Lecture notes for PM 464/764 – Week Seven

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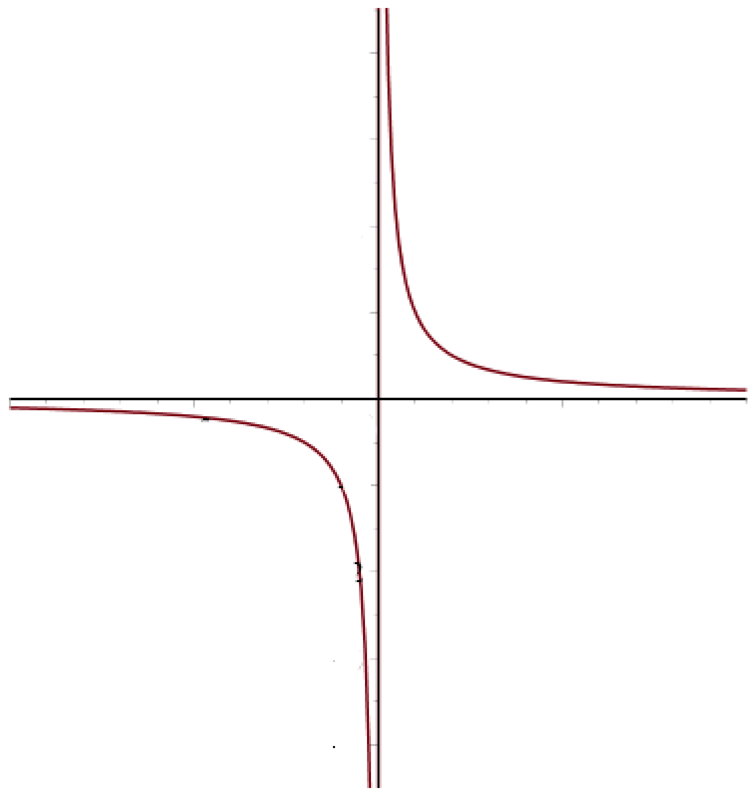
1 Projective space

The cast of characters now appears to be set. We know the objects we're dealing with (algebraic varieties), and the morphisms. There is even a supporting cast, including local rings, rational functions, dimension, and a few others. But ... there's something missing.

It may not feel that way to you, of course. Let me show you what I mean.

Let V be the hyperbola $xy = 1$. Beautiful variety. Let W be the x-axis. Another beautiful variety.



Define $f\colon V \to W$ by $f(x,y) = x$. It's an isomorphism ... almost. It's got an almost-inverse, defined by $g(x,0) = (x, 1/x)$. The "almost" part is, of course, that g is undefined at $x = 0$.

So there's a one-to-one correspondence between every point of W and every point of V *except one*. That doesn't seem fair. In particular, it suggests that there might be something extra that we're not seeing.

Let's think about what that might be. If you look at the picture, you can see that the curve V goes off to infinity as $x \to 0$. So the missing point is ... infinity?

Actually, yes.

Let's write $x = X/Z$ and $y = Y/Z$, so that the denominator (you know, the one that's going to zero to take y off to infinity) is easier to see. Then the equation for V becomes

$$XY = Z^2$$

after we clear denominators. Now, $x = 0$ is the same as $X = 0$, which doesn't present a problem – we just take $Z = 0$ too. The variable Y can be anything you like.

Well ... except that by writing $x = X/Z$, we're implicitly agreeing that the actual *values* of the new variables X and Z don't matter – it's just their *ratio* that matters. After all, $X = Z = 4$ and $X = Z = 2$ both correspond to the value $x = 1$ for our original variable.

This is where projective space comes from.

Definition 1.1. *Let n be a positive integer.* **Complex projective space** $\mathbb{P}^n$ *is the set of nonzero $(n+1)$-tuples of complex numbers, modulo the equivalence that $\mathbf{v} \sim \mathbf{w}$ if and only if $\mathbf{v} = \lambda \mathbf{w}$ for some $\lambda \in \mathbb{C}$.*

Let's unpack this. Let's start with $\mathbb{P}^1$, the projective line.

