PMath 464/764 – Homework 6
Due Monday, June 21

1. Let $V = V(Y - X^2, Z - X^3) \subset \mathbb{A}^3$. Find a vector (a, b, c) such that the tangent space to V at $(1, 1, 1)$ is

$$T_{(1,1,1)}(V) = \{(1, 1, 1) + t(a, b, c)\}$$

2. Let $V = V(X^2 - Y^3, X - Z^3) \subset \mathbb{A}^3$. Compute the dimension of the Zariski tangent space to V at $(0, 0, 0)$.

3. Let V be a one-dimensional variety, and let $W \subset V$ be any algebraic subset. Prove that if $W \neq V$, then W is finite.

4. Let $W = V(F)$ be an irreducible curve in $\mathbb{A}^2$, and let S be an infinite subset of W. Prove that any rational function z which vanishes on S (that is, whose value at every point of S is zero) must be identically zero.

5. Let $V = V(F)$ be the zero set of a polynomial of degree 2 in $\mathbb{C}[X, Y]$. If V has a singular point, prove that V is reducible.

[Hint: After a translation, you can assume that the singular point is $(0, 0)$.]

6. Let $V = V(F_1, \ldots, F_r) \subset \mathbb{A}^n$ be a variety, and let $P \in V$ be a point. Let $H = V(L_1, \ldots, L_s)$ be the tangent space to V at P, defined by linear polynomials L_i, and let $W = V \cap H$.

If $(L_1, \ldots, L_s, F_1, \ldots, F_r)$ is a prime ideal of $\mathbb{C}[X_1, \ldots, X_n]$, and if W is a proper subvariety of V, prove that P is a singular point of W.