PMath 464/764 – Homework 3 solutions

1. Decompose $V(Y^4-X^2, Y^4-X^2Y^2+XY^2-X^3) \subset \mathbb{A}^2(\mathbb{C})$ into irreducible components.

Solution: Write $D = V(Y^4-X^2, Y^4-X^2Y^2+XY^2-X^3)$. If we factor the two polynomials involved, we get:

$$Y^4 - X^2 = (Y^2 - X)(Y^2 + X)$$
$$Y^4 - X^2Y^2 + XY^2 - X^3 = (Y - X)(Y + X)(Y^2 + X)$$

So we immediately see that the parabola $Y^2 + X = 0$ is a subset of D. It's irreducible because the polynomial $Y^2 + X$ is irreducible: considered as a polynomial in X, it has degree one, and its coefficients Y^2 and 1 have no common factors. Therefore, since $\mathbb{C}[X, Y]$ is a UFD, the ideal $(Y^2 + X)$ is prime and the parabola is irreducible.

Let's now try to find all the points of D which do not lie on this parabola.

If (x, y) is on D but not on the parabola $Y^2 + X = 0$, then the first equation gives $y^2 = x$, and the second gives $y^2 = x^2$ (since $y^2 + x \neq 0$). Thus, $x^2 = x$, and hence $x = 0$ or $x = 1$. Plugging these values into $y^2 = x$ gives three points, $(0, 0)$, $(1, 1)$, and $(1, -1)$. But $(0, 0)$ lies on the parabola $Y^2 + X = 0$, so we can delete it from our list of irreducible components. Thus, we have the following irreducible decomposition of D:

$$D = \{(1, -1)\} \cup \{(1, 1)\} \cup \{Y^2 + X = 0\}$$

All three sets are irreducible, the first two because they're points (or because $\mathbb{C}[X, Y]/(X - 1, Y \pm 1) \cong \mathbb{C}$, which is a domain and hence $(X - 1, Y \pm 1)$ is prime, so $(1, 1)$ and $(1, -1)$ are irreducible), and the second because we proved it. ♣

2. Let D be the algebraic set $V(2X^3 - X^2Y - 2XY + Y^2) \subset \mathbb{A}^2$. Find all the irreducible components of D. [Hint: Use the quadratic formula.]

Solution: If we use the quadratic formula on $2X^3 - X^2Y - 2XY + Y^2$ as a polynomial in Y, we come up with the following factorisation:

$$2X^3 - X^2Y - 2XY + Y^2 = (Y - 2X)(Y - X^2)$$

Thus, D is the union of the line $Y = 2X$ and the parabola $Y = X^2$ – that is, $D = V(Y - 2X) \cup V(Y - X^2)$. It looks like these two curves ought to be the irreducible components of D, and indeed they are! To prove it, we just need to show that $V(Y-2X)$ and $V(Y-X^2)$ are irreducible. It suffices to show that the ideals $(Y-2X)$ and $(Y-X^2)$ are prime.

Anyway, let's show that $(Y-2X)$ and $(Y-X^2)$ are prime. We could argue the same way we did in the previous problem, but we'll do it a different way this time, just for kicks.

Let's take the quotients of $\mathbb{C}[X,Y]$ by the ideals $(Y-2X)$ and $(Y-X^2)$, and check that we end up with domains. This will prove that the ideals $(Y-2X)$ and $(Y-X^2)$ are prime, and that the sets $V(Y-2X)$ and $V(Y-X^2)$ are irreducible, as desired.

In both cases, we end up with a ring isomorphic to $\mathbb{C}[X]$, since the homomorphisms ϕ, $\psi\colon \mathbb{C}[X,Y] \to \mathbb{C}[X]$ defined by $\phi(F(X,Y)) = F(X,2X)$ and $\psi(F(X,Y)) = F(X,X^2)$ are surjective – their images contain X! – with kernels $(Y-2X)$ and $(Y-X^2)$, respectively. By the First Isomorphism Theorem, those kernels are both prime ideals, just like we wanted. ♣

3. Let $V = V(Y^2 - X^2(X+1)) \subset \mathbb{A}^2$, and let $\overline{X}$ and $\overline{Y}$ be the residues of X and Y, respectively, in $\Gamma(V)$. Let $z = \overline{Y}/\overline{X} \in K(V)$. Find the pole sets of z and z^2.

Solution: The pole set of a rational function z is the set of points P where z is not defined. That means that for any pair of polynomials f and g satisfying $z = f/g$, we have $g(P) = 0$.