Points in $\mathbb{P}^1$ are 2-tuples (better known as "pairs"):

$$[x : y]$$

But the only thing that matters here is the ratio of x and y, not their individual values. So $[1 : 1] = [2 : 2]$, for example, and $[3 : 0] = [155 : 0]$. And $[0 : 0]$ is not allowed, because the tuples have to be nonzero.

In fact, the point $[x : y]$ really just represents the fraction x/y. And if $y = 0$, then it represents ... infinity!

Let's explore that a little more. If x is a complex number, then it represents a point in projective space by $[x : 1]$. And the point $[x : y] \in \mathbb{P}^1$ represents the complex number x/y ... unless $y = 0$.

But there's only one point in $\mathbb{P}^1$ with $y = 0$, because $[x : 0] = [x' : 0]$ for all x and x'. (Except zero, of course, which isn't allowed.) So really, in a moral sense at least, we have

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$$

Hang on, though – how did we know that y was supposed to be the denominator? Couldn't we just as easily have turned $[x : y]$ into y/x?

Why, yes we could! And if we did that, then $[1 : 0]$ would turn into 0 instead of ∞, and $[0 : 1]$ would be the odd point out.

So really, $\mathbb{P}^1$ is two copies of $\mathbb{C}$ glued together, where a nonzero complex number z is glued to $1/z$.

In fact, there's even more to it than that ... but we'll leave it there for now. Let's go on to $\mathbb{P}^2$.

In the projective plane $\mathbb{P}^2$, we've now got three coordinates, so that a typical point looks like $[x : y : z]$. In this case, it's only the ratios that matter, but they *all* matter. So $[2 : 4 : 1] = [4 : 8 : 2]$, but $[2 : 4 : 1] \neq [4 : 8 : 1]$.

Similarly to $\mathbb{P}^1$, there's a copy of $\mathbb{A}^2$ sitting inside $\mathbb{P}^2$, via

$$(x, y) \mapsto [x : y : 1]$$

and any $[x : y : z]$ with $z \neq 0$ corresponds to the point $(x/z, y/z)$ in $\mathbb{A}^2$. So the new points are the ones with $z = 0$.

Unlike in $\mathbb{P}^1$, though, there are lots of points with $z = 0$. In fact, $[x : y : 0] = [x' : y' : 0]$ if and only if there is some $\lambda \in \mathbb{C}$ with $\lambda(x, y) = (x', y')$. (Remember that $[0 : 0 : 0]$ is not allowed!) But that's *exactly* what we need for $\mathbb{P}^1$! So, in fact, there's a little copy of $\mathbb{P}^1$ sitting inside $\mathbb{P}^2$. We call that copy the "line at infinity". Thus, morally speaking:

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$$

But, just like in $\mathbb{P}^1$, there's no reason to single out z as the special "denominator coordinate". We could equally pick x or y to be the sacrificial coordinate, and embed $\mathbb{A}^2$ in $\mathbb{P}^2$ as $[x : 1 : y]$ or $[1 : x : y]$. So $\mathbb{P}^2$ is actually three copies of $\mathbb{A}^2$ glued together, with (x, y) glued to $(x/y, 1/y)$ and $(1/x, y/x)$.

In general, that's how things work. Projective n-space $\mathbb{P}^n$ is just $n+1$ copies of $\mathbb{A}^n$ glued together, and in each case, $\mathbb{P}^n - \mathbb{A}^n$ is just a copy of $\mathbb{P}^{n-1}$.

The subset $U_i = \{[x_0 : \ldots : x_n] \mid x_i \neq 0\}$ is the image of $\mathbb{A}^n$ via

$$(x_0, \ldots, \widehat{x_i}, \ldots, x_n) \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]$$

where the hat over x_i means that you leave it out. This subset is called the ith standard open affine subset of $\mathbb{P}^n$.

(Remember that if $x_i \neq 0$, then you can divide all the coordinates by x_i – which doesn't change the point in projective space! – and you end up with an ith coordinate of one.)

Great. But we're doing algebraic geometry, so we want algebraic sets, varieties, morphisms, rational functions, all that jazz. So ... let's have it.

To define algebraic subsets of $\mathbb{P}^n$, we will take advantage of the definition of algebraic subset in $\mathbb{A}^n$ that we already have.

