PMath 464/764 – Homework 9
Due Monday, July 12

1. In each part, compute $\operatorname{ord}_P(f)$ on the curve X.

(a) $X = Z(x+y)$, $P = (0,0)$, $f(x,y) = x^5 - y^5$.

(b) $X = Z(y^2 - x^3 - x)$, $P = (0,0)$, $f(x,y) = (x^2 - y)/(xy)$.

(c) $X = Z(x^2 + y^2 - 1)$, $P = (0,1)$, $f(x,y) = (x+y-1)/(2x+y-1)$.

2. Let X be the closed subset $V(x^3 + y^3 - z^3 - w^3, xy + zw) \subset \mathbb{P}^3$. Show that $\phi(x : y : z : w) = [x : z]$ or $[-w : y]$ defines a morphism from X to $\mathbb{P}^1$.

3. Let X be the closed subset $V(x^3+y^3-z^3-w^3, xy+zw) \subset \mathbb{P}^3$, and define $\phi\colon X \to \mathbb{P}^1$ by $\phi(x : y : z : w) = [x : z]$ or $[-w : y]$. Compute the degree of ϕ. [You may assume that X is irreducible.]

4. Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map defined by $\phi(x : y : z) = [yz : xz : xy]$.

(a) Show that ϕ is birational, and its own inverse.

(b) Find open sets U and V in $\mathbb{P}^2$ such that $\phi\colon U \to V$ is an isomorphism.

(c) Find the largest open set U where ϕ is defined.

5. Let R be a DVR with maximal ideal M, and quotient field K. Suppose that R contains a field k such that the composition:

$$k \to R \to R/M$$

is an isomorphism of k with R/M. Verify the following assertions:

(a) For any $z \in R$, there is a unique $\lambda \in k$ such that $z - \lambda \in M$.

(b) Let t be a uniformising parameter for R, and let $z \in R$. Then for any $n \geq 0$ there are unique $\lambda_0, \ldots, \lambda_n \in k$ and $z_n \in R$ such that:

$$z = \lambda_0 + \lambda_1 t + \ldots + \lambda_n t^n + z_n t^{n+1}$$

6. Let $\phi\colon X \to \mathbb{P}^1$ be a surjective morphism of projective varieties, and assume that it has degree two – that is, assume that $[K(X) : \phi^*(K(\mathbb{P}^1))] = 2$.

The branch locus of ϕ is the set:

$$B = \{P \in \mathbb{P}^1 \mid \#\phi^{-1}(P) = 1\}$$

If $B = \{[1:0], [0:1]\}$, prove that X is birational to $\mathbb{P}^1$.