s+1}(\tau_0)$.)

Now, given any short exact sequence of groups

$$1 \to A \to B \to C \to 1$$

we have

$$\chi(B) = \chi(A)\chi(C).$$

This means that for each g, Theorem 2.1 only needs to be proven for one value of s. We will, in fact, show it for all $s \geq 1$, as this is the more natural approach. The short exact sequence is necessary, however, to establish the case where $s = 0$.

For the involutions τ which have fixed points, let us consider separately the cases where the fixed point sets do or do not separate F. If $\mathsf{Fix}(\tau)$ consists of m disjoint simple closed curves which together do not separate F, then F/τ is a non-orientable surface with n boundary components. The Dehn twist on a boundary curve of F/τ is isotopic to the identity since it lifts to two Dehn twists on isotopic curves with opposite twist directions. In addition, elements can permute the components of $\mathsf{Fix}(\tau)$. Therefore $\Gamma_{g-m}^{s+m}(\tau_0)$ may be identified with a subgroup of index $m!$ in the mapping class group of $(F/\tau, \{\pi(q_i)\})$. The Euler characteristic of the former is given by Theorem 2.1 above, that of the latter is obtained by dividing by $m!$

If $\mathsf{Fix}(\tau)$ consists of m disjoint simple closed curves which together do separate F, then F/τ is an orientable surface with m boundary components of genus h, where $g = 2h + m - 1$. In this case the ordinary mapping class group Γ_h^{s+m} may be identified with a subgroup of index $m!$ in the mapping class group of $(F/\tau, \{\pi(q_i)\})$. The Euler characteristic of Γ_h^{s+m} was computed in [9]. Combining all of these results we have the following general statement.

Corollary 2.2. *If $\mathsf{Fix}(\tau)$ is non-separating and consists of m simple closed curves then*

$$\chi(\Gamma_g^s(\tau)) \quad = \quad (-2)^{s+m-1}(1 - 2^{g-m-1})\frac{(g+s-2)!}{m!(g-m)!}B_{g-m}.$$

If $\mathsf{Fix}(\tau)$ is separating and consists of m simple closed curves then $g - m + 1$ must be even and

$$\chi(\Gamma_g^s(\tau)) = (-1)^{s+m}\frac{(g-m+s-2)!}{m!(g-m+1)(g-m-1)!}B_{g-m+1}$$

2.4 Moduli spaces and polygonal identifications

We embark on the proof of the main result Theorem 2.1, by first expressing $\chi(\mathcal{M}_g^s(\tau_0))$ in terms of pairings of sides of polygons. Recall [26] that for every k-pointed Riemann surface $(X, \{p_1, \ldots, p_k\})$ and every collection of weights $(t_1, \ldots, t_k)$ with $t_i > 0$ and $t_1 + \ldots + t_k = 1$, there is a unique quadratic differential which has double poles at the points p_i of type $-dz^2/z^2$, no other poles, closed real trajectories and connected singular trajectories with the distance (in the norm defined by the quadratic differential) from p_i to the singular trajectory equal to t_i. This concept was first used in [8] (see also [7]) to describe cells for the moduli space of curves. The singular trajectories of the differential then decompose X into k disks which, after scaling X, can be taken to be radius t_i, centered at each p_i. This exhibits $(X, \{p_1, \ldots, p_k\})$ as obtained in a unique manner by identifying the sides of k disks D_i in $\mathbb{C}$ (compare [7] and [8]). Fixing an identification pattern with a marking and varying the edge lengths determines an open cell in $\mathcal{T}_g^k \times \Delta^{k-1}$, where Δ^{k-1} is a $k-1$ simplex.

Suppose now that X admits a fixed point free involution τ_0 which interchanges the points p_{2i-1} and p_{2i} for $i = 1, \ldots, s$. It follows directly from Strebel's work that the real and imaginary trajectories of the differential are invariant under τ_0. (This fact was first pointed out to us by Seppälä.) This means that $\mathcal{T}_g^s(\tau_0) \times \Delta^{s-1}$ admits an ideal triangulation with cells determined as follows: Choose positive integers $n_1, \ldots, n_s$ with $n_1 + \ldots + n_s = 2n$. For each i take two copies P_i^+ and P_i^- of a polygon with n_i sides. Also take a pairing ω of the sides of $\cup_{1 \le i \le s} P_i^+ \cup P_i^-$ which gives an oriented, connected surface of genus g and is compatible with the interchange maps $P_i^+ \longleftrightarrow P_i^-$. These maps together induce an involution σ_ω of $(\cup_{1 \le i \le s} \partial P_i^+ \cup \partial P_i^-)/\omega$ We assume that σ_ω is orientation reversing. Finally, this pairing (as in [8] and [9]) must satisfy the requirement that the valence of each vertex of the boundary graph $(\cup_{1 \le i \le s} \partial P_i^+ \cup \partial P_i^-)/\omega$ is at least 3. Finally, choose a marking f_ω of the result that identifies the center of P_i^+ with q_{2i-1} and the center of P_i^- with q_{2i}. Varying the edge lengths determines an open cell whose dimension is $n-1$. Orbits of cells are determined by the identification alone (without the marking). This gives an *ideal* triangulation in the following sense: Each marked identification determines an open cell, corresponding to positive lengths on each edge. By allowing an edge length to go to 0, we obtain a cell of one lower dimension as long as the resulting identification still gives a surface of genus g. If, however, the result has smaller genus, the face lies outside of $\mathcal{T}_g^s(\tau_0) \times \Delta^{s-1}$.

By forming the quotient of X_ω by τ_0, the identification ω induces an identification ω' of the union of s polygons P_i having n_i edges to obtain a connected, non-orientable surface X'_ω of Euler characteristic $1-g$. (The boundary graph has n edges and $1+n-g-s$ vertices and the same valence condition as above, since τ_0 is fixed point free.) The marking f_ω induces a marking f'_ω of X'_ω which pairs the ordered center points $\{\overline{p}_1, \ldots, \overline{p}_s\}$ with $\{\overline{q}_1, \ldots, \overline{q_s}\}$.

Conversely, given an identification ω' of the edges of $\cup_{1 \le i \le s} P_i$ to obtain a non-orientable surface X'_ω of Euler characteristic $1-g$, with special points $\{\overline{p}_1, \ldots, \overline{p}_s\}$, and given a marking f'_ω of X'_ω with $f'_\omega(\overline{p}_i) = \overline{q}_i$, there are 2^{s-1} corresponding choices of the pairing ω and marking f_ω. To see this, visualize the disks P_i^+ and P_i^- in a common plane so that assigning U for an up normal or D for a down normal orients them. Fix U as the orientation for P_1^+ and D for P_1^-. Arbitrarily assign U or D to P_i^+ and the opposite to P_i^- for $i = 2, \ldots, s$. Now there is one and only one identification ω lifting ω' which gives a surface on which the chosen orientations agree:

Think of the disks P_i as also lying in a common plane with a U orientation on each. Let e_1 be an edge of P_i and e_2 an edge of P_j (perhaps $i = j$) and suppose that ω' pairs e_1 with e_2. Let e_1^U be the edge corresponding to E_1 in either P_i^+ or P_i^-, whichever is labeled U. Similarly define e_1^D, e_2^U and e_2^D. If ω' pairs e_1 and e_2 so that the orientations on P_i and P_j agree, then e_1^U is paired with e_2^U and e_1^D with e_2^D. Otherwise they switch. (Notice that we could have performed the entire process by first starting with any P_i^+ as either up or down and we would have the same 2^{s-1} possibilities.) The marking f'_ω lifts to a unique f_ω once we specify which point of $\pi^{-1}(\overline{q}_1)$ to call q_1. Therefore, given (ω', f'_ω), there are exactly 2^{s-1} distinct lifts (ω, f_ω). Since elements of the mapping class group $\Gamma_g^s(\tau_0)$ do not permute the points $\{q_i\}$, there are similarly 2^{s-1} distinct ω.