Definition 1.2. *An algebraic subset of $\mathbb{P}^n$ is a subset $X \subset \mathbb{P}^n$ such that for all i, $X \cap U_i$ is an algebraic subset of $U_i \cong \mathbb{A}^n$.*

Before we do any examples, let's think a little about what this definition amounts to.

Say $V \subset \mathbb{P}^n$ is a projective algebraic set. Let $X_0, \ldots, X_n$ be the coordinates in projective space. Then $V \cap U_0$ is an algebraic subset of $\mathbb{A}^n$. By the Nullstellensatz, this means that V is the zero set of a finite set of polynomials $f_1, \ldots, f_r$ in the variables $x_1, \ldots, x_n$ where $x_i = X_i/X_0$. So in projective language, $V \cap U_0$ is given by:

$$f_1(X_1/X_0, \ldots, X_n/X_0) = \ldots = f_r(X_1/X_0, \ldots, X_n/X_0) = 0$$

except maybe for points where $X_0 = 0$, where this doesn't really make sense. But that's ok, because those points don't show up in U_0 anyway.

If we take a common denominator in $f_i(X_1/X_0, \ldots, X_n/X_0) = 0$, we'll get some power of X_0. So we can write that equation as

$$\frac{F_i(X_0, \ldots, X_n)}{X_0^d} = 0$$

for some non-negative integer d. Multiplying through by X_0^d – which is harmless, because $X_0 \neq 0$ on U_0 – gives us a different representation of V:

$$F_1(X_0, \ldots, X_n) = \ldots = F_r(X_0, \ldots, X_n) = 0$$

Now, X_j/X_0 is an expression of total degree zero. (Degree one on the top, degree one on the bottom – they cancel out.) So every term in $f_i(X_1/X_0, \ldots, X_n/X_0) = 0$ has degree zero. That means that every term in $F_i(X_0, \ldots, X_n)$ has degree d.

Definition 1.3. *A polynomial $f(X_0, \ldots, X_n)$ is* **homogeneous** *if and only if every term of f has the same degree.*

So V is actually the zero set of a finite set of homogeneous polynomials! I mean, except possibly for some messing around with points satisfying $X_0 = 0$ ... but we can deal with that by applying the same logic to U_i for every i. (Every point in projective space lies in at least one U_i, because every point in projective space has at least one nonzero coordinate.)

Thus, we've deduced the following theorem:

Theorem 1.4. *A subset $V \subset \mathbb{P}^n$ is algebraic if and only if it is the zero set $V(F_1, \ldots, F_r)$ of a finite set of homogeneous polynomials F_i.*

It's worth a moment to point out here that polynomials on projective space *are not functions*.

I mean, what's the value of X^2 on the point $[1 : 1]$? Looks like "1", right? But $[1 : 1] = [3 : 3]$, so it's also "9". Uh oh.

But to define algebraic sets, we only care whether or not a polynomial is zero. And *that* makes sense, even for points in projective space. Let say $F(X_0, \ldots, X_n) = 0$, for some representation of the point $[X_0 : \ldots : X_n]$ in $\mathbb{P}^n$. And now let's say some jerk comes along and multiplies all our coordinates by λ.

Well, we'll just compute

$$F(\lambda X_0, \ldots, \lambda X_n) = \lambda^d F(X_0, \ldots, X_n) = 0$$

where d is the degree of F. So F is still zero. (If you're wondering about that middle step there, remember that each term of F has degree d, so you can factor exactly d copies of λ out of it.)

There's a projective Nullstellensatz, too. Like the affine one, we won't be proving it here. Unlike the affine one, there's a slight extra wrinkle in it.

Definition 1.5. *The* **irrelevant ideal** *of* $\mathbb{C}[X_0, \ldots, X_n]$ *is the ideal* $(X_0, \ldots, X_n)$*.*

Theorem 1.6. *Let n be a positive integer. There is a bijection*

$$\{\textit{algebraic subsets of } \mathbb{P}^n\} \longleftrightarrow \{\textit{RRH ideals of } \mathbb{C}[x_0, \ldots, x_n]\}$$

where "RRH" stands for "relevant radical homogeneous". The bijection is given by $X \mapsto I(X)$ and $I \mapsto V(I)$.

