PMath 464/764 – Homework 5 solutions

1. Compute the dimension of $B = V(Y - X^2, Z - W^2) \subset \mathbb{A}^4$.

Solution: The dimension of B is two.

The dimension of $\mathbb{A}^4$ is four. All we need to do is find a sequence of subvarieties of $\mathbb{A}^4$, including B, that has four links in it. Here's my sequence:

$$\{(0,0,0,0)\} \subsetneq V(X, Y, Z - W^2) \subsetneq B \subsetneq V(Y - X^2) \subsetneq \mathbb{A}^4$$

It's pretty easy to see that all these inclusions are strict, so all we need to do is to justify the irreducibility of each algebraic set.

This is easy for $\{(0,0,0,0)\}$, $V(Y - X^2)$, and $\mathbb{A}^4$. To check the irreducibility of $V(X, Y, Z - W^2)$ and $V(Y - X^2, Z - W^2)$, we'll show that the corresponding ideals are prime.

The homomorphism $\phi\colon \mathbb{C}[X, Y, Z, W] \to \mathbb{C}[W]$ given by

$$\phi(f(X, Y, Z, W)) = f(0, 0, W^2, W)$$

is clearly onto, and its kernel is the ideal $(X, Y, Z - W^2)$. The First Isomorphism Theorem then shows that $(X, Y, Z - W^2)$ is prime.

The homomorphism $\psi\colon \mathbb{C}[X, Y, Z, W] \to \mathbb{C}[X, W]$ given by

$$\psi(f(X, Y, Z, W)) = f(X, X^2, W^2, W)$$

is again clearly onto, and its kernel is the ideal $(Y - X^2, Z - W^2)$. Since $\mathbb{C}[X, W]$ is a domain, the First Isomorphism Theorem means that the ideal $(Y - X^2, Z - W^2)$ is prime.

So the sequence described above is good enough to compute the dimension of B! Which is therefore two. ♣

2. Find the singular points of the curve $X^4 + Y^4 - X^2Y^2 = 0$.

Solution: The only singular point is $(0, 0)$.

We'll set the partial derivatives to zero, and solve. We have the following system:

$$\begin{aligned} 4X^3 - 2XY^2 &= 0 \\ 4Y^3 - 2X^2Y &= 0 \\ X^4 + Y^4 - X^2Y^2 &= 0 \end{aligned}$$

From the first equation, either $X = 0$ or $2X^2 = Y^2$. From the second equation, either $Y = 0$ or $2Y^2 = X^2$. Taking these together shows that if either of X or Y is 0, then the other must be as well, so possible singular points are (0,0) (which is easily seen to be a singular point), or solutions of the following system:

$$
\begin{aligned}
2X^2 - Y^2 &= 0 \\
2Y^2 - X^2 &= 0 \\
X^4 + Y^4 - X^2Y^2 &= 0
\end{aligned}
$$

But the first two equations already imply that $3X^2 = 3Y^2 = 0$, so (0,0) is the only singular point. ♣

3. Let $F(X,Y) = Y^2 - f(X)$ for some polynomial $f(X)$. Show that the singular points of the curve $F = 0$ are precisely the points $(a, 0)$, where $X = a$ is a multiple root of $f(X) = 0$.

Solution: We get the following system of equations:

$$
\begin{aligned}
f'(X) &= 0 \\
2Y &= 0 \\
Y^2 - f(X) &= 0
\end{aligned}
$$

So a point (X,Y) on $F = 0$ is singular if and only if $Y = f(X) = f'(X) = 0$. Solutions of this system are pairs of the form $(a, 0)$, where $X = a$ is a common root of $f(X) = 0$ and $f'(X) = 0$. Such a are precisely the multiple roots of $f(X) = 0$. ♣

4. Let C be a one-dimensional subvariety of $\mathbb{A}^2$. Prove that $C = V(F)$ for some nonzero polynomial F.

Solution: Let G be any nonzero element of $I(C)$. Since C is a variety, its ideal is prime, and so some irreducible factor F of G is also in $I(C)$. We will show that $C = V(F)$.

Certainly $C \subset V(F)$. So if $P \in C$ is any point, this gives us the following chain of subvarieties of $\mathbb{A}^2$:

$$\{P\} \subsetneq C \subset V(F) \subsetneq \mathbb{A}^2$$

The dimension of $\mathbb{A}^2$ is two, so the containment $C \subset V(F)$ has to be an equality. ♣

5. Show that an irreducible plane curve has only a finite number of singular points.

Solution: Let $F(X,Y)=0$ be an irreducible plane curve. If we were to look for singular points, we'd solve the following system of polynomial equations:

$$\begin{aligned} F_X(X,Y) &= 0 \\ F_Y(X,Y) &= 0 \\ F(X,Y) &= 0 \end{aligned}$$

We wish to show that $V(F_X, F_Y, F)$ is finite.

Let V be any irreducible component of $W = V(F_X, F_Y, F)$. If we can show that V is a single point, then we are done, because any algebraic set has only finitely many irreducible components.

If W is infinite, then its dimension must be at least one. (Every point of W gives a chain of length one with W.) Since W can't have dimension two – it's a proper subset of $\mathbb{A}^2$ – this means that $W = V(H)$ for some irreducible polynomial H.

But $W \subset V(F)$, so $V(H) \subset V(F)$ and therefore H is a factor of F. Since F is irreducible, this means that we can take $H = F$.

It's equally true, however, that H must be a factor of F_X and F_Y as well. So F is a factor of F_X and F_Y as well. Which, since F has larger degree than F_X and F_Y, means that $F_X = F_Y = 0$.

This means F is constant, and hence $V(F)$ is not a curve (it's empty). So W wasn't really infinite, and so $F = 0$ has a finite set of singular points after all. ♣

6. Let $f\colon X \to Y$ be a morphism.

(a) If f is injective, prove that $\dim X \leq \dim Y$.