We now consider polygonal identifications induced by ω'. Define $\lambda_g^N(n_1, \ldots, n_s)$ to be the number of identifications ω of the edges of $\cup_{1\leq i\leq s} P_i$ that give a non-orientable surface of genus g (Euler characteristic $1 - g$) and so that the valence of each vertex of the boundary graph $(\cup_{1\leq i\leq s}\partial P_i)/\omega$ is at least 3. Here we are assuming that an initial edge of each P_i is fixed so that, for example when $s = 1$, the identifications $aabb$ and $abba$ would be counted as different ways of constructing a Klein bottle. As an illustration,

$$\lambda_1^N(2) = 4, \tag{2}$$

corresponding to the four patterns:

$$a\,a\,b\,b, \quad a\,b\,b\,a, \quad a\,b\,a\,b^{-1}, \quad a\,b\,a^{-1}\,b.$$

Notice that if the surface has Euler characteristic $1 - g$ then the number of vertices of the boundary graph is $1 + n - g - s$.

Similarly, we define $\lambda_g^O(n_1, \ldots, n_s)$ to be the number of identifications of the edges of $\cup_{1\leq i\leq s} P_i$ that give a *orientable* surface of genus g (Euler characteristic $2-2g$) with the same valence condition. Finally, define $\lambda_g(n_1, \ldots, n_s)$ to be the number of identifications of the edges of $\cup_{1\leq i\leq s} P_i$ that give a surface of Euler characteristic $1-g$, *orientable or not*, with the same valence condition.

Now set

$$\lambda_g^s(n) = \sum_{n_1+\cdots+n_s=2n} \lambda_g(n_1, \ldots, n_s),$$

and let $\lambda_g^{s,N}(n)$ and $\lambda_{(g+1)/2}^{s,O}(n)$ be the sums, over the same range, with summands $\lambda_g^N(n_1, \ldots, n_s)$ and $\lambda_{(g+1)/2}^O(n_1, \ldots, n_s)$, respectively. Finally, let

$$\Lambda_g^s = \sum_{n=g+s}^{3g+3s-3} \frac{(-1)^{n-s}}{2n} \lambda_g^s(n) \tag{3}$$

and let $\Lambda_g^{s,N}$ and $\Lambda_{(g+1)/2}^{s,O}$ be the analogous sums in which $\lambda_g^s(n)$ is replaced by $\lambda_g^{s,N}(n)$ and $\lambda_{(g+1)/2}^{s,O}(n)$, respectively.

Next we express $\chi(\mathcal{M}_g^s(\tau_0))$ in terms of one of the quantities defined above.

Lemma 2.3.

$$\chi(\mathcal{M}_g^s(\tau_0)) = 2^{s-1}\Lambda_g^{s,N}.$$

Proof: Recall that $\mathcal{M}_g^s(\tau_0))$ is an orbifold and by definition, each point of an orbifold $\mathcal{M}$ has a neighborhood that is locally modeled on $\mathbb{R}^{\dim(\mathcal{M})}$ modulo a finite group. An orbifold triangulation of $\mathcal{M}$ is a triangulation by simplices ω which have the property that the finite group associated to each point of the interior of ω is the same. If we call this group $G(\omega)$, the Euler characteristic on $\mathcal{M}$ is computed with the formula:

$$\chi(\mathcal{M}) = \sum_{\omega} \frac{(-1)^{\dim(\omega)}}{o(G(\omega))}.$$

To compute the Euler characteristic of $\mathcal{M}_g^s(\tau_0)$, consider the ideal triangulation of $\mathcal{M}_g^s(\tau_0) \times \Delta^{s-1}$ described by identification patterns. We cannot apply the formula above directly because cells of the triangulation must be closed. Instead we use the dual complex described in [8]: $\mathcal{M}_g^s(\tau_0) \times \Delta^{s-1}$ retracts onto a spine complex Y of dimension $2g-3+3s$ which has a k-cell for each identification pattern ω with $n-1 = 3g-3s-4-k$. The cells are the identification patterns and the local groups are the symmetries of the configuration which determines the cell. Since these must fix the points q_i, they are all cyclic. Counting each identification pattern of (with no initial edges) weighted by the reciprocal of the order of the cyclic symmetry group is the same as counting each pattern with a choice of initial edge weighted by $1/2n$ (compare [9]). The result follows. $\square$

Now clearly

$$\Lambda_g^s = \Lambda_g^{s,N} + \Lambda_{(g+1)/2}^{s,O}, \tag{4}$$

where the second term on the right is naturally zero when g is even. Then Lemma 2.3 and (4) enable us to establish the main result by determining Λ_g^s and $\Lambda_{(g+1)/2}^{s,O}$. This will be carried out in the next section, where we actually determine a parametrized quantity that specializes to each of Λ_g^s and $\Lambda_{(g+1)/2}^{s,O}$ at different values of the parameter.

Previously Harer and Zagier [9] showed that $\chi(\mathcal{M}_g^s) = \Lambda_{(g+1)/2}^{s,O}$ and thus determined that

$$\chi(\mathcal{M}_g^s) = (-1)^s \frac{(g+s-2)!}{(g+1)(g-1)!} B_{g+1},$$

when g is odd (of course, $\chi(\mathcal{M}_g^s) = 0$ for g even).

3 Matrix models and the counting series for embedded graphs

3.1 Polygonal identifications and graph embeddings

To determine Λ_g^s and $\Lambda_{(g+1)/2}^{s,O}$ we first identify $\lambda_g^s(n)$ and $\lambda_{(g+1)/2}^{s,O}(n)$ as solutions of equivalent enumerative questions for combinatorial maps, since appropriate forms for their generating series are already available, from current work in algebraic combinatorics. A *map* is an embedded graph in a surface with the property that each complementary region is a 2-cell. In topology one says the graph fills the surface and in combinatorics ones calls this a 2-cell embedding. The graph divides the surface into regions called *faces*. In an orientable surface a map is *rooted* by distinguishing an edge, and an end of this edge. In nonorientable surfaces an edge is distinguished together with an end of this edge, and a side of this end of the edge. Rooted maps have only the trivial automorphism. Collectively, orientable and nonorientable surfaces are referred to as *locally* orientable. (Of course every surface is locally orientable, but the terminology is provided to make it clear that we are considering both orientable and non-orientable surfaces at once.) From this point of view, an identification ω of the edges of P_n with boundary graph $(\partial P_n)/\omega$ corresponds precisely to a two-cell embedding of the boundary graph with a single face; the rooting allows us to uniquely recover the labels of the sides and the orientation of P_n. Thus we have the following combinatorial identifications:

- $\lambda_g^{s,O}(n)$ is $s!$ times the number of rooted maps with s faces, n edges and no vertices of valences 1 or 2, in orientable surfaces of genus g (Euler characteristic $2-2g$),
- $\lambda_g^{s}(n)$ is $s!$ times the number of rooted maps with s faces, n edges and no vertices of valences 1 or 2, in locally orientable surfaces of Euler characteristic $1-g$.

