Lecture notes for PM 464/764 – Week Six

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1 The Zariski tangent space

The rowspace of the Jacobian matrix is the span of the gradient vectors $\nabla f_i(P)$. Which is perpendicular to the nullspace of the Jacobian matrix. Which we just showed was isomorphic to M/M^2.

So M/M^2 is isomorphic to the orthogonal complement of the span of all the normal vectors to V at P. Perpendicular to normal is tangent, so we have the following definition:

Definition 1.1. *Let $V \subset \mathbb{A}^n$ be a variety, $P \in V$ a point. Let $\mathcal{O}_P(V)$ be the local ring at P, and let $\mathcal{M} = \mathcal{M}_P(V)$ be the maximal ideal of $\mathcal{O}_P(V)$. The* **Zariski tangent space** *to V at P is*

$$T_P(V) = \left(\mathcal{M}/\mathcal{M}^2\right)^*$$

This is, $T_P(V)$ is the dual $\mathbb{C}$-vector space to $\mathcal{M}_P(V)/\mathcal{M}_P(V)^2$.

The **tangent space** *to V at P is the set*

$$T_P(V) = P + \ker J_P(V) = \{Q \in \mathbb{A}^n \mid Q - P \in \ker J_P(V)\}$$

where $J_P(V)$ is the Jacobian matrix of V at P.

In other words, the tangent space is the translate by P of all the vectors that are perpendicular to the normal vectors to V at P. Which makes sense.

Few things here. First, notice that the Zariski tangent space and the tangent space (vanilla flavour) correspond if you translate the tangent space by $-P$, and then turn a vector $Q = (a_1, \ldots, a_n)$ into the dual to the class $a_1x_1 + \ldots a_nx_n \mod \mathcal{M}^2$. (Note that the dual of an element of $\mathcal{M}/\mathcal{M}^2$ is well defined because we chose a basis $\{x_1, \ldots, x_n\}$ for $\mathcal{M}/\mathcal{M}^2$.)

More puzzlingly, we used $\mathcal{M}$ instead of M. And then we threw in a gratuitous-looking dual. How come?

Well, the dual is there because M/M^2 is a collection of functions. The tangent space is a collection of vectors. The two are dual to one another, so we take a dual.

The answer to the other question is "peer pressure":

Theorem 1.2. *Let D be a domain, D_M the localization of D at a maximal ideal M. For any positive integer n, the natural inclusion $M \hookrightarrow \mathcal{M}$ induces an isomorphism*

$$M^{n-1}/M^n \cong \mathcal{M}^{n-1}/\mathcal{M}^n$$

where $\mathcal{M}$ denotes the ideal of D_M generated by M. In particular, for $n = 1$ and $n = 2$, we have

$$D/M \cong D_M/\mathcal{M}$$

and

$$M/M^2 \cong \mathcal{M}/\mathcal{M}^2$$

Proof: As required in the statement of the theorem, define $f\colon M^{n-1}/M^n \to \mathcal{M}^{n-1}/\mathcal{M}^n$ by $f(a+M^n) = a+\mathcal{M}^n$. It's pretty easy to see that f is a well defined homomorphism (since $M \subset \mathcal{M}$), so we just need to show that it's one-to-one and onto.

Say $f(a+M) = 0+\mathcal{M}$. We want to show that $a \in M^n$.

Well, $a+\mathcal{M}^n = \mathcal{M}^n$, so $a \in \mathcal{M}^n$, giving $a = m/b$, where $m \in M^n$ and $b \notin M$.

Since b is nonzero mod M, it's got an inverse c mod M, so you figure m/b should be the same thing as mc, at least mod M. Let's verify that.

Since $D/M \cong \mathbb{C}$ is a field, there is a $c \in D$ satisfying $bc \equiv 1 \pmod{M}$, or $bc = 1 + m'$ for $m' \in M$. We have

$$ab = m \in M^n$$

and so

$$abc = mc \in M^n$$

giving

$$a = mc - m'a \in M^n$$

because $m' \in M$ and $a \in M^{n-1}$. Thus, f is injective. (Everything in the kernel of f is congruent to zero mod M^n.)

Now for surjectivity. Let $a/b \in \mathcal{M}^{n-1}$ be any element. We want to find some $x \in M^{n-1}$ such that $x + \mathcal{M}^n = a/b + \mathcal{M}^n$. Again, since b is nonzero mod M, it's got an inverse c mod M, and you'd expect that a/b would be the same – mod M – as ac. Let's verify that.

Since $b \notin M$, then as before, there is some $c \in D$ with $bc =$

$1 + m'$ for some $m' \in M$. Then

$$ac - a/b = \frac{abc - a}{b} = \frac{a + m'a - a}{b} = m'\frac{a}{b} \in \mathcal{M}^n$$

and so $f(ac + M) = a/b + \mathcal{M}^n$, as desired. ♣