PMath 464/764 – Homework 10 solutions

1. Let $X = C = V(x) \subset \mathbb{P}^2_k$. Compute $\operatorname{div} y/z$.

*Solution:* The only zero of $x_1/x_2 = y/z$ is when $y = 0$, which is to say at the point $[0 : 0 : 1]$. Since $y$ is not the tangent line to $C$ at $[0 : 0 : 1]$, it follows that $\operatorname{ord}_P(y/z) = 1$ for $P = [0 : 0 : 1]$. Similarly, the only pole of $y/z$ is when $z = 0$, which is at the point $[0 : 1 : 0]$, and since $z = 0$ is not the tangent line, it follows that $\operatorname{ord}_P(y/z) = -1$ when $P = [0 : 1 : 0]$. In summary, then, we've computed:

$$\operatorname{div}(y/z) = [0 : 0 : 1] - [0 : 1 : 0]$$

2. Let $X = C = V(Y^2Z - X(X - Z)(X - \lambda Z)) \subset \mathbb{P}^2$, and let $\lambda \in k$ satisfy $\lambda \neq 0, 1$. Let $x = X/Z$ and $y = Y/Z$ be elements of $K = k(x, y)$. Calculate $\operatorname{div}(x)$ and $\operatorname{div}(y)$.

*Solution:* We get $\operatorname{div}(x/z) = \operatorname{div}(x) - \operatorname{div}(z)$ and $\operatorname{div}(y/z) = \operatorname{div}(y) - \operatorname{div}(z)$, so it suffices to compute $\operatorname{div}(x)$, $\operatorname{div}(y)$, and $\operatorname{div}(z)$.

To compute $\operatorname{div}(x)$, we need to compute the intersection points, with multiplicity, of $x = 0$ with $C$. If $x = 0$ on $C$, then we get $y^2z = 0$, so the only intersection points are $[0 : 0 : 1]$ and $[0 : 1 : 0]$. (Remember that if $x = y = 0$, then since we're in projective space, $z$ isn't allowed to be 0, and we can scale it down to 1.) By Bezout's Theorem, $\operatorname{div}(x)$ has degree 3, so one of these points is an intersection of multiplicity 2. Since $x = 0$ is indeed the tangent line to $C$ at $[0 : 0 : 1]$, that point must be the double intersection, so we get:

$$\operatorname{div}(x) = 2[0 : 0 : 1] + [0 : 1 : 0]$$

The other two divisors are easier. The intersection points of $y = 0$ and $C$ are clearly $[0 : 0 : 1]$, $[1 : 0 : 1]$, and $[\lambda : 0 : 1]$ (set $y = 0$ and solve for $x$ and $z$). By Bezout's Theorem, $\operatorname{div}(y)$ has degree 3, so these must all be simple intersections, so we get:

$$\operatorname{div}(y) = [0 : 0 : 1] + [1 : 0 : 1] + [\lambda : 0 : 1]$$

By contrast, $z = 0$ intersects $C$ in the single point $[0 : 1 : 0]$ ($z = 0$ implies $x = 0$), so by Bezout's Theorem we get:

$$\operatorname{div}(z) = 3[0 : 1 : 0]$$

We can now compute the two divisors:

$$\begin{aligned}\operatorname{div}(x/z) &= 2[0:0:1] - 2[0:1:0] \\ \operatorname{div}(y/z) &= [0:0:1] + [1:0:1] + [\lambda:0:1] - 3[0:1:0]\end{aligned}$$

and we're done. ♣

3. Let $C = V(yw - x^2, wz - xy, y^2 - xz) \subset \mathbb{P}^3$; you may assume that $C$ is an irreducible curve. Compute $\operatorname{div}(w/z)$.

*Solution:* First, let's find all the points on $C$ where $w$ or $z$ vanish, because those are the only points that are going to matter for our calculation. If $w = 0$, then certainly $x = 0$, and thus $y = 0$ as well, so the only point where $w$ vanishes is the point $[0:0:0:1]$. Similarly, if $z = 0$, then $y = 0$, so $x = 0$ as well, so we must be dealing with the point $[1:0:0:0]$.

So, let's compute the order of vanishing of $w/z$ at these two points. At $P = [0:0:0:1]$, we can restrict to the affine piece $z = 1$, so $w/z$ becomes just $w$, and we're dealing with the affine curve:

$$V(yw - x^2, w - xy, y^2 - x) \subset \mathbb{A}^3$$

at the origin. The first task is to find a uniformizer for $C$ at this point. That'll be the same as a uniformizer for the affine curve at the origin, which can be any plane not containing the tangent line. Which means we really ought to compute the tangent line first. This is easy; we compute the intersection of the tangent planes of all the surfaces which define $C$. This gives us three planes (well, two really):

$$\begin{aligned}0w + 0x + 0y &= 0 \\ w + 0x + 0y &= 0 \\ 0w - x + 0y &= 0\end{aligned}$$

So our tangent line is $w = x = 0$. Thus, $y$ is a very good-looking uniformizer indeed. Since $x = y^2$, that means that $\operatorname{ord}_P(x) = 2$, and so $\operatorname{ord}_P(w/z) = \operatorname{ord}_P(w) = \operatorname{ord}_P(xy) = 3$.