For $\mathbf{i}=(i_1,i_2,\ldots)$, let $m(\mathbf{i},j,n)$ and $m^O(\mathbf{i},j,n)$ be, respectively, the numbers of rooted maps in locally orientable and orientable surfaces, with n edges, j faces and i_k vertices of valence k for $k\geq 1$. Thus $\sum_{k\geq 1} ki_k = 2n$, and $\sum_{k\geq 1} i_k = |V|$, the number of vertices. But from the Euler-Poincaré theorem, for the maps counted by $\lambda^O_{(g+1)/2}(n)$ and $\lambda_g(n)$ we have $|F|-|E|+|V| = s-n+|V| = 1-g$ so $|V| = n-g-s+1$. Thus we can write

$$\lambda_g^s(n) = s!\sum_{\mathbf{i}} m(\mathbf{i},s,n), \qquad \lambda^{s,O}_{(g+1)/2}(n) = s!\sum_{\mathbf{i}} m^O(\mathbf{i},s,n), \tag{5}$$

where, in both cases the sum is over $\mathbf{i}$ such that

$$i_1=i_2=0, \qquad \sum_{k\geq 3} ki_k = 2n, \qquad \sum_{k\geq 3} i_k = n-g-s+1. \tag{6}$$

The vector $\mathbf{i}$ is called the *vertex distribution* of such maps. Let the generating series for maps with respect to vertex distribution be

$$M(\mathbf{y},x,z) = \sum_{\mathbf{i},j,n} m(\mathbf{i},j,n)\mathbf{y}^{\mathbf{i}}x^jz^n, \quad M^O(\mathbf{y},x,z) = \sum_{\mathbf{i},j,n} m^O(\mathbf{i},j,n)\mathbf{y}^{\mathbf{i}}x^jz^n, \tag{7}$$

where $\mathbf{y}=(y_1,y_2,\ldots)$, and $\mathbf{y}^{\mathbf{i}} = \prod_{k\geq 1} y_k^{i_k}$.

3.2 A pair of matrix integrals

These two series have the following integral representations, as given in Jackson [13], for M^O, and Goulden and Jackson [4], for M:

$$\begin{aligned}
M^O(\mathbf{y},N,z) &= 2z\frac{\partial}{\partial z}\log\left(\frac{\int_{\mathcal{V}_N} e^{\sum_{k\geq 1}\frac{1}{k}y_k\sqrt{z}^k \operatorname{trace}\mathbf{M}^k} e^{-\frac{1}{2}\operatorname{trace}\mathbf{M}^2}\,d\mathbf{M}}{\int_{\mathcal{V}_N} e^{-\frac{1}{2}\operatorname{trace}\mathbf{M}^2}\,d\mathbf{M}}\right),\\
M(\mathbf{y},N,z) &= 4z\frac{\partial}{\partial z}\log\left(\frac{\int_{\mathcal{W}_N} e^{\frac{1}{2}\sum_{k\geq 1}\frac{1}{k}y_k\sqrt{z}^k \operatorname{trace}\mathbf{M}^k} e^{-\frac{1}{4}\operatorname{trace}\mathbf{M}^2}\,d\mathbf{M}}{\int_{\mathcal{W}_N} e^{-\frac{1}{4}\operatorname{trace}\mathbf{M}^2}\,d\mathbf{M}}\right),
\end{aligned}$$

where $\mathcal{V}_N$ and $\mathcal{W}_N$ are, respectively, the vector spaces of Hermitian complex and real symmetric $N \times N$ matrices. The reader who wishes to have greater detail is directed to the above sources, and to the discussion in Section 5 of this paper.

The integrals over $\mathcal{V}_N$ and $\mathcal{W}_N$ can be transformed by the Weyl integration theorems [10] to the following integrals over the reals. For a positive integer N, let $\lambda = (\lambda_1, \dots, \lambda_N)$, let $V(\lambda)$ be the Vandermonde determinant $\prod_{1\le i<j\le N}(\lambda_j - \lambda_i)$, let $p_k(\lambda) = \lambda_1^k + \cdots + \lambda_N^k$, the power sum symmetric function of degree k, for $k \ge 1$, and $d\lambda = d\lambda_1 \cdots d\lambda_N$:

$$M^O(\mathbf{y}, N, z) = 2z\frac{\partial}{\partial z}\log\left(\frac{\int_{\mathbb{R}^N} |V(\lambda)|^2 e^{\sum_{k\ge1}\frac{1}{k}\sqrt{z}^k y_k p_k(\lambda)} e^{-\frac{1}{2}p_2(\lambda)} d\lambda}{\int_{\mathbb{R}^N} |V(\lambda)|^2 e^{-\frac{1}{2}p_2(\lambda)} d\lambda}\right), \tag{8}$$

$$M(\mathbf{y}, N, z) = 4z\frac{\partial}{\partial z}\log\left(\frac{\int_{\mathbb{R}^N} |V(\lambda)| e^{\frac{1}{2}\sum_{k\ge1}\frac{1}{k}\sqrt{z}^k y_k p_k(\lambda)} e^{-\frac{1}{4}p_2(\lambda)} d\lambda}{\int_{\mathbb{R}^N} |V(\lambda)| e^{-\frac{1}{4}p_2(\lambda)} d\lambda}\right). \tag{9}$$

A joint generalization of these is afforded by

$$M_\gamma(\mathbf{y}, x, z) = \frac{2}{\gamma} z \frac{\partial}{\partial z} \log Z_\gamma(\mathbf{y}, x, z), \tag{10}$$

where

$$Z_\gamma(\mathbf{y}, N, z) = \frac{\int_{\mathbb{R}^N} |V(\lambda)|^{2\gamma} e^{\gamma\sum_{k\ge1}\frac{1}{k}\sqrt{z}^k y_k p_k(\lambda)} e^{-\frac{1}{2}\gamma p_2(\lambda)} d\lambda}{\int_{\mathbb{R}^N} |V(\lambda)|^{2\gamma} e^{-\frac{1}{2}\gamma p_2(\lambda)} d\lambda}, \tag{11}$$

since

$$M(\mathbf{y}, N, z) = M_{\frac{1}{2}}(\mathbf{y}, N, z), \quad M^O(\mathbf{y}, N, z) = M_1(\mathbf{y}, N, z). \tag{12}$$

Thus both M and M^O can be obtained by specializing the parameter γ in M_γ, and it is the series M_γ that we therefore study in detail. We now make a few comments about the interpretation of the above series. First, the numerator of (11) is an even function of $\sqrt{z}$ (to see this, consider the simultaneous substitutions $\lambda_j \to -\lambda_j$, $j = 1, \dots, N$), so the occurrence of $\sqrt{z}$ is of no concern; M_γ really is a power series in z. Second, the second argument in these map series is an indeterminate, x, yet in each case the integral representation evaluates the series at any positive integer value N (which is the dimension of the region of integration). However, the number of rooted maps with n edges is finite for each n, so M_γ can be viewed as a power series in z with coefficients that are polynomials in $y_1, \dots, y_{2n}, x$. Thus, since this holds for infinitely many N, a polynomiality argument allows us to obtain $M_\gamma(\mathbf{y}, x, z)$ from $M_\gamma(\mathbf{y}, N, z)$, in principle, by deriving an explicit presentation of $M_\gamma(\mathbf{y}, N, z)$ as a power series in z, and formally replacing N by x in the resulting (polynomial) coefficients of each power of z.

At this point we do not elaborate on the way in which these coefficients depend on the parameter γ. The nature of this dependence and an associated combinatorial conjecture are discussed in Section 5 of the paper. In the following result, we obtain preliminary expressions for Λ_g^s and $\Lambda_{(g+1)/2}^{s,O}$, as specializations of the parameter in a quantity $\xi_g^s(\gamma)$, which is therefore referred to as the parametrized Euler characteristic. Here we use the notation $[A]B$ for the coefficient of A in the expansion of B.