Now consider $z = \overline{Y}/\overline{X}$. Written that way, we see that z is defined at every point whose X-coordinate is nonzero. Since the only point of V whose X-coordinate is 0 is the point $(0,0)$, that means that z is defined at every point of V except possibly $(0,0)$. Let's try to prove that z is not defined at $(0,0)$.

Say that $z = f/g$ with $g(0,0) \neq 0$. Then we'd have

$$\overline{X}f(\overline{X},\overline{Y}) = \overline{Y}g(\overline{X},\overline{Y})$$

in $\Gamma(V)$. Lifting this to $\mathbb{C}[X,Y]$ gives:

$$Xf(X,Y) = Yg(X,Y) + (Y^2 - X^2(X+1))h(X,Y)$$

where $h(X,Y)$ is some polynomial. Since $g(0,0) \neq 0$, this means that the constant term in $g(X,Y)$ is nonzero, and hence there is a term of the form αY on the right side of the above equation. (It can't be cancelled by any term of $(Y^2 - X^2(X+1))h(X,Y)$ because every term of that has degree at least 2.) But every term on the left side of the equation is divisible by X, so we've derived a contradiction. Therefore, the pole set of z is exactly the point $(0,0)$.

For z^2, we simply note that $(\overline{Y}/\overline{X})^2 = X+1$ in $K(V)$, so the pole set of z^2 is clearly empty. Which, to be honest, is kind of freaky, given that z had a pole there. ♣

4. Let $F(X,Y) = Y^2 - X^3 + X \in \mathbb{C}[X,Y]$, and let a and b be constants (elements of $\mathbb{C}$). Write $W = V(F)$, and let P be the point $(0,0)$ on W.

(a) Show that $aX + bY$ is an element of the maximal ideal $M = M_P(W)$ of the local ring $\mathcal{O}_P(W)$ at the point $P = (0,0)$.

(b) Show that $aX + bY$ is an element of M^2 if and only if the line $aX + bY = 0$ is tangent to W at $(0,0)$. (M is the same as in part (a).)

Solution: (a) We have $a(0) + b(0) = 0$ for any a and b, so it instantly follows that $aX + bY \in M$.

(b) First, we need to compute the tangent line to W at the point $(0,0)$. We do this using implicit differentiation:

$$\begin{aligned} Y^2 &= X^3 - X \\ 2Y\,dY &= (3X^2 - 1)dX \\ dX &= 0 \end{aligned}$$

and so the tangent line is $X = 0$. Thus, we can restate the question:

Prove that $aX + bY \in M^2$ if and only if $b = 0$.

To show that $X \in M^2$, we need only produce two elements r and s in M such that $X = rs$. So, for example, we could set $r = Y$ and $s = Y/(X^2+1)$, and that would do the trick. (Both r and s are in M because their values are 0.)

Now assume that $b \neq 0$. We wish to show that $aX + bY \notin M^2$. It suffices to show that $Y \notin M$, because if $Y \notin M$, then $bY \notin M$ for $b \neq 0$, and hence $aX + bY \notin M$, because we already know that $X \in M^2$.

Thus, assume $Y \in M^2$. Then we can find elements $r_1, \ldots, r_n, s_1, \ldots, s_n \in M$ such that $Y = \sum_i r_i s_i$. By clearing denominators and lifting to $\mathbb{C}[X,Y]$, we can find polynomials f_i, g_i, h, and P in $\mathbb{C}[X,Y]$ such that:

$$Yh(X,Y) = f(X,Y)g(X,Y) + (Y^2 - X^3 + X)P(X,Y)$$

where $f_i(0,0) = g_i(0,0) = 0$, but $h(0,0) \neq 0$. Let's now take the partial derivative of both sides with respect to Y:

$$h(X,Y) + Yh_Y(X,Y) = \sum_i (f_i)_Y(X,Y)g_i(X,Y) + f_i(X,Y)(g_i)_Y(X,Y) \\ + 2YP(X,Y) + (Y^2 - X^3 + X)P_Y(X,Y)$$

Plugging in $X = Y = 0$ gives:

$$h(0,0) = 0$$

which is a contradiction! So $Y \notin M^2$, as advertised. ♣

5. Let $W = V(Y^2 - X^2(X+1)) \subset \mathbb{A}^2(\mathbb{C})$, $P = (0,0)$. Show that for every pair of complex numbers a and b that $aX + bY$ is not an element of $(M_P(W))^2$ unless $a = b = 0$.

