PMath 464/764 – Homework 4
Due Monday, June 7

1. Let $V = V(Y - X^2) \subset \mathbb{A}^2(\mathbb{C})$, and $P = (1,1) \in V$. Which of the following three rational functions are equal in $\mathcal{O}_P(V)$?

1. $F_1(X,Y) = \frac{1}{X+1}$
2. $F_2(X,Y) = \frac{X}{X+Y}$
3. $F_3(X,Y) = \frac{X^2}{X+Y^2}$

2. Let V be a variety, $P \in V$ any point, $\mathcal{O}_P(V)$ the local ring of V at P. Let $\phi\colon \mathcal{O}_P(V) \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism. Prove that $\phi(f) = f(P)$ for all $f \in \mathcal{O}_P(V)$.

3. Let $F(X,Y) \in \mathbb{C}[X,Y]$ be a nonzero polynomial. Let $U \subset \mathbb{A}^2$ be the Zariski open subset $U = \mathbb{A}^2 - V(F)$. (In other words, it's the set of points where $F \neq 0$.) Prove that $\Gamma(U) = \mathbb{C}[X, Y, 1/F]$.

4. Let $V = V(Y^2 - X^3 - X) \subset \mathbb{A}^2$ be a curve, and let $P = (0,0)$. Let $\mathcal{M} = \mathcal{M}_P(V) \subset \mathcal{O}_P(V)$ be the maximal ideal of the local ring of V at P. Prove that $\dim_{\mathbb{C}} \mathcal{M}/\mathcal{M}^2 = 1$.

5. Let $V = V(X^2 - Y^3, Y^2 - Z^3) \subset \mathbb{A}^3$, and let $P = (0,0,0)$. Write $\mathcal{M} = \mathcal{M}_P(V)$. Show that $\dim_{\mathbb{C}}(\mathcal{M}/\mathcal{M}^2) = 3$.

6. Let $\phi\colon \mathbb{A}^2 \to \mathbb{A}^2$ be a polynomial map, given by: $\phi(X,Y) = (f(X,Y), g(X,Y))$. Define the Jacobian matrix of ϕ to be the following matrix:

$$J = \begin{bmatrix} \partial f/\partial X & \partial f/\partial Y \\ \partial g/\partial X & \partial g/\partial Y \end{bmatrix}$$

Show that if ϕ is an isomorphism, then $\det J$ is a nonzero constant. [Hint: What is the Jacobian of ϕ^{-1}, and how does it relate to J?]

Note: The converse of this problem is unknown, and is the $n = 2$ case of the Jacobian Conjecture. (The general case of the Conjecture is exactly the same as the $n = 2$ case, only with n polynomials and variables instead of 2.) If you prove it, let me know.