Now for the point $Q = [1:0:0:0]$. This time, the shoe's on the other foot, as we restrict to the affine piece $w = 1$. Our rational function becomes $1/z$, our point $Q$ becomes the origin, and our curve becomes:

$$V(y - x^2, z - xy, y^2 - xz)$$

I'll skip the calculations this time around (they're very similar), but our tangent line is $y = z = 0$, so $x$ is a uniformiser. That means that $\mathrm{ord}_Q(y) = 2$, and so $\mathrm{ord}_Q(w/z) = \mathrm{ord}_Q(1/z) = \mathrm{ord}_Q(1/xy) = -3$.

So, in short, our divisor is $\mathrm{div}(w/z) = 3[0:0:0:1] - 3[1:0:0:0]$. ♣

4. Compute $I_P(F, G)$, where $F$ is the curve $y^2x - z^3 = 0$, $G$ is the curve $yz + xy + 2xz = 0$, and $P$ is the point $[1:0:0]$.

*Solution:* We must work in the affine piece $x = 1$, in which case the equations boil down to $F = Y^2 - Z^3$ and $G = YZ + Y + 2Z$, and the point $P$ is the origin. Now everything is just like chapter 3, only with $Z$ instead of $X$. We see that $G$ is smooth at $P$ (since it has multiplicity 1), so:

$$\begin{aligned} I_P(F,G) &= \mathrm{ord}_P^G(Y^2 - Z^3) \\ &= \mathrm{ord}_P^G(Y^2(1 + (Z/Y)^2 Z)) \\ &= \mathrm{ord}_P^G\left(Y^2\left(1 + \left(\frac{1}{2Z+2}\right)^2 Z\right)\right) \\ &= 2 \end{aligned}$$

because $Y$ is a uniformizer for $G$ at $P$ and $1 + Z/(2Z+2)^2$ is a unit in the local ring of $G$ at $P$ (it has value 1 there).

5. Let $X$ be a smooth projective curve, and $f \in K(X)$ a rational function such that $\mathrm{div}\, f = P - Q$ for points $P$ and $Q$ in $X$. Prove that $X$ is isomorphic to $\mathbb{P}^1$.

*Solution:* Certainly $f$ defines a rational map $\phi(P) = [f(P) : 1]$ from $X$ to $\mathbb{P}^1$, which since $X$ is smooth must be a morphism. Consider the set $\phi^{-1}(0:1)$. It consists precisely of the zeroes of $f$, clearly, which means $\phi^{-1}(0:1) = \{P\}$. Thus, the ramification index of $\phi$ at $P$ is equal to the degree of $\phi$.

Consider the pullback morphism $\phi^*\colon \mathcal{O}_{[0:1]}(\mathbb{P}^1) \to \mathcal{O}_P(X)$. If we restrict to the affine piece $y = 1$, then we know that $x$ is a uniformizer at 0 (the affine reincarnation of $[0:1]$). But $\phi^* x = f$, and $\mathrm{ord}_P(f) = 1$, so $f$ is a uniformizer at $P$! Thus, the degree of $\phi$ is 1. This means that $\phi$ is actually a birational map of smooth curves and therefore an isomorphism, and so $X$ is isomorphic to $\mathbb{P}^1$, as desired. ♣

6. Let $C \subset \mathbb{A}^2$ be a smooth plane curve, $P = (0,0) \in C$ a point on $C$. Let $L$ be the tangent line to $C$ at $P$, and assume that $L \neq C$. Prove that $\mathrm{ord}_P(L) \geq 2$.

*Solution:* Say $C = V(F)$, where $F$ is an irreducible polynomial. Then we can write

$$F(X, Y) = L(X, Y) + Q(X, Y)$$

where $L$ is the linear homogeneous polynomial defining the tangent line $L$, and $Q$ is a polynomial with no terms of degree less than 2. (To see that you can do this, write $F(X, Y) = a_0 + a_X X + a_Y Y + Q(X, Y)$, and see what happens when you plug $X = Y = 0$ into $F$, $F_X$, and $F_Y$.)

From this, we get

$$L \equiv -Q \pmod{()F}$$

But $-Q$ has no terms of degree less than two, so $\text{ord}_P(T) \geq 2$ for each term $T$ of $Q$. (Note that $Q$ is nonzero because $C \neq L$.) And if you add a bunch of things together, the order of the sum cannot be less than the minimum order of a summand, meaning that $\text{ord}_P(-Q) \geq 2$, and hence also $\text{ord}_P(L) \geq 2$, as desired. ♣