Proposition 3.1. *Let*

$$\xi_g^s(\gamma) = s!(-1)^s[x^s t^{g+s-1}]\,\frac{1}{\gamma}\log W_\gamma(x,t), \tag{13}$$

where

$$W_\gamma(N,t) = \frac{\displaystyle\int_{\mathbb{R}^N} |V(\lambda)|^{2\gamma}\left(\prod_{j=1}^{N} e^{-i\gamma\lambda_j/\sqrt{t}}(1-i\sqrt{t}\lambda_j)^{-\gamma/t}\right)d\lambda}{\displaystyle\int_{\mathbb{R}^N} |V(\lambda)|^{2\gamma} e^{-\gamma p_2(\lambda)/2}d\lambda}. \tag{14}$$

Then, for $s, g \geq 1$,

$$\Lambda_g^s = \xi_g^s(\tfrac{1}{2}), \qquad \Lambda_{(g+1)/2}^{s,O} = \xi_g^s(1). \tag{15}$$

Proof: Let $\mathbf{u}(t) = (u_1, u_2, \ldots)$, where $u_1 = u_2 = 0$, $u_k = -(i\sqrt{t})^{k-2}$, $k \geq 3$ (here i is $\sqrt{-1}$, and is not an index), and let Ψ be the operator whose action is defined by

$$\Psi f(\mathbf{y}, x, z) = \tfrac{1}{2}\int_0^1 f(\mathbf{u}(t), x, z)\frac{dz}{z}.$$

Now, for $\mathbf{i}$ satisfying the conditions in (6), under the specialization $\mathbf{y} = \mathbf{u}(t)$, we obtain

$$\mathbf{y}^{\mathbf{i}} = (-1)^{n-g-s+1+\frac{1}{2}\{2n-2(n-g-s+1)\}}\, t^{\frac{1}{2}\{2n-2(n-g-s+1)\}} = (-1)^n t^{g+s-1}.$$

Then we obtain $\Lambda_g^s = s!(-1)^s[x^s t^{g+s-1}]\Psi M$ and $\Lambda_{(g+1)/2}^{s,O} = s!(-1)^s[x^s t^{g+s-1}]\Psi M^O$, for $g \geq 1, s \geq 1$, from (3), (5) and (7). But, from (10) and (11),

$$\Psi M_\gamma = \frac{1}{\gamma}\int_0^1 \left(\frac{\partial}{\partial z}\log Z_\gamma(\mathbf{u}(t), N, z)\right)dz = \frac{1}{\gamma}\log Z_\gamma(\mathbf{u}(t), N, 1),$$

since $Z_\gamma(\mathbf{u}(t), N, 0) = 1$. Now let $W_\gamma(N,t) = Z_\gamma(\mathbf{u}(t), N, 1)$. The result follows from (12) and the simplification

$$\begin{aligned}
-\tfrac{1}{2}p_2(\lambda) + \sum_{k\geq 1}\frac{1}{k}u_k p_k(\lambda) &= -\sum_{k\geq 2}\frac{(i\sqrt{t})^{k-2}}{k}p_k(\lambda) \\
&= \frac{1}{i\sqrt{t}}p_1(\lambda) + \frac{1}{t}\sum_{k\geq 1}\frac{(i\sqrt{t})^k}{k}p_k(\lambda) \\
&= \sum_{j=1}^{N}\left(\frac{-i\lambda_j}{\sqrt{t}} + \frac{1}{t}\log(1-i\sqrt{t}\lambda_j)^{-1}\right).
\end{aligned}$$

Note that the application of Ψ to M_γ is justified, since the number of maps with no vertices of valences 1 or 2 (this is forced by the specialization $y_1 = y_2 = 0$), and $|E| - |V| = g + s - 1$ is finite for each g and s (e.g., these conditions imply that there are at most $3g + 3s - 3$ edges, as in the upper bound for the summation in (3)). Thus $W_\gamma(x,t)$ can be viewed as a power series in t with coefficients that are polynomials in x. $\square$

4 The main result

4.1 Selberg integration

We now determine $W_\gamma(x,t)$ from the integral representation (14) for $W_\gamma(N,t)$ by giving an explicit presentation of $W_\gamma(N,t)$ as a series in t, with coefficients that are polynomials in N, and then, appealing to polynomiality, formally replacing N by x. To evaluate $W_\gamma(N,t)$ we use the following theorem due to Mehta [20]. It is derived in a similar fashion to Selberg's theorem, but based on an integral of Cauchy instead of the beta integral (note that again $i=\sqrt{-1}$, and is not an index).

Theorem 4.1. *For* $\mathrm{Re}\, a, \mathrm{Re}\, b, \mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0, \mathrm{Re}\,(\alpha+\beta) > 1$ *and*

$$-\frac{1}{N} < \mathrm{Re}\,\gamma < \min\left(\frac{\mathrm{Re}\,\alpha}{N-1}, \frac{\mathrm{Re}\beta}{N-1}, \frac{\mathrm{Re}\,(\alpha+\beta+1)}{2(N-1)}\right)$$

then

$$\int_{\mathbb{R}^N} |V(\lambda)|^{2\gamma} \prod_{j=1}^{N}\left(1+i\frac{\lambda_j}{a}\right)^{-\alpha}\left(1-i\frac{\lambda_j}{b}\right)^{-\beta} d\lambda = \kappa \prod_{j=0}^{N-1} \frac{\Gamma(\alpha+\beta-(N+j-1)\gamma-1)}{\Gamma(\alpha-\gamma j)\Gamma(\beta-\gamma j)}$$

where

$$\kappa = \left(\frac{2\pi\, a^\alpha b^\beta}{(a+b)^{(\alpha+\beta)-\gamma(N-1)-1}}\right)^N \prod_{j=0}^{N-1} \frac{\Gamma(1+(1+j)\gamma)}{\Gamma(1+\gamma)}.$$

In proving the following result we use Mehta's integration theorem to evaluate $\log W_\gamma(N,t)$, and then apply the polynomiality argument to give an explicit (asymptotic) power series presentation of $\log W_\gamma(x,t)$.

Theorem 4.2. *As an asymptotic expansion in* $1/t$ *we have*

$$\log W_\gamma(x,t) = -\gamma x \sum_{k\geq 1} \frac{B_{2k} t^{2k-1}}{2k(2k-1)} + \sum_{\delta\geq 1} \frac{t^\delta}{\delta(\delta+1)} \sum_{r=1}^{\delta+1} \binom{\delta+1}{r} B_{\delta+1-r}$$
$$\times \left(\frac{x^r(-1)^{\delta+1-r}}{\gamma^{\delta-r}} - \sum_{m=1}^{r+1}\binom{r+1}{m}\frac{B_{r+1-m}}{r+1}\frac{x^m}{\gamma^{r-m}}(-1)^{\delta-m}\right).$$

Proof: From (14), $W_\gamma(x,t) = F_\gamma(x,t)/G_\gamma(x,t)$, where

$$F_\gamma(N,t) = \int_{\mathbb{R}^N} |V(\lambda)|^{2\gamma}\left(\prod_{j=1}^{N} e^{-i\gamma\lambda_j/\sqrt{t}}(1-i\sqrt{t}\lambda_j)^{-\gamma/t}\right) d\lambda,$$

and

$$G_\gamma(N,t) = \int_{\mathbb{R}^N} |V(\lambda)|^{2\gamma} e^{-\gamma p_2(\lambda)/2} d\lambda.$$

We begin by determining the integrals $F_\gamma(N,t)$ and $G_\gamma(N,t)$ as different specializations of Mehta's integral in Theorem 4.1 above.
To determine G_γ: We can rewrite G_γ in the form

$$G_\gamma(N,t) = \lim_{L\to\infty} \int_{\mathbb{R}^N} |V(\lambda)|^{2\gamma} \prod_{j=1}^{N} \left(1 + i\frac{\lambda_j}{\sqrt{L}}\right)^{-\frac{1}{2}\gamma L} \left(1 - i\frac{\lambda_j}{\sqrt{L}}\right)^{-\frac{1}{2}\gamma L} d\lambda,$$

so applying Theorem 4.1, with $a = b = \sqrt{L}$ and $\alpha = \beta = \gamma L/2$, gives

$$G_\gamma = \left(\prod_{j=0}^{N-1} \frac{\Gamma(1+(1+j)\gamma)}{\Gamma(1+\gamma)}\right) \lim_{L\to\infty} \left(\frac{2\pi\sqrt{L}^{\gamma L}}{(2\sqrt{L})^{\gamma L-\gamma(N-1)-1}}\right)^N \prod_{j=0}^{N-1} \frac{\Gamma(\gamma L-(N+j-1)\gamma-1)}{\Gamma(\frac{1}{2}\gamma L-\gamma j)^2}.$$

