PMath 464/764 – Homework 9 solutions

1. In each part, compute $\operatorname{ord}_P(f)$ on the curve X.

(a) $X = Z(x+y)$, $P = (0,0)$, $f(x,y) = x^5 - y^5$.

(b) $X = Z(y^2 - x^3 - x)$, $P = (0,0)$, $f(x,y) = (x^2 - y)/(xy)$.

(c) $X = Z(x^2 + y^2 - 1)$, $P = (0,1)$, $f(x,y) = (x+y-1)/(2x+y-1)$.

Solution: (a) First off, we can simplify $f(x,y) = x^5 - y^5 = 2x^5$ on X, so that $\operatorname{ord}_P(f) = 5\operatorname{ord}_P(x)$. Since x is a uniformizer at P, we conclude that $\operatorname{ord}_P(f) = 5$. ♣

(b) Note that y is a uniformizer at P (the tangent line is $x = 0$), and we can write $x = (1/(x^2+1))y^2$, so in particular $\operatorname{ord}_P(x) = 2$. We can rewrite:

$$f(x,y) = \frac{(1/(x^2+1)^2)y - 1}{(1/(x^2+1))y^2}$$

where the numerator is in $\mathcal{O}_P^*$, and the denominator vanishes to order 2 at P. Thus, we compute that $\operatorname{ord}_P(f) = 0 - 2 = -2$. ♣

(c) The tangent line to X at P is the line $y = 1$, so that any other line through P corresponds to a uniformizer. In particular, $2x+y-1$ and $x+y-1$ are both uniformizers at P, so their quotient is in $\mathcal{O}_P^*$. Thus, we conclude that $\operatorname{ord}_P(f) = 0$. ♣

2. Let X be the closed subset $V(x^3 + y^3 - z^3 - w^3, xy + zw) \subset \mathbb{P}^3$. Show that $\phi(x : y : z : w) = [x : z]$ or $[-w : y]$ defines a morphism from X to $\mathbb{P}^1$.

Solution: First, we show that for any point $P \in X$, some representation of ϕ is defined there. The only way the representation $[x : z]$ is not defined at P is if $x = z = 0$. But if $[a : b : c : d] \in X$ and $a = c = 0$, then $b^3 = d^3$, so b and d are both nonzero. Thus, $\phi(x : y : z : w) = [-w : y]$ is well defined on all the points of X for which $[x : z]$ is not well defined.

All that's left is to show that the two representations of ϕ agree wherever they are both defined. Since $xy + zw = 0$ for all points of X, it follows that $(x,z) = (z/y)(-w,y)$ on X, and so both $[x : z]$ and $[-w : y]$ represent ϕ on X. ♣

3. Let X be the closed subset $V(x^3+y^3-z^3-w^3, xy+zw) \subset \mathbb{P}^3$, and define $\phi\colon X \to \mathbb{P}^1$ by $\phi(x:y:z:w) = [x:z]$ or $[-w:y]$. Compute the degree of ϕ. [You may assume that X is irreducible.]

Solution: To compute the degree, let's restrict to the affine piece $z=1$ of $X \subseteq \mathbb{P}^3$. Then ϕ becomes the dominant rational map $\phi(x,y,w) = x$ from X to $\mathbb{A}^1$. Thus, the degree of $[K(X) : \phi^* K(\mathbb{P}^1)]$ is just the degree $[\mathbb{C}(x,y,w) : \mathbb{C}(x)]$, where x, y, and w satisfy the relations in the definition of X, modulo $z=1$.

First, notice that $w = -xy \in \mathbb{C}(x,y)$, so that $\mathbb{C}(x,y,w) = \mathbb{C}(x,y)$. Thus, the degree of ϕ is just the degree of a minimal polynomial of y over $\mathbb{C}(x)$. Eliminating w from $x^3+y^3 = 1+w^3$ gives us $x^3+y^3 = 1-x^3y^3$, or $y^3 = (1-x^3)/(x^3+1)$. The rational function on the right is not a perfect cube in $\mathbb{C}(x)$, and so y has degree three over $\mathbb{C}(x)$. We conclude that the degree of ϕ is three. ♣

4. Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map defined by $\phi(x:y:z) = [yz:xz:xy]$.

(a) Show that ϕ is birational, and its own inverse.

(b) Find open sets U and V in $\mathbb{P}^2$ such that $\phi\colon U \to V$ is an isomorphism.

(c) Find the largest open set U where ϕ is defined.

Solution: (a) This is straightforward algebra. (Don't forget you can divide out by common factors – it's only a rational map!) ♣

(b) Choose U and V both to be the subset $\{x_0x_1x_2 \neq 0\}$ of $\mathbb{P}^2$. Then we can rewrite ϕ as:

$$\phi([x_0:x_1:x_2]) = \left[\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right]$$

which is clearly an isomorphism from U to V. ♣

(c) The largest open set on which ϕ is defined is the set $\mathbb{P}^2 - \{[1:0:0],[0:1:0],[0:0:1]\}$.

As presented, ϕ is well defined at every point with at least two nonzero coordinates – that is, at every point other than $P_2 = [0:0:1]$, $P_1 = [0:1:0]$, and $P_0 = [1:0:0]$. But rational maps with domain $\mathbb{P}^n$ have only one representation, up to multiplication by a common factor. Since xy, xz, and yz have no common factor, it follows that there is no way to improve on this representation, so ϕ cannot be defined on any of the three points P_i.