Notice that the irrelevant ideal is the only radical ideal left out of this correspondence. This is because its zero set is empty, despite not being the unit ideal: there are no points of $\mathbb{P}^n$ with all coordinates zero.

Let's do some examples of projective algebraic sets now.

The set $X + Y = 0$ in $\mathbb{P}^1$ is the single point $[1 : -1]$. Yes, it's also the single point $[2 : -2]$ – they're the same point. It's also the single point $[-1 : 1]$.

We've already seen the projective algebraic set V, given by $XY = Z^2$. It's the hyperbola $xy = 1$ when you restrict it to the standard affine $Z \neq 0$. But if you restrict it to the standard affine $X \neq 0$, you divide both side by X^2 to get $(Y/X) = (Z/X)^2$, and thus $y = z^2$. Which is a parabola!

So which is it? A hyperbola, or a parabola?

Well, it turns out a hyperbola and a parabola are very similar curves.

The set $V(XY - Z^2)$ is a projective curve. Its intersection with $Z = 0$ is two points: $[1 : 0 : 0]$ and $[0 : 1 : 0]$. And its intersection with $Z \neq 0$ is a hyperbola.

The intersection of V with $X = 0$ is just one point: $[0 : 1 : 0]$. And its intersection with $X \neq 0$ is a parabola.

Turns out this is the real difference between a hyperbola and a parabola: one of them is V with two points missing, and the other is V with only one point missing.

So what's an ellipse, then? Is it V with no points missing?

Alas, no. It's V with two points missing. But the two points aren't defined over the real numbers. (Check it out with $X^2 + Y^2 = Z^2$: when $Z = 0$ you get $X^2+Y^2 = 0$, or $X = \pm iY$, giving you the two points $[1 : i : 0]$ and $1 : -1 : 0]$.)

One last example, in $\mathbb{P}^3$. Let $V = V(X^3 - Y^3, WX - YZ)$. Notice that the two homogeneous polynomials don't have the same degree – that's ok. They just have to be individually homogeneous.

Now, $X^3 - Y^3 = (X - Y)(X - \gamma Y)(X - \gamma^2 Y)$, where γ is a primitive cube root of unity. So we get

$$V = V(X-Y, WX-YZ) \cup V(X-\gamma Y, WX-YZ) \cup V(X-\gamma^2 Y, WX-YZ)$$

so V is the union of three smaller algebraic sets. This reminds me of a definition we should make:

Definition 1.7. *A nonempty projective algebraic set is* **reducible** *if and only if it is the union of two proper projective algebraic* *subsets. It is* **irreducible** *if and only if it is not reducible. The empty set is neither irreducible nor reducible.*

Theorem 1.8. *A projective algebraic set V is irreducible if and only if the ideal $I(V)$ is prime.*

Proof: Say $V = W \cup W'$ for proper algebraic subsets W and W'. Then by the projective Nullstellensatz, there are homogeneous polynomials f and f' that are identically zero on W and W', respectively, but not identically zero on all of V. But since $W \cup W' = V$, it follows that ff' is identically zero on all of V. Thus, $I(V)$ is not prime.

Conversely, if $I(V)$ is not prime, then there are homogeneous polynomials such that f and f' do not vanish identically on all of V, but their product ff' does. Then $V = (V \cap V(f)) \cup (V \cap V(f'))$ is reducible. ♣

Let's look at the first of the three smaller subsets, namely $V(X-Y, WX-YZ)$. If you have a point $[W : X : Y : Z]$ with $X = Y$ and $WX = YZ$, then certainly $WX = XZ$. But this means that either $X = 0$ or $W = Z$, so we have a further decomposition

$$V(X-Y, WX-YZ) = V(X-Y, W-Z) \cup V(X,Y)$$

Similarly, we have

$$V(X-\gamma Y, WX-YZ) = V(X-\gamma Y, \gamma W-Z) \cup V(X,Y)$$

and

$$V(X-\gamma^2 Y, WX-YZ) = V(X-\gamma^2 Y, \gamma^2 W-Z) \cup V(X,Y)$$

So in fact, $V(X^3-Y^3, WX-YZ)$ is the union of four subsets:

$$V(X,Y) \cup V(X-Y, W-Z) \cup V(X-\gamma Y, \gamma W-Z) \cup V(X-\gamma^2 Y, \gamma^2 W-Z)$$

And each of these four sets is irreducible, because the various ideals are all prime. The picture here is that $V(X^3 - Y^3)$ is the union of three planes that intersect along the line $X = Y = 0$, which is contained in $V(WX-YZ)$. So each of the three planes intersects the surface $V(WX-YZ)$ in a pair of lines, but in each case $X = Y = 0$ is one of the pair of lines.