But the duplication theorem for the gamma function gives

$$\Gamma(\gamma L-(N+j-1)\gamma-1) = \frac{1}{\sqrt{\pi}} 2^{\gamma L-\gamma(N+j-1)-2}\Gamma(\tfrac{1}{2}(\gamma L-(N+j-1)\gamma-1))\Gamma(\tfrac{1}{2}(\gamma L-(N+j-1)\gamma)),$$

and it is straightforward to establish by standard properties of the gamma function that

$$\lim_{L\to\infty} \frac{\Gamma(\frac{1}{2}(\gamma L-(N+j-1)\gamma-1))}{\Gamma(\frac{1}{2}\gamma L-\gamma j)\ (\frac{1}{2}\gamma L)^{-\frac{1}{2}(N-j-1)\gamma-\frac{1}{2}}} = 1 = \lim_{L\to\infty} \frac{\Gamma(\frac{1}{2}(\gamma L-(N+j-1)\gamma))}{\Gamma(\frac{1}{2}\gamma L-\gamma j)\ (\frac{1}{2}\gamma L)^{-\frac{1}{2}(N-j-1)\gamma}}.$$

It follows that

$$G_\gamma(N,t) = \left(\frac{\sqrt{2\pi}}{\gamma^{\frac{1}{2}((N-1)\gamma+1)}}\right)^N \prod_{j=0}^{N-1} \frac{\Gamma(1+(1+j)\gamma)}{\Gamma(1+\gamma)}.$$

To determine F_γ: We can rewrite F_γ in the form

$$F_\gamma(N,t) = \lim_{L\to\infty} \int_{\mathbb{R}^N} |V(\lambda)|^{2\gamma} \prod_{j=1}^{N} \left(1 + i\frac{\sqrt{t}}{L}\lambda_j\right)^{-\gamma L/t} (1 - i\sqrt{t}\lambda_j)^{-\gamma/t} d\lambda$$

so applying Theorem 4.1 with $a = L/\sqrt{t}$, $b = 1/\sqrt{t}$, $\alpha = \gamma L/t$ and $\beta = \gamma/t$, gives

$$F_\gamma = \left(\frac{2\pi}{\sqrt{t}^{\gamma(N-1)+1}}\right)^N A_\gamma(N,t)\, B_\gamma(N,t) \prod_{j=0}^{N-1} \frac{\Gamma(1+(1+j)\gamma)}{\Gamma(1+\gamma)\Gamma\left(\frac{\gamma}{t}-\gamma j\right)},$$

where

$$A_\gamma(N,t) = \lim_{L\to\infty} \left(\frac{L^{\frac{\gamma L}{t}} \left(\frac{\gamma L}{t}\right)^{\frac{\gamma}{t}-\gamma(N-1)-1}}{(L+1)^{\frac{\gamma}{t}(L+1)-\gamma(N-1)-1}}\right)^N = \left(\left(\frac{\gamma}{t}\right)^{\frac{\gamma}{t}-\gamma(N-1)-1} e^{-\frac{\gamma}{t}}\right)^N,$$

and, again by standard properties of the gamma function,

$$B_\gamma(N,t) = \lim_{L\to\infty} \prod_{j=0}^{N-1} \left(\frac{\Gamma\left(\frac{\gamma}{t}(L+1) - (N+j-1)\gamma - 1\right)}{\Gamma\left(\frac{\gamma L}{t} - \gamma j\right)\left(\frac{\gamma L}{t}\right)^{\frac{\gamma}{t}-\gamma(N-1)-1}} \right) = 1.$$

Thus

$$F_\gamma(N,t) = \left(\frac{2\pi}{t^{\frac{1}{2}(\gamma(N-1)+1)}} e^{-\gamma/t} \left(\frac{\gamma}{t}\right)^{\frac{\gamma}{t}-\frac{1}{2}(\gamma(N-1)+1)} \right)^N \prod_{j=0}^{N-1} \frac{\Gamma(1+(1+j)\gamma)}{\Gamma(1+\gamma)\Gamma\left(\frac{\gamma}{t}-\gamma j\right)}.$$

These evaluations for F_γ and G_γ immediately yield

$$W_\gamma(N,t) = \frac{F_\gamma(N,t)}{G_\gamma(N,t)} = \left(\sqrt{2\pi}\ e^{-\gamma/t} \left(\frac{\gamma}{t}\right)^{\frac{\gamma}{t}-\frac{1}{2}((N-1)\gamma+1)} \right)^N \prod_{j=0}^{N-1} \frac{1}{\Gamma\left(\frac{\gamma}{t}-\gamma j\right)}. \tag{16}$$

We now exhibit $\log W_\gamma(N,t)$ explicitly as a power series in t, with coefficients that are polynomials in N. Let $N = qK$ and $\gamma = 1/q$ where q and K are positive integers. This specialization of N does not affect the recoverability of $\log W_\gamma(x,t)$ from $\log W_\gamma(N,t)$, since there is an infinite set of such K, so the polynomiality argument still holds. With these choices for N and γ we have

$$\prod_{j=0}^{qK-1} \Gamma\left(\frac{1}{tq} - \frac{j}{q}\right) = \frac{\Gamma(\frac{1}{tq})}{\Gamma(\frac{1}{tq}-K)} \prod_{j=1}^{K} \left(\prod_{k=0}^{q-1} \Gamma\left(\frac{1}{tq} - j + \frac{k}{q}\right)\right),$$

and the product over k can be simplified by Gauss' multiplication theorem for the gamma function, whence

$$\prod_{j=0}^{qK-1} \Gamma\left(\frac{1}{tq} - \frac{j}{q}\right) = \frac{\Gamma(\frac{1}{tq})}{\Gamma(\frac{1}{tq}-K)} \left(\frac{\sqrt{2\pi}^{q-1}}{q^{\frac{1}{t}-\frac{1}{2}-q(K+1)}} \right)^K \prod_{j=1}^{K} \Gamma\left(\frac{1}{t} - qj\right). \tag{17}$$

But

$$\prod_{j=1}^{K} \Gamma\left(\frac{1}{t} - qj\right) = \frac{\Gamma(\frac{1}{t})^K t^{qK(K+1)/2}}{\prod_{j=1}^{K}\prod_{l=1}^{qj}(1-lt)}, \quad \text{and} \quad \frac{\Gamma(\frac{1}{tq})}{\Gamma(\frac{1}{tq}-K)} = \frac{1}{(tq)^K}\prod_{j=1}^{K}(1-tqj).$$

Combining these results with (17) and then with (16) gives

$$W_{\frac{1}{q}}(qK,t) = \left(\frac{\sqrt{2\pi t}}{\Gamma(\frac{1}{t})\,(et)^{\frac{1}{t}}} \right)^K \frac{\prod_{l=1}^{K}\prod_{j=1}^{ql}(1-jt)}{\prod_{j=1}^{K}(1-tqj)}. \tag{18}$$

