Lecture notes for PM 464/764 – Week Eight

David McKinnon
Department of Pure Mathematics
University of Waterloo

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1 Projective morphisms

We've defined projective algebraic sets. Now we need to define projective morphisms.

You'd think it would be as easy as an $(n+1)$-tuple of homogeneous polynomials, but no. Problem is, $[0 : \ldots : 0]$ isn't a point in projective space. So the tuple of polynomials can't all vanish at the same time, or the image point won't be defined. We also need the polynomials all to have the same degree, or else rescaling the input variables won't leave the destination point unchanged.

There's another problem, too, which is less of a problem than an opportunity. Let $V = V(Y^2Z - X^3 - XZ^2)$ be a curve in $\mathbb{P}^2$. Define a morphism $\phi\colon V \to \mathbb{P}^1$ by

$$\phi(X : Y : Z) = [X : Y]$$

This is perfectly well defined everywhere except at $[0 : 0 : 1]$, which is a point of V. But, as points in $\mathbb{P}^2$, we have:

$$[X : Y] = [YZ : X^2 + Z^2]$$

for all points on V. (Check it out! It's a bit freaky. And should remind you of that example from before ...) But then we also have

$$\phi(X : Y : Z) = [YZ : X^2 + Z^2]$$

which has the perfectly respectable value $[0 : 1]$ at the point $[0 : 0 : 1]$.

So we don't want our projective morphisms to be mere tuples of polynomials.

Definition 1.1. *Let $V \subset \mathbb{P}^n$ and $W \subset \mathbb{P}^m$ be projective algebraic sets. A* **morphism** *from V to W is a function $f : V \to W$ such that for every point $P \in V$, there is an $(m+1)$-tuple $[f_0 : \ldots : f_m]$ of homogeneous polynomials of the same degree such that $f_i(P) \neq 0$ for some i, and*

$$f(Q) = [f_0(Q) : \ldots : f_m(Q)]$$

for all $Q \in V$ with $f_i(Q) \neq 0$ for some i.

An **isomorphism** *is a morphism with an inverse morphism.*

The most important examples of projective isomorphisms are **projective changes of coordinates**.

That is, let $L_0, \ldots, L_n$ be a linearly independent set of linear homogeneous polynomials. Like $\{x_0 + x_1, x_1 + x_2 \ldots, x_{n-1} + x_n, x_n\}$, for example. Then we can define a morphism from $\mathbb{P}^n$ to $\mathbb{P}^n$ by

$$T(a) = [L_0(a) : \ldots : L_n(a)]$$

Since the set of homogeneous polynomials in $n+1$ variables is of dimension $n+1$, it follows that the L_i are a basis. In particular, the common zero locus of the L_i must be empty! (Remember that the zero vector is not a point in projective space!) So T is actually a morphism.

But better than that, T is an isomorphism! The matrix whose rows are the coefficients of the L_i must be invertible, because the L_i are a basis of the space of homogeneous linear polynomials. So there is an inverse matrix, whose entries we can use to get another set $L'_0, \ldots, L'_n$ of linear homogeneous polynomials that will define an inverse morphism to T.

We can also do the rational map thing.

Definition 1.2. *Let $V \subset \mathbb{P}^n$ and $W \subset \mathbb{P}^m$ be projective algebraic sets. A* **rational map** *from V to W is a function $f : U \to W$ for some nonempty open subset $U \subset V$ such that for every point $P \in U$, there is an $(m+1)$-tuple $[f_0 : \ldots : f_m]$ of homogeneous polynomials of the same degree such that $f_i(P) \neq 0$ for some i, and*

$$f(Q) = [f_0(Q) : \ldots : f_m(Q)]$$

for all $Q \in V$ with $f_i(Q) \neq 0$ for some i.

This is the same thing as a morphism, except that it doesn't have to be defined everywhere.

Definition 1.3. *Let $V \subset \mathbb{P}^n$ be a projective variety. The* **function field** *$K(V)$ of V is the field $K(U)$, where U is any affine piece of V.*

That is, U is the affine variety obtained by intersecting V with any standard affine piece of $\mathbb{P}^n$.

Moreover, if $P \in V$ is any point, then the **local ring** *of V at P is the local ring $\mathcal{O}_P(U)$, where U is any affine piece of V.*

I'll point out that it doesn't matter which affine piece you pick, because all the affine pieces of V are birational to each other (using the inclusion map, where defined!), so their function fields are isomorphic. And if that birational map makes P correspond with Q, then the pullback induces an isomorphism of local rings too. And these facts make the following definition sensible too, because singularity depends only on the local ring:

Definition 1.4. *Let V be a projective variety, $P \in V$ any point. Then P is a* **singular point** *of V if and only if P is a singular point of U, where U is any affine piece of V. We say that P is a* **smooth point** *of V otherwise. We say that V is* **smooth** *if and only if every point of V is smooth.*

This leads us to a definition of tangent space to a projective variety too.