Notice that by part (b), ϕ is an isomorphism on $U = \{x_0 x_1 x_2 \neq 0\}$. If $x_i = 0$ for some i, then $\phi([x_0 : x_1 : x_2]) = P_i$, so ϕ "blows down" the three coordinate axes to three points. (Except, of course, for the three points at which it is undefined.) ♣

5. Let R be a DVR with maximal ideal M, and quotient field K. Suppose that R contains a field k such that the composition:

$$k \to R \to R/M$$

is an isomorphism of k with R/M. Verify the following assertions:

(a) For any $z \in R$, there is a unique $\lambda \in k$ such that $z - \lambda \in M$.

(b) Let t be a uniformising parameter for R, and let $z \in R$. Then for any $n \geq 0$ there are unique $\lambda_0, \ldots, \lambda_n \in k$ and $z_n \in R$ such that:

$$z = \lambda_0 + \lambda_1 t + \ldots + \lambda_n t^n + z_n t^{n+1}$$

Solution: (a) This follows immediately from the hypothesis on k and R/M: if $z \in R$, then z is congruent to some element $\lambda \in k$ modulo M, so $z - \lambda = 0$ in R/M. ♣

(b) Choose a non-negative integer n and an element $z \in R$. We'll induce on n. For the $n = 0$ case, note that from part (a), there is an element $\lambda_0 \in k$ such that $z - \lambda_0 \in M$, so $z = \lambda_0 + z_0 t$ for some $z_0 \in M$.

Now assume that there are elements $\lambda_0, \ldots, \lambda_{n-1} \in k$ and $z_{n-1} \in R$ such that:

$$z = \lambda_0 + \lambda_1 t + \ldots + \lambda_{n-1} t^{n-1} + z_{n-1} t^n$$

From part (a), we can write $z_{n-1} = \lambda_n + \alpha t$ for some elements $\lambda_n \in k$ and $\alpha \in R$, so we get:

$$z = \lambda_0 + \lambda_1 t + \ldots + \lambda_n t^n + (\alpha z_{n-1}) t^{n+1}$$

which proves the existence of such an expression.

For uniqueness, assume that we had two expressions:

$$\begin{aligned} z &= \lambda_0 + \ldots + \lambda_n t^n + z_n t^{n+1} \\ z &= \lambda'_0 + \ldots + \lambda'_n t^n + z'_n t^{n+1} \end{aligned}$$

By subtraction, we get:

$$(\lambda_0 - \lambda_0') + \ldots + (\lambda_n - \lambda_n')t^n + (z_n - z_n')t^{n+1} = 0$$

If this series isn't just a bunch of zeroes added together, then there's some first nonzero term $(\lambda_i = \lambda_i')t^i$. This term is in M^i but not M^{i+1}, because $\lambda_i - \lambda_i' \neq 0$, and is therefore a unit in R.

But the ith term is just the negative of the sum of all the following terms, which all lie in M^{i+1}! So there can't be any nonzero term in the sum, and hence the two expressions are identical. ♣

6. Let $\phi\colon X \to \mathbb{P}^1$ be a surjective morphism of projective varieties, and assume that it has degree two – that is, assume that $[K(X) : \phi^*(K(\mathbb{P}^1))] = 2$. The branch locus of ϕ is the set:

$$B = \{P \in \mathbb{P}^1 \mid \#\phi^{-1}(P) = 1\}$$

If $B = \{[1:0], [0:1]\}$, prove that X is birational to $\mathbb{P}^1$.

Solution: Identify $K(\mathbb{P}^1)$ with $\mathbb{C}(t)$, and write $x = \phi^*(t)$. Then $K(X) = k(x, \sqrt{p(x)})$ for some squarefree polynomial $p(x) \in \mathbb{C}[x]$. (This is by the quadratic formula.)

The rational function t on $\mathbb{P}^1$ has a zero at $[0:1]$ and a pole at $[1:0]$, and has a well defined nonzero value at all other points of $\mathbb{P}^1$, where we are choosing homogeneous coordinates u and v on $\mathbb{P}^1$ such that $t = u/v$. Thus, the rational function $p(t)$ has zeroes at $[\alpha:1]$ for each root α, a pole at $[1:0]$, and a well defined nonzero value at all other points of $\mathbb{P}^1$.

Now view X as birational to a curve in $\mathbb{A}^2$, defined by the polynomial $y^2 - p(x) = 0$. When restricted to this affine piece of X, the morphism ϕ becomes $\phi(x,y) = x$. The branch locus of ϕ then becomes the set of $x \in \mathbb{A}^1$ such that $p(x) = 0$ ... but this set is precisely the set $\{0\}$, by hypothesis! (The other point is at infinity in this setup.) Thus, $p(x) = x^n$ for some non-negative integer n.

Clearly $n \neq 0$, since then p is constant and ϕ has degree 1, not 2. The only other squarefree possibility for $p(x)$ is x, in which case $K(X)$ is isomorphic to $\mathbb{C}(x, \sqrt{x})$. This is isomorphic to $\mathbb{C}(x)$, and we obtain that X is birational to $\mathbb{P}^1$, as desired. ♣