For the Bernoulli numbers B_j given by (1), recall that

$$\log\left(\frac{\Gamma(\frac{1}{t})\,(et)^{\frac{1}{t}}}{\sqrt{2\pi t}} \right) = \sum_{k\geq 1} \frac{B_{2k}t^{2k-1}}{2k(2k-1)}$$

as an asymptotic series in $1/t$, and

$$\sum_{j=1}^{n} j^k = \frac{1}{k+1}\sum_{r=1}^{k+1}\binom{k+1}{r}B_{k+1-r}(-1)^{k+1-r}n^r,$$

for $k \geq 0$ and $n \geq 1$. Then applying these results to (18) gives, as an asymptotic expansion in $1/t$,

$$\begin{aligned}\log W_{\frac{1}{q}}(qK,t) &= -K\sum_{k\geq 1}\frac{B_{2k}t^{2k-1}}{2k(2k-1)} + \sum_{j=1}^{K}\sum_{\delta\geq 1}\frac{t^\delta}{\delta}q^\delta j^\delta - \sum_{l=1}^{K}\sum_{j=1}^{ql}\sum_{\delta\geq 1}\frac{t^\delta}{\delta}j^\delta \\ &= -K\sum_{k\geq 1}\frac{B_{2k}t^{2k-1}}{2k(2k-1)} + \sum_{\delta\geq 1}\frac{t^\delta}{\delta(\delta+1)}\sum_{r=1}^{\delta+1}\binom{\delta+1}{r}B_{\delta+1-r}(-1)^{\delta+1-r} \\ &\quad \times\left(q^\delta K^r - \frac{q^r}{r+1}\sum_{m=1}^{r+1}\binom{r+1}{m}B_{r+1-m}(-1)^{r+1-m}K^m\right).\end{aligned}$$

The result follows by polynomiality, upon replacing q by $1/\gamma$ and K by $N\gamma$, thus giving an explicit power series presentation of $\log W_\gamma(x,t)$. □

We are now in a position to give an explicit expression for the coefficient $\xi_g^s(\gamma)$ given in (13).

Corollary 4.3. *For $g\geq 1$, $s\geq 1$,*

$$\xi_g^s(\gamma) = \begin{cases} \dfrac{(g+s-2)!}{g!}(-1)^s\dfrac{B_g}{2}\left(\dfrac{1}{\gamma^g}-\dfrac{1}{\gamma}\right), & \textit{if } g \textit{ is even,} \\ \dfrac{(g+s-2)!(-1)^{s+1}}{(g+1)!}\left\{\dfrac{(g+1)B_g}{\gamma^g} + \displaystyle\sum_{r=0}^{g+s}\binom{g+1}{r+1-s}B_{g+s-r}\dfrac{B_{r+1-s}}{\gamma^{r+1-s}}\right\}, & \textit{if } g \textit{ is odd.}\end{cases}$$

Proof: The ring of asymptotic series in $1/t$ is a subring of the ring of formal power series in t. Thus $\log W_\gamma$ has an asymptotic expansion that is the generating series for $\xi_g^s(\gamma)$ in $1/t$. Moreover, if a function has an asymptotic series then it is unique. Thus from Proposition 3.1 and Theorem 4.2 we obtain, for $g \geq 1$ and $s \geq 2$,

$$\begin{aligned}\xi_g^s(\gamma) &= s!(-1)^s[x^s t^{g+s-1}]\frac{1}{\gamma}\log W_\gamma(x,t) \\ &= \frac{s!(-1)^{g+s}}{(g+s)(g+s-1)}\left\{\binom{g+s}{s}\frac{B_g}{\gamma^g} + \sum_{r=1}^{g+s}\binom{g+s}{r}B_{g+s-r}\binom{r+1}{s}\frac{B_{r+1-s}}{(r+1)\gamma^{r+1-s}}\right\} \\ &= \frac{(g+s-2)!(-1)^{g+s}}{(g+1)!}\left\{\frac{(g+1)B_g}{\gamma^g} + \sum_{r=1}^{g+s}\binom{g+1}{r+1-s}B_{g+s-r}\frac{B_{r+1-s}}{\gamma^{r+1-s}}\right\}.\end{aligned}$$

For $g \geq 1$ and $s = 1$, we obtain

$$\xi_g^s(\gamma) = \frac{(-1)^{g+1}}{g(g+1)}\left\{B_{g+1} + \frac{(g+1)B_g}{\gamma^g} + \sum_{r=1}^{g+1}\binom{g+1}{r}B_{g+1-r}\frac{B_r}{\gamma^r}\right\}, \tag{19}$$

where, for the first term in (19), we have used the fact that the Bernoulli number B_j is 0 for j odd and greater than 1.

Thus, for all $g \geq 1, s \geq 1$, we have

$$\xi_g^s(\gamma) = \frac{(g+s-2)!(-1)^{g+s}}{(g+1)!} \left\{ \frac{(g+1)B_g}{\gamma^g} + \sum_{r=0}^{g+s} \binom{g+1}{r+1-s} B_{g+s-r} \frac{B_{r+1-s}}{\gamma^{r+1-s}} \right\}. \tag{20}$$

(For $s \geq 2$, the summand in (20) corresponding to $r = 0$ is zero, so (20) agrees with (19) in this case. For $s = 1$, the summand in (20) corresponding to $r = 0$ is B_{g+1}, since $B_0 = 1$, so (20) agrees with (19) in this case also.) This gives the result immediately for g odd.

For g even, the zero values for Bernoulli numbers of odd subscript bigger than 1 mean that for the summation in (20), only the terms corresponding to $r = s$ and $r = g+s-1$ are non-zero, so for g even we obtain

$$\xi_g^s(\gamma) = \frac{(g+s-2)!(-1)^s}{(g+1)!} \left(\frac{(g+1)B_g}{\gamma^g} + (g+1)B_g B_1 \frac{1}{\gamma} + (g+1)B_1 B_g \frac{1}{\gamma^g} \right),$$

and the result follows in this case, since $B_1 = -\frac{1}{2}$. □

4.2 Proof of the main result

Next we prove the main result, Theorem 2.1, evaluating Λ_g^s and $\Lambda_{(g+1)/2}^{s,O}$ by specializing the parameter γ in Corollary 4.3. This is straightforward, though it requires additional properties of the Bernoulli numbers.

Corollary 4.4. *For $g, s \geq 1$,*

$$\Lambda_g^s = \begin{cases} \dfrac{(g+s-2)!}{g!}(-1)^s(2^{g-1}-1)B_g, & \text{if } g \text{ is even,} \\ \dfrac{(g+s-2)!(-1)^s}{(g+1)(g-1)!} B_{g+1}, & \text{if } g \text{ is odd;} \end{cases}$$

$$\Lambda_{(g+1)/2}^{s,O} = \begin{cases} 0 \quad, & \text{if } g \text{ is even,} \\ \dfrac{(g+s-2)!(-1)^s}{(g+1)(g-1)!} B_{g+1}, & \text{if } g \text{ is odd.} \end{cases}$$

Proof: From Proposition 3.1 we have $\Lambda_g^s = \xi_g^s(\frac{1}{2})$ and $\Lambda_{(g+1)/2}^{s,O} = \xi_g^s(1)$, so for g even the result follows immediately from Corollary 4.3.