It's worth taking a bit of time to describe the relationship between a projective algebraic set and its affine pieces.

Definition 1.9. *Let $V \subset \mathbb{A}^n$ be an affine algebraic set, and consider $\mathbb{A}^n$ as the subset $x_0 \neq 0$ in $\mathbb{P}^n$. The* **projective closure** *of V in $\mathbb{P}^n$ is defined to be the intersection W of all projective algebraic sets containing V.*

Since the intersection of algebraic sets is again an algebraic set, it follows that the projective closure is an algebraic set. How do we figure out what it is?

Theorem 1.10. *Let $V \subset \mathbb{A}^n$ be an affine algebraic set, $W \subset \mathbb{P}^n$ its projective closure. If $V = V(F)$ for some polynomial F of degree d in $\mathbb{C}[X_1, \ldots, X_n]$, then $W = V(f)$, where $f = x_0^d F(x_1/x_0, \ldots, x_n/x_0)$. (This f is called the* **homogenization** *of F, and F is called the* **dehomogenization** *of f.)*

Proof: Notice that f is actually a polynomial, because the x_0^d exactly cancels out the denominators. In particular, it's homogeneous, because each variable x_i/x_0 has degree zero.

First, notice that if $P = (a_1, \ldots, a_n) \in V$, then we embed P in $\mathbb{P}^n$ as $[1 : a_1 : \ldots : a_n]$, and compute

$$f(P) = 1^d F(a_1/1, \ldots, a_n/1) = F(a_1, \ldots, a_n) = 0$$

so V is indeed contained in $V(f)$. By definition of the projective closure, this means that $W \subset V(f)$.

Now we need to show that $V(f) \subset W$. To do this, we must show that $V(f)$ is contained in every projective algebraic set that contains V. To do *that*, it's enough to show that every homogeneous polynomial g that vanishes on V must also vanish on $V(f)$.

Thus, let g be a homogeneous polynomial that vanishes on V. Then the dehomogenized polynomial

$$G(X_1, \ldots, X_n) = g(1, X_1, \ldots, X_n)$$

must vanish on V, and therefore must have a factor of F:

$$G = QF$$

But if d' is the degree of g, we have:

$$g = x_0^{d'} G(x_1/x_0, \ldots, x_n/x_0) = x_0^{d'} Q(x_1/x_0, \ldots, x_n/x_0) F(x_1/x_0, \ldots, x_n/x_0)$$

which since $d' \geq d$, is in the homogeneous ideal (f). So g vanishes on $V(f)$, and we're done. ♣

In general, we have the following, less useful theorem:

Theorem 1.11. *Let V be an affine algebraic set, W its projective closure. Then $I(W)$ is the ideal generated by the homogenizations of elements of $I(V)$.*

Proof: We have $W = \bigcap_f V(f)$, where the intersection ranges over all homogeneous polynomials $f \in I(W)$. But $f \in I(W)$ if and only if its dehomogenization F lies in $I(V)$. Since two homogeneous polynomials have the same dehomogenization if and only if their ratio is a power of x_0, we see that $f \in I(W)$ if and only if $f = x_0^d g$ for some homogenization g of a polynomial in $I(V)$. And we're done. ♣

Sadly, it's not true that the projective closure of $V(F_1, \ldots, F_n)$ is the set $V(f_1, \ldots, f_n)$, where f_i is the homogenization of F_i. You have to include *all* the elements of $I(V)$, not just the generators.

Well. Unless you pick the right set of generators. But that's a topic for a different course.

We still need morphisms and stuff. Well, we gotta have something to do next week.