Definition 1.5. *Let P be a point on a projective variety V. The* **Zariski tangent space** *to V at P is the Zariski tangent space to any affine piece of V that contains P, namely the dual of $\mathcal{M}_P(V)/\mathcal{M})P(V)^2$.*

The **tangent space** *to V at P is the projective closure of the tangent space to V at P in any affine piece of V containing P.*

And we can define dimension for projective varieties. It never ends!

Definition 1.6. *Let V be a projective variety. The* **dimension** *of V is the dimension of any nonempty affine piece of V.*

It's not too hard to see that this is the same as the length of the longest chain of nonempty projective subvarieties, just like in the affine case:

$$V_0 \subsetneq \ldots \subsetneq V_d = V$$

Back in Affineland, we had a beautiful correspondence between morphisms and homomorphisms of the coordinate rings of the two varieties. This reminds us that we haven't talked about the algebra side of projective geometry very much yet! We glossed over the ideal stuff – because it's all basically the same as in the affine case – and we haven't talked about the coordinate ring stuff.

That's because the homogeneous coordinate ring is terrible.

Definition 1.7. *Let $V \subset \mathbb{P}^n$ be a projective algebraic set. The* **homogeneous coordinate ring** *of V is the ring $\mathbb{C}[X_0, \ldots, X_n]/I(V)$.*

So far, so good. Here's the problem.

Let V be the line $S = 0$ in $\mathbb{P}^2(S : T : U)$, and let W be the curve $V(XY - Z^2)$, in $\mathbb{P}^2(X : Y : Z)$.

First, notice that $V \cong W$, because of the morphism

$$\phi(0 : T : U) = [T^2 : U^2 : TU]$$

whose inverse is

$$\phi^{-1}(X : Y : Z) = [0 : X : Z] = [0 : Z : Y]$$

Notice that we need two different representations of the inverse to cover all the points of W.

And if you doubt my word that these are inverses:

$$\phi(\phi^{-1}(X : Y : Z)) = \phi(0 : X : Z) = [X^2 : Z^2 : XZ]$$

$$= [X^2 : YZ : XZ] = [X : Y : Z]$$

and similarly for the other representation of ϕ^{-1}. The other way around:

$$\phi^{-1}(\phi(0 : T : U)) = \phi^{-1}(T^2 : U^2 : TU) = [0 : T^2 : TU] = [0 : T : U]$$

and similarly for the other representation of ϕ^{-1}.

Anyway, so V is isomorphic to W. But the homogeneous coordinate ring of V is

$$\mathbb{C}[S, T, U]/(S) \cong \mathbb{C}[T, U]$$

but the homogeneous coordinate ring of W is

$$\mathbb{C}[X, Y, Z]/(XY - Z^2)$$

which isn't a UFD. (I mean, $XY = Z^2$ is two different factorizations into irreducibles. "How do you know that X, Y, and Z are irreducible and non-associate?" you ask. "Shut up," I reply, helpfully.) So it's a pretty frickin' different ring from $\mathbb{C}[T, U]$, and certainly not isomorphic to it.

So we won't be talking about homogeneous coordinate rings any more. They're not useless – far from it! – but they're not the tool we need right now.

It will actually take some time to build the tool we need to keep track of morphisms. Worse yet, we won't build it for varieties of dimension bigger than one. So, for the remainder of the course, we're going to specialize to the case of curves – algebraic varieties of dimension one.

2 Curves

Definition 2.1. *A* **curve** *is an algebraic variety of dimension one. A projective curve is a projective algebraic variety of dimension one, and an affine curve is an affine algebraic variety of dimension one.*

In particular, a curve is irreducible. I say this because not everyone holds this particular philosophical belief, and I believe it's best to prepare you for the harsh, cold world outside this course.

The aspect of curves I particularly want to deal with first is its local rings. A smooth point on a curve has a very special kind of local ring.

Definition 2.2. *A* **discrete valuation ring** *(or* **DVR***, for short) is a Noetherian local domain whose maximal ideal is principal and nonzero. A generator for the maximal ideal is called a* **uniformizing parameter***, or* **uniformizer***, for short.*

The prime geometric example of this is the ring

$$\mathbb{C}[t]_{(t)} = \left\{ \frac{f(t)}{g(t)} \mid g(0) \neq 0 \right\}$$

which is the local ring of $\mathbb{A}^1$ at the point 0. The maximal ideal is all the non-units, which are exactly the fractions f/g where $f(0) = 0$. (These are the ones whose reciprocals don't lie in the ring.)