For g odd, we obtain from Corollary 4.3 that

$$\xi_g^s(\gamma) = \frac{(g+s-2)!(-1)^{s+1}}{(g+1)!} \left(\frac{(g+1)B_g}{\gamma^g} + \left[\frac{t^{g+1}}{(g+1)!} \right] B(t)B(\frac{t}{\gamma}) \right), \tag{21}$$

where $B(t)$ is the exponential generating series for Bernoulli numbers, given in (1). Now the following differential equations can be easily verified:

$$\begin{aligned} B(t)^2 &= (1-t)B(t) - t\frac{t}{dt}B(t), \\ B(t)B(2t) &= (1-t)B(t) - t\frac{t}{dt}B(t) - \frac{t}{2}B(2t). \end{aligned}$$

Thus (for g odd), applying the first of these differential equations to (21) gives

$$\begin{aligned}\xi_g^s(1) &= \frac{(g+s-2)!(-1)^{s+1}}{(g+1)!}\{(g+1)B_g+(B_{g+1}-(g+1)B_g-(g+1)B_{g+1})\} \\ &= \frac{(g+s-2)!(-1)^s}{(g+1)(g-1)!}B_{g+1}.\end{aligned}$$

Applying the second differential equation gives

$$\begin{aligned}\xi_g^s(\tfrac{1}{2}) &= \frac{(g+s-2)!(-1)^{s+1}}{(g+1)!}\left\{(g+1)B_g2^g+\left(B_{g+1}-(g+1)B_g-(g+1)B_{g+1}-(g+1)B_g2^{g-1}\right)\right\} \\ &= \frac{(g+s-2)!(-1)^{s+1}}{(g+1)!}\left\{-gB_{g+1}+(g+1)B_g(2^g-2^{g-1}-1)\right\},\end{aligned}$$

and the result follows, since $B_g=0$ for g odd except when $g=1$, and $2^g-2^{g-1}-1=0$ when $g=1$. □

As discussed at the end of Section 2, Corollary 4.4 completes the proof of the main result, Theorem 2.1. Note that the natural zero value for $\Lambda^{s,O}_{(g+1)/2}$ for g even is in agreement with the computed value for $\xi_g^s(1)$ in Corollary 4.4.

5 A conjectured refinement of the virtual Euler characteristic

The argument γ of the parametrized Euler characteristic $\xi_g^s(\gamma)$ was introduced as an artifact for interpolating between complex curves ($\gamma=1$) and real curves ($\gamma=\frac{1}{2}$), as specialized in (15). However, we conjecture that the parameter γ itself has geometric significance. In particular, $\xi_g^s(\gamma)$ has been evaluated as a polynomial in $1/\gamma$ in Corollary 4.3, and the coefficients in this polynomial have a geometric interpretation.

The evidence for this is indirect, but can be lifted from a combinatorial treatment of maps in Goulden and Jackson [3], that specializes to the series expansion (compare with (7)),

$$M_\gamma(\mathbf{y},x,z)=\sum_{\mathbf{i},j,n}m_\gamma(\mathbf{i},j,n)\mathbf{y}^{\mathbf{i}}x^jz^n, \tag{22}$$

of $M_\gamma(\mathbf{y},x,z)$, defined in terms of an integral in (10). Note that $M_\gamma(\mathbf{y},x,z)$ is in fact a parametrized map series because of the specializations in (12). We conjecture that $m_\gamma(\mathbf{i},j,n)$ is a polynomial in $1/\gamma$, with integer coefficients, and that these coefficients have combinatorial significance. Moreover,

$$\xi_g^s(\gamma)=s!(-1)^s[x^st^{g+s-1}]\Psi M_\gamma,$$

from the proof of Proposition 3.1, so we deduce that $\xi_g^s(\gamma)$ is a finite alternating summation of $m_\gamma(\mathbf{i},j,n)$'s, and thus we can lift the investigation of the significance of $m_\gamma(\mathbf{i},j,n)$ to $\xi_g^s(\gamma)$ itself.

The lifting from maps is through properties of symmetric functions, and Jack functions in particular, which we now summarize (for complete details see Macdonald [19]).

5.1 Jack symmetric functions

We say that $\mu = (\mu_1, \ldots, \mu_k)$ is a partition of n, written $\mu \vdash n$ or $|\mu| = n$, with k parts, written $l(\mu) = k$, if $\mu_1 \geq \ldots \geq \mu_k \geq 1$, and $\mu_1 + \cdots + \mu_k = n$. The μ_i's are the parts of μ. The monomial symmetric function m_μ in a countable set of algebraically independent indeterminates $\mathbf{x} = (x_1, \ldots)$, is the sum of all distinct monomials in the x_i's whose exponents are the parts of μ, with repetition, in some permuted order. If $\mathcal{P}$ is the set of all partitions (including a single, empty, partition of 0), then $\{m_\mu\}_{\mu\in\mathcal{P}}$ forms a basis for the ring of symmetric functions. If $p_\mu = p_{\mu_1} \cdots p_{\mu_k}$, where p_m is the mth power sum (symmetric function) in $\mathbf{x}$, then $\{p_\mu\}_{\mu\in\mathcal{P}}$ is also a basis, and hence we can define an inner product $\langle\ ,\ \rangle_\alpha$ for an indeterminate α, by the orthogonality

$$\langle p_\lambda, p_\mu\rangle_\alpha = \frac{|\mu|!}{|\mathcal{C}_\mu|}\alpha^{l(\mu)}\delta_{\lambda,\mu},$$

where $\mathcal{C}_\mu$ is the conjugacy class in the symmetric group on $|\mu|$ symbols with disjoint cycle lengths given by the parts of μ. Then with respect to this inner product, the Jack symmetric functions $\{J_\mu(\mathbf{x};\alpha)\}_{\mu\in\mathcal{P}}$ also form a basis for symmetric functions in $\mathbf{x}$ (depending on the parameter α), defined by the conditions

$$\langle J_\lambda, J_\mu\rangle_\alpha = 0 \text{ for } \lambda \neq \mu; \qquad [m_\mu]J_\lambda = 0 \text{ unless } \mu \preceq \lambda; \qquad [x_1\cdots x_n]J_\mu = n!, \text{ where } |\mu| = n,$$

imposing *orthogonality, triangularity* and *normalization,* respectively, where $\preceq$ denotes reverse lexicographic order. The Cauchy theorem for Jack functions is

$$\prod_{i,j\geq 1}(1 - x_iy_j)^{-1} = \sum_{\theta\in\mathcal{P}} \frac{J_\theta(\mathbf{x};\alpha)J_\theta(\mathbf{y};\alpha)}{\langle J_\theta, J_\theta\rangle_\alpha}.$$

Finally, let

$$\langle f(\lambda)\rangle_{\mathbb{R}^N} = \frac{\int_{\mathbb{R}^N}|V(\lambda)|^{2\gamma}e^{-\frac{\gamma}{2}p_2}f(\lambda)d\lambda}{\int_{\mathbb{R}^N}|V(\lambda)|^{2\gamma}e^{-\frac{\gamma}{2}p_2}d\lambda}.$$

Then, as a connection between integration and Jack symmetric functions, it has been previously conjectured in Goulden and Jackson [4] that

$$\langle J_\theta(\lambda;\alpha)\rangle_{\mathbb{R}^N} = J_\theta(1_N;\alpha)[p_2^m]J_\theta, \tag{23}$$

where 1_N is the vector with N 1's, and $\theta \vdash 2m$.

5.2 Lifting from the combinatorial conjecture

We now give a symmetric function representation for $M_\gamma(\mathbf{p}(\mathbf{y}), x, z)$, which is dependent on the truth of the integration conjecture above.