(The difference between this question and the previous one is that $Y^2 - X^3 + X = 0$ is smooth at (0,0), whereas $Y^2 - X^2(X+1)$ has a node at (0,0). Draw the pictures and check it out, if you like.)

Solution: As in the previous problem, assume that there are elements r_i and s_i in M such that $aX + bY = \sum_i r_i s_i$. Then, as in the previous problem, we get polynomials f_i, g_i, h, and P in $\mathbb{C}[X,Y]$ satisfying:

$$(aX + bY)h(X,Y) = \sum_i f_i(X,Y)g_i(X,Y) + (Y^2 - X^3 - X^2)P(X,Y)$$

where $f(0,0) = g(0,0) = 0$, but $h(0,0) \neq 0$. Let's now take the partial derivative of both sides with respect to X:

$$ah + (aX + bY)h_X = \sum_i ((f_i)_X g_i + f_i(g_i)_X) + (-3X^2 - 2X)P + (Y^2 - X^3 - X^2)P_X$$

and hence by plugging in $X = Y = 0$:

$$ah(0,0) = 0$$

which means $a = 0$. Now that $a = 0$, we take the partial derivative with respect to Y:

$$aXh_Y + bYh_Y + bh = \sum_i ((f_i)_Y g_i + f_i(g_i)_Y) + (2Y)P + (Y^2 - X^3 - X^2)P_Y$$

$$bh(0,0) = 0$$

which means $b = 0$. So $aX + bY \in (M_P(W))^2$ if and only if $a = b = 0$, as desired. ♣

6. Let $C = V(Y^2 - X^3)$ in $\mathbb{A}^2(\mathbb{C})$. Show that the function field $K(C)$ of C is isomorphic to $\mathbb{C}(T)$, the field of rational functions in one variable, but that $\Gamma(C)$ is not isomorphic to $\mathbb{C}[T]$.

Solution: For the first part, we need to find an isomorphism $\phi\colon \mathbb{C}(T) \to K(C)$. The element $\phi(T)$ has to have the property that any element of $K(C)$ is a rational function of $\phi(T)$. That is, every element of $K(C)$ has to be obtainable from $\phi(T)$ via a combination of addition, multiplication, division, and multiplication by constants.

Actually, there are lots of choices for ϕ that work, so let's just pick one. If $f(T)$ is a rational function of T, then let $\phi(f) = f(Y/X)$.

It's trivial to check that ϕ is a homomorphism, so let's not bother. It's injective because every homomorphism from a field to a nonzero ring is injective. All that's left is to show that ϕ is surjective.

Every element of $K(C)$ is a rational function of X and Y. Therefore, if we can show that X and Y are in the image of ϕ, then we'll know that ϕ is surjective. Consider the following:

$$\begin{aligned} X &= X^3/X^2 = Y^2/X^2 = \phi(T^2) \\ Y &= (Y/X)X = \phi(T)\phi(T^2) = \phi(T^3) \end{aligned}$$

so we're done.

Now for the other part. Assume that we have an homomorphism $\phi\colon \Gamma(C) \to \mathbb{C}[T]$. (We will show that ϕ is not an isomorphism.) Then we can write:

$$\begin{aligned} \phi(X) &= p(T) \\ \phi(Y) &= q(T) \end{aligned}$$

for polynomials $p(T)$ and $q(T)$. However, we also know that $\phi(Y^2 - X^3) = 0$, so we deduce that $(q(T))^2 = p(T))^3$, and hence that:

$$\left(\frac{q(T)}{p(T)}\right)^2 = p(T)$$

In particular, $q(T)/p(T)$ is a polynomial! So $p(T)$ is a factor of $q(T)$. Thus, we can write $q(T) = r(T)p(T)$ for some polynomial $r(T)$. Thus, every polynomial $\phi(F)$ in the image of ϕ can be written in the following form:

$$\phi(F) = a + g(T)p(T)$$

where $g(T)$ is a polynomial and a is a constant.

Now then, $p(T)$ is a perfect square, so its degree is even. If its degree is 0, then $p(T)$ is constant and the image of ϕ is just $\mathbb{C}$, so ϕ is not an isomorphism. Thus, the degree of $p(T)$ must be at least two. But then the degree of $\phi(F)$ can't be one! So ϕ is not surjective, and cannot be an isomorphism. ♣