But if $f(0) = 0$, then $f(t) = tq(t)$ for some polynomial q, so we get

$$\frac{f(t)}{g(t)} = t\frac{q(t)}{g(t)}$$

and so f/g is in the principal ideal (t). Conversely, anything in (t) is a non-unit, so the maximal ideal of $\mathbb{C}[t]_{(t)}$ is just (t). Which is, as you can see, principal. So t is a uniformizer for $\mathbb{C}[t]_{(t)}$. (As is $25t$ or πt, for example, or even $t^3 + 2t$. Because $t^3 + 2t = t(t^2 + 2)$, and $t^2 + 2$ is a unit.)

In fact, there's more structure there. For any rational function f/g, you can separate all the factors of t in top and bottom (well, ok, presumably not both), and write:

$$\frac{f(t)}{g(t)} = t^a \frac{p(t)}{q(t)}$$

where a is an integer (possibly negative, if f/g isn't in the local ring!) and p/q is a unit of the local ring $\mathbb{C}[t]_{(t)}$.

Which means that every ideal of $\mathbb{C}[t]_{(t)}$ is the ideal (t^a) for some (non-negative) integer a. Neat!

Turns out this pattern isn't special to this ring.

Theorem 2.3. *Let $C \subset \mathbb{A}^n$ be a curve, and $P \in C$ a smooth point. Then the local ring $\mathcal{O}_P(C)$ is a DVR, and any linear function f whose zero set is not tangent to C at P is a uniformizer for $\mathcal{O}_P(C)$.*

By "tangent to C at P", I mean "containing the tangent space $T_P(C)$".

Proof: We'll only prove this for plane curves (that is, curves in $\mathbb{A}^2$). Sadly, a proof for a general smooth curve is beyond the scope of these notes, but you can find one pretty easily online.

So let P be a smooth point on a curve $C \subset \mathbb{A}^2$. Since the local ring at P is invariant under isomorphisms, we can change coordinates in $\mathbb{A}^2$ so that $P = (0, 0)$.

We need to use the fact that C is smooth at P. Well, the curve C is defined by a single polynomial equation $f(x, y) = 0$. Since $P = (0, 0)$ is on C, we also know that $f(0, 0) = 0$, so that

$$f(x, y) = ax + by + q(x, y)$$

where a and b are constants, and q is a polynomial whose terms all have degree 2 or higher.

Since C is smooth at P, we know that $J_P(f) = (f_x(P), f_y(P))$ is not the zero matrix. In other words, either $f_x(0, 0) = a$ or $f_y(0, 0) = b$ is nonzero. By another judiciously chosen change of coordinates, we can keep the point P right where it is, but change the nonzero $ax + by$ into the much neater y.

So now our equation looks like

$$f(x, y) = y + r(x, y)$$

where $r(x, y)$ has only terms of degree 2 or higher. And $y = 0$ is the tangent line to C at P (go back and check the definition if this isn't clear.)

To prove that $\mathcal{O}_P(C)$ is a DVR, we need to show that it's Noetherian, local, a domain, and that the maximal ideal is principal and nonzero. The first three of those we already know, so we just need to check that the maximal ideal is principal. (It's pretty obviously nonzero.)

Thus, let $cx + dy$ be any linear form in x and y with $c \neq 0$. (This is the same as picking a line through P that does not contain the tangent line $y = 0$.) After the linear transformation

$$\begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix}$$

our linear form $cx + dy$ becomes x, but P and the line $y = 0$ are unchanged. (The restriction $c \neq 0$ ensures that the matrix is invertible.)

We will show that x generates the maximal ideal of $\mathcal{O}_P(C)$.

Well, we already know that the maximal ideal of $\mathcal{O}_P(C)$ is generated by x and y. And we have the equation

$$f(x, y) = y + r(x, y)$$

and both sides equal zero in the ring $\mathcal{O}_P(C)$. If we regroup the terms of $f(x, y)$, we can get a new equation

$$f(x, y) = yg(x, y) + xp(x)$$

where $g(x, y)$ is a polynomial with $g(0, 0) \neq 0$ (because of that linear y term), and $p(x)$ is a polynomial in x. (There's no constant term because $f(0, 0) = 0$.)

But this enables the following equation in $\mathcal{O}_P(C)$, because $f(x, y) = 0$ there:

$$y = -x\frac{p(x)}{g(x, y)} \in x\mathcal{O}_P(C)$$

Note that this is indeed an equality in $\mathcal{O}_P(C)$ because $g(P) = g(0, 0)$ is nonzero.

So $y \in (x)$, and so $(x, y) = (x)$, just like we wanted! Meaning that the maximal ideal $(x, y) = (x)$ of $\mathcal{O}_P(C)$ is principal, and so $\mathcal{O}_P(C)$ is a DVR with x as its generator, as advertised. ♣