(b) If f is surjective, prove that $\dim X \geq \dim Y$.

Solution: (a) Let $P_0 \subsetneq P_1 \subsetneq \ldots \mathbb{P}_{\dim X}$ be a sequence of prime ideals in the coordinate ring $\Gamma(X)$. We first assume that X is a subvariety of Y, and prove the result in that case, by constructing a sequence of prime ideals of $\Gamma(Y)$ of the same length.

If X is a subvariety of Y, then there is a quotient map $q\colon \Gamma(Y) \to \Gamma(X) \cong \Gamma(Y)/I(X)$. Since the quotient map is surjective, we get a nice and easy sequence of prime ideals of $\Gamma(Y)$:

$$q^{-1}(P_0) \subsetneq \ldots \subsetneq q^{-1}(P_{\dim X})$$

It is straightforward to check that $q^{-1}(P_i)$ is a prime ideal, and that the inclusions are all strict. (That last one is because q is onto.)

Notice too that if X is a subvariety of Y, then the only case in which $\dim X = \dim Y$ is if $X = Y$. In every other case, we could add Y to the end of any chain of subvarieties of X to get a strictly longer chain.

Now for the general case. In light of what we've just proven, we can replace Y with $V(I(f(X)))$, and assume that f is dominant as well as injective.

In that case, we have a well defined homomorphism of function fields $f^*\colon K(V(I(f(X)))) \to K(X)$. (Note that $V(I(f(X)))$ is irreducible because its preimage under f is the irreducible X.) Since f^* is a homomorphism of fields, we know that f^* is injective, and so we can regard $K(X)$ as a field extension of $f^*K(V(I(f(X))))$.

Here's where the "too hard" thing kicks in: it turns out that because f is injective and dominant, those two fields are actually *equal*:

$$K(X) = f^*K(V(I(f(X))))$$

This means, from what we already know, that X and $V(I(f(X)))$ are birational to one another, and therefore there are open subsets U of X and U' of $V(I(f(X)))$ on which f is an isomorphism. That is:

$$f(U) = U'$$

and the rational map inverse f^{-1} of f is defined on U' and satisfies

$$f^{-1}(U') = U$$

Let's now extend (slightly) the definition of dimension to arbitrary Zariski open subsets of varieties. It's exactly the same as the usual definition:

Definition 1 *Let V be a Zariski open subset of a variety X. Let*

$$V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V$$

be a chain of maximal length such that each V_i is the intersection with V of a subvariety of X. Then the **dimension** *of V is n. The empty set does not have a dimension.*

We will now show that U has the same dimension as X, and U' has the same dimension as $V(I(f(X)))$. Since U and U' are isomorphic, they also have the same dimension, and we'll be done.

Theorem 1 *Let X be a variety, and U a non-empty open subset of X. Then*

$$\dim X = \dim U$$

Proof: We first show that $\dim U \leq \dim X$. Let

$$V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = U$$

be a chain such that each V_i is the intersection with U of a subvariety X_i of X. Then we have a chain

$$X_0 \subset X_1 \subset \ldots \subset X_n = X$$

of subvarieties of X, with strict inclusions because $V_i \neq V_{i+1}$ immediately implies that $X_i \neq X_{i+1}$.

Now for $\dim X \leq \dim U$. This we will prove by induction on the dimension of X.

If the dimension of X is zero, the claim is trivial. If the dimension is one, then the claim is again trivial. Thus, we may assume that the dimension of X is at least two.

Say $X \subset \mathbb{A}^n$, and let $P \in U$ be any point. Let H be a linear subspace of $\mathbb{A}^n$ (a subvariety defined by a linear polynomial) that contains P. Then $\dim H = \dim X - 1$, by Krull's Hauptidealsatz.

But $H \cap U$ is a nonempty open subset of H, so by induction, the dimension of $H \cap U$ is also $\dim X - 1$. This means, by adding U to the end of any chain in $H \cap U$, we have $\dim U \geq \dim X$. Which is what we wanted. ♣

So now we have $\dim X = \dim U = \dim U' = \dim V(I(f(X)))$, and we're done. ♣

(b) There is a chain of subvarieties of Y:

$$V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_{\dim Y} = Y$$

It would be great if $f^{-1}(V_i)$ were irreducible for all i, because then we could just prove that and be done.

Sadly, it's not true. But it's close: for each i, there is an irreducible component Z_i of $f^{-1}(V_i)$ such that $V(I(f(Z_i))) = V_i$.

(The set $V(I(f(Z_i)))$ is called the **closure** of $f(Z_i)$. It's not the same as $f(Z_i)$ because $f(Z_i)$ might not be an algebraic set, but it is contained in every algebraic set that contains $f(Z_i)$.)

To prove the existence of this Z_i, pick a V_i, and write $f^{-1}(V_i) = W_1 \cup \ldots \cup W_r$ as a union of irreducible components. Then $V_i = f(W_1) \cup \ldots \cup f(W_r)$, and so

$$V_i = V(I(f(W_1))) \cup \ldots \cup V(I(f(W_r)))$$

is a decomposition of V_i as a union of Zariski closed sets. Since V_i is irreducible, this means that $V_i = V(I(f(W_j)))$ for some j. Define $Z_i = W_j$.

We now have a sequence of subvarieties of X:

$$Z_0 \subset Z_1 \subset \ldots \subset Z_{\dim Y}$$

Since $V_i \neq V_{i+1}$, it follows that $V(I(f(W_j))) \neq V(I(f(W_{j+1})))$, and therefore $Z_j \neq Z_{j+1}$. So this sequence has strict inclusions, and we deduce that $\dim X \geq \dim Y$, as desired. ♣