Proposition 5.1. *If (23) holds, and* $\mathbf{p}(\mathbf{y}) = (p_1(\mathbf{y}), \ldots)$, *then*

$$M_\gamma(\mathbf{p}(\mathbf{y}), x, z) = \frac{2}{\gamma}z\frac{\partial}{\partial z}\log\sum_{\theta\in\mathcal{P}} z^{\frac{|\theta|}{2}}\frac{J_\theta(\mathbf{y};\frac{1}{\gamma})J_\theta(1_N;\frac{1}{\gamma})}{\langle J_\theta, J_\theta\rangle_{\frac{1}{\gamma}}}[p_2^{\frac{|\theta|}{2}}]J_\theta. \tag{24}$$

Proof: If we replace y_i by $p_i(\mathbf{y})$, $i \geq 1$ (this presents no difficulties since the $p_i(\mathbf{y})$ are algebraically independent for countable $\mathbf{y}$), then from (10) and (11) we obtain

$$\begin{aligned} M_\gamma(\mathbf{p}(\mathbf{y}), x, z) &= \frac{2}{\gamma} z \frac{\partial}{\partial z} \log \langle e^{\gamma \sum_{k\geq 1} \frac{1}{k} \sqrt{z}^k p_k(\mathbf{y}) p_k(\lambda)} \rangle_{\mathbb{R}^N} \\ &= \frac{2}{\gamma} z \frac{\partial}{\partial z} \log \langle \prod_{i\geq 1} \prod_{j=1}^{N} (1 - \sqrt{z} y_i \lambda_j)^{-\gamma} \rangle_{\mathbb{R}^N} \\ &= \frac{2}{\gamma} z \frac{\partial}{\partial z} \log \langle \sum_{\theta \in \mathcal{P}} z^{\frac{|\theta|}{2}} \frac{J_\theta(\mathbf{y}; \frac{1}{\gamma}) J_\theta(\lambda; \frac{1}{\gamma})}{\langle J_\theta, J_\theta \rangle_{\frac{1}{\gamma}}} \rangle_{\mathbb{R}^N}, \end{aligned}$$

from Cauchy's theorem above, and the result follows from the conjectured integral evaluation (23). □

Jack function series like that on the right hand side of (24) have been considered in [3], where a combinatorial conjecture is made about their coefficients. The conjecture specialized to $m_\gamma(\mathbf{i}, j, n)$ is that $m_{1/(b+1)}(\mathbf{i}, j, n)$ is a polynomial in b with nonnegative integer coefficients. Note from (7), (12) and (22) that these coefficients sum to $m(\mathbf{i}, j, n)$ (here $b = 1$), and the constant term is $m(\mathbf{i}, j, n)$ (here $b = 0$). Thus b in this context is a parameter of nonorientability.

Table 1 gives the values of $m_{1/(b+1)}(\mathbf{i}, j, n)$ for maps with at most three edges. For example, the number 4 in (2) of Section 1 is consistent with the value $m((0, 0, 0, 1), 1, 2; b) = 1 + b + 3b^2$ in the table. The constant term 1 identifies the single orientable rooted map (in the torus) with 2 edges, a single face and a single vertex. The other terms b and $3b^2$ mean that there are indeed 4 nonorientable rooted maps (in the Klein bottle) with 2 edges, 1 face and 1 vertex. In this case, these 4 maps would be further subdivided into two classes, of size 1 and 3, with different values, 1 and 2, respectively, of the unknown measure of nonorientability recorded by this parameter.

We conclude that γ, which interpolates between complex curves and real curves in the context of Euler characteristics, has a separate and classical existence in terms of the Jack parameter. It should be noted that, even in the formal study of Jack functions themselves (rather than, for example, the complicated summation we are considering here), a combinatorial interpretation of the Jack parameter has been sought over the last decade (see Hanlon [6] and Stanley [25]), with recent success reported by Lapointe and Vinet [18] and Knop and Sahi [16].

5.3 Schur symmetric and zonal polynomials

We have introduced symmetric functions here indirectly, for technical reasons. However, the special cases $\gamma = \frac{1}{2}, 1$ of the series M_γ give the map series M, M^O, respectively, from (12). Moreover, the special cases $\gamma = \frac{1}{2}, 1$ of Jack functions with parameter $1/\gamma$ are zonal polynomials, Schur symmetric functions, respectively (in the latter case there is also a known scalar introduced). Applying these specializations to the Jack function expression (24) therefore yields expressions with zonal polynomials and Schur symmetric functions, respectively. These expressions have been obtained directly by Jackson and Visentin [15] (for $\gamma = 1$) and Goulden and Jackson [5] (for $\gamma = \frac{1}{2}$). (This means that (23) and (24) have been proved in the cases $\alpha = 2, 1$ and $\gamma = \frac{1}{2}, 1$, respectively.) The combinatorial encodings providing these direct derivations is quite different from the Wick's lemma methodology that yielded the integral expressions (9) for M and (8) for M^O, in [4] and [13], respectively. We conclude with a sketch of these combinatorial encodings. For more complete details see the above references.

A graph embedded in an orientable surface is rooted, and the two ends of its n edges are labelled from 1 to $2n$, with the root end of the root edge labelled 1. The map is encoded by two permutations in $\mathfrak{S}_{2n}$. In the first, the disjoint cycles are the cyclic lists of the labels at the ends of edges incident with a vertex, in clockwise order, one for each vertex. In the second, the cycles are transpositions interchanging the labels at the two ends of a vertex, on for each edge. The cycles of the product of these permutations then specify the labels of edges in the boundary of each face of the map, and the determination of M^O is therefore reduced to a computation in the centre of the group algebra of $\mathfrak{S}_{2n}$), derivable from the connection coefficients of this subalgebra. The generating series for these yields the Schur symmetric function expression that follows from the summation over θ in (24) by setting $\gamma = 1$. The logarithm is applied by the standard combinatorial construction to recover the connected components in this enumeration (since an arbitrary pair of permutations in the group algebra corresponds in general to an unordered collection of labelled maps). The operator $2z\partial/\partial z$ is applied to account for the rooting of the map.

For M, a graph embedded in a locally orientable surface has the four side-end positions of its n edges labelled from 1 to $4n$, with the root side and end of the root edge labelled 1. The map is encoded by three matchings (pairings) on the $4n$ symbols. In the first, the pairs consist of the labels that appear at the two ends of the same side of an edge. In the second, the pairs consist of the labels that appear at the two sides of the same end of an edge. In the third, the pairs consist of the labels that appear in a corner of a face. It can then be shown that the determination of M can therefore be reduced to a computation in the double coset algebra of the hyperoctahedral group embedded in $\mathfrak{S}_{4n}$ as the centralizer of a fixed fixed point free involution. The generating series that follows from this yields the zonal polynomial expression that follows from the summation over θ in (24) by setting $\gamma = \frac{1}{2}$. The logarithm and application of $4z\partial/\partial z$ occur as in the determination of M^O.

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n	j	$\mathbf{i}$	$m_{1/(b+1)}(\mathbf{i}, j, n)$
1	1	(2)	1
		$(0,1)$	b
	2	$(0,1)$	1
2	1	$(2,1)$	2
		$(0,2)$	b
		$(1,0,1)$	$4b$
		$(0,0,0,1)$	$1+b+3b^2$
	2	$(0,2)$	1
		$(1,0,1)$	4
		$(0,0,0,1)$	$5b$
	3	$(0,0,0,1)$	2
3	1	$(2,2)$	3
		$(0,3)$	b
		$(3,0,1)$	2
		$(1,1,1)$	$12b$
		$(0,0,2)$	$1+b+5b^2$
		$(2,0,0,1)$	$9b$
		$(0,1,0,1)$	$3+3b+9b^2$
		$(1,0,0,0,1)$	$6+6b+18b^2$
		$(0,0,0,0,0,1)$	$13b+13b^2+15b^3$
	2	$(0,3)$	1
		$(1,1,1)$	12
		$(0,0,2)$	$9b$
		$(2,0,0,1)$	9
		$(0,1,0,1)$	$15b$
		$(1,0,0,0,1)$	$30b$
		$(0,0,0,0,0,1)$	$10+10b+32b^2$
	3	$(0,0,2)$	4
		$(0,1,0,1)$	6
		$(1,0,0,0,1)$	12
		$(0,0,0,0,0,1)$	$22b$
	4	$(0,0,0,0,0,1)$	5

Table 1: The refined map numbers $m_{1/(b+1)}(\mathbf{i}, j, n)$, for $n \leq 3$ edges.