PMath 464/764 – Homework 2
Due Monday, May 24

1. Let $F(X,Y) = X^2 + Y^2 - 1$ and $G(X,Y) = X^2 - Y^2 - 1$ be polynomials in $\mathbb{C}[X,Y]$. Show that $V(F)$ and $V(G)$ are isomorphic.

2. Let $\phi\colon \mathbb{A}^1 \to V$ be a map with $V = V(Y^2 - X^3)$ and $\phi(t) = (t^2, t^3)$. Show that although ϕ is a polynomial map which is one-to-one and onto, ϕ is not an isomorphism.

3. Let $\phi\colon \mathbb{A}^1 \to V$ be a map with $V = V(Y^2 - X^2(X+1))$ and $\phi(t) = (t^2 - 1, t(t^2-1))$. Show that ϕ is one-to-one and onto, except that $\phi(\pm 1) = (0,0)$.

4. Let $V = V(X^2 - Y^3, Y^2 - Z^3) \subset \mathbb{A}^3$, and let $\overline{\alpha}\colon \Gamma(V) \to \mathbb{C}[T]$ be given by $\overline{\alpha}(X) = T^9$, $\overline{\alpha}(Y) = T^6$, and $\overline{\alpha}(Z) = T^4$. [You may assume that $\overline{\alpha}$ is well defined.]

(a) What is the polynomial map $f\colon \mathbb{A}^1 \to V$ such that $f^* = \overline{\alpha}$?

(b) Show that f is one-to-one and onto, but not an isomorphism.

5. Let $\phi\colon V \to W$ be an onto polynomial map. If X is an algebraic subset of W, show that $\phi^{-1}(X)$ is an algebraic subset of V. If $\phi^{-1}(X)$ is irreducible, show that X is irreducible. This gives a useful test for irreducibility.

6. Let $V \subset \mathbb{A}^n$ be a variety. Show that the following are equivalent:

(i) V is a point.

(ii) $\Gamma(V) = \mathbb{C}$.

(iii) $\dim_{\mathbb{C}} \Gamma(V)$ is finite.