PMath 464/764 – Homework 7 solutions

1. Which points of $\mathbb{P}^2$ belong to only one of the three sets $X \neq 0$, $Y \neq 0$, and $Z \neq 0$?

Solution: There are three such points: $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. Every other point with two coordinates equal to zero is the same as one of these three, up to multiplying by a scalar. (Remember that $[0 : 0 : 0]$ isn't a point!) ♣

2. Show that any two distinct lines in $\mathbb{P}^2$ intersect in a point.

[A line in $\mathbb{P}^2$ is a projective algebraic set defined as the zero set of a single nonzero linear polynomial.]

Solution: Two distinct lines in $\mathbb{P}^2$ are defined by two linear equations $L_1 = 0$ and $L_2 = 0$ in three variables, and since the lines are distinct, we know that L_1 and L_2 are not scalar multiples of one another. This means they have to be linearly independent.

The intersection points of L_1 and L_2 are the points P in projective space whose homogeneous coordinates satisfy $L_1(P) = L_2(P) = 0$. Since L_1 and L_2 are linearly independent, it follows that this linear system has rank 2, so since there are 3 variables, the solution space has dimension 3-2=1. In other words, there is a nonzero solution $[x : y : z]$ of this system such that any solution of this system is a multiple of $[x : y : z]$. But points in projective space are only defined up to scalar multiplication, so we conclude that $[x : y : z]$ is the unique point of intersection. ♣

3. Find all the intersection points of the two projective curves $x^2 - y^2 + xz + z^2 = 0$ and $x + y + z = 0$. [Hint: Compute on an affine piece first, then check the rest.]

Solution: First, let's find the intersection points lying in the affine piece $z = 1$, and then check the line at infinity $z = 0$. If we set $z = 1$, we get the affine system:

$$\begin{aligned} x^2 - y^2 + x + 1 &= 0 \\ x + y + 1 &= 0 \end{aligned}$$

Solving the second equation for y and plugging it into the first gives $-x = 0$, so $x = 0$ and hence $y = -1$. Thus, the point $[0 : -1 : 1]$ is the only intersection point of the two curves in the affine piece $z = 1$.

If $z = 0$, then we get the following system of equations:

$$\begin{aligned} x^2 - y^2 &= 0 \\ x + y &= 0 \end{aligned}$$

The first equation is clearly redundant, so we just have $x + y = z = 0$. This might appear to define lots of different points, but in fact since a point in projective space is determined only up to scalar multiple, it determines the unique point $[1 : -1 : 0]$. Thus, there are two intersection points of the two curves: $[1 : -1 : 0]$ and $[0 : -1 : 1]$. ♣

4. Find an algebraic subset $V \subset \mathbb{P}^2$ such that $V \cap \{x \neq 0\}$, $V \cap \{y \neq 0\}$, and $V \cap \{z \neq 0\}$ are all irreducible, but V is reducible. [Hint: Question 1.]

Solution: Let $V = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. Then the intersection of V with each affine piece of $\mathbb{P}^2$ is a single point – and therefore irreducible – but the three-point V is obviously reducible, with three components.

Oddly enough, because the empty set is not irreducible, this is the only possible example. (See if you can prove that – it's fun!) ♣

5. Let V be an algebraic subset of $\mathbb{A}^n$, embedded as $x_0 \neq 0$ in $\mathbb{P}^n$. If V is irreducible, prove that its projective closure W is also irreducible.

Solution: Let $W = W_1 \cup \ldots \cup W_r$ be the irreducible decomposition of W. Then

$$V = (V \cap W_1) \cup \ldots \cup (V \cap W_r)$$

expresses V as a union of closed sets. Since V is irreducible, we must have

$$V = V \cap W_i$$

for some i. But then W_i is a closed subset of $\mathbb{P}^n$ that contains V, so we have $W \subset W_i$, by definition of the projective closure. Which means $W = W_i$, and W is irreducible, as desired. ♣

6. Find all the irreducible components of the projective algebraic set $V = V(xw - yz, z^2 - yw) \subset \mathbb{P}^3$.

Solution: There are two components: the line $z = w = 0$, and the irreducible curve $xw - yz = z^2 - yw = y^2 - xz = 0$, which is the projective closure of $V(X - YZ, Z^2 - Y)$.

Let's work in affine pieces, because it's easier. First, let's work in $w \neq 0$, whereupon our algebraic set becomes

$$V(X - YZ, Z^2 - Y) \subset \mathbb{A}^3$$

Let's show that this is an irreducible subset of $\mathbb{A}^3$, by computing the quotient

$$\mathbb{C}[X, Y, Z]/(X - YZ, Y - Z^2)$$

Well, the first step is to notice that

$$\mathbb{C}[X, Y, Z]/(X - YZ, Y - Z^2) \cong \mathbb{C}[Y, Z]/(Y - Z^2)$$

via the isomorphism $p(X, Y, Z) \mapsto p(YZ, Y, Z)$. Similarly, we compute

$$\mathbb{C}[Y, Z]/(Y - Z^2) \cong \mathbb{C}[Z]$$

via the isomorphism $p(Y, Z) \mapsto p(Z^2, Z)$.

Since $\mathbb{C}[Z]$ is a domain, the ideal $(X - YZ, Z^2 - Y)$ is prime, so the algebraic set $V(X-YZ, Z^2-Y)$ is irreducible. Since the projective closure of an irreducible algebraic set is itself irreducible, we have found one irreducible component of V, namely, the projective closure D of $V(X - YZ, Z^2 - Y)$.

That just leaves the points where $w = 0$. Any point with $w = z = 0$ satisfies $xw - yz = z^2 - yw = 0$, so it lies on V. That means the line $w = z = 0$ is contained in V. And those are the only points of V with $w = 0$, so every point of V either lies on $w = z = 0$, or else it's in the set D.

The next question is: is $z = w = 0$ contained in D? Intuitively, you expect the answer to be "no", because it's a one-dimensional algebraic set, and V is itself a one-dimensional algebraic set, so you expect $w = z = 0$ to be an irreducible component of V. Let's prove that intuition.

We know $Y = Z^2$ in the affine ($w \neq 0$) part of V. That means $Y^2 = YZ^2$. But $YZ = X$, so we get $Y^2 = XZ$. In Affineland, this isn't news – we did all that just by fooling around with equations. But in projective space, we've come up with a new relation! The equation $y^2 = xz$ isn't satisfied by all the points of $w = z = 0$, so that line is not contained in the projective closure D of the other piece. So, V is the union of the irreducible D and the irreducible $w = z = 0$.

The only thing left to do is to prove that $D = V(xw-yz, z^2-yw, y^2-xz)$. Well, because D is contained in $V(xw-yz, z^2-yw, y^2-xz)$, we know that the D has at most one point that isn't in the set $V(X-YZ, Z^2-Y)$, and that's the point $P = [1:0:0:0]$. All we have to do is show that P is in the projective closure of $V(X - YZ, Z^2 - Y)$, and we're done. But this is easy: the affine piece $x \neq 0$ of $V(xw - yz, z^2 - yw, y^2 - xz)$ is $V(W - YZ, Z^2 - YW, Y^2 - Z)$, which is irreducible, contains more than one point, and contains the point $(0, 0, 0)$, which corresponds to P. ♣