Lecture notes for PM 464/764 – Week Five

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1 Dimension

It would be nice to define the dimension of an algebraic set. Turns out it's a giant pain in the neck. So of course we're going to do it.

First problem: Let V be the union of the y-axis and the point $(3,5)$ in the xy-plane. How should we define the dimension of V?

This is a whopping problem, actually, because it doesn't make sense to define the dimension of V. It has a zero-dimensional part, and a one-dimensional part, and that's as good as you can do, really.

Which means that I lied. We're only going to define dimension for irreducible things.

Before I give you the definition, I'm going to try to explain why the definition I haven't given yet is actually pretty good.

Here's the intuition. Let's say you have a variety V. Any subvariety of V (that is, an irreducible algebraic set that is contained in V) should, intuitively, be of a smaller dimension that V, because otherwise you could use it to break V into two algebraic pieces.

So if you want to know the dimension of V, create the longest chain you can $V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V$ of nonempty varieties V_i, and then let n be the dimension of V. That is, V_0 is zero-dimensional (a point), V_1 is one-dimensional, and so on up the chain.

Definition 1.1. *Let V be a variety. Let*

$$V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n = V$$

be a chain of maximal length such that each V_i is a variety. Then the **dimension** *of V is n. The empty set does not have a dimension.*

Second problem: This is an enormous pain in the neck. I mean, how do you know you've found a chain of maximal length? How do you know there *is* a chain of maximal length?

The next step is to translate all this into algebra. Which, in the interest of full disclosure, does not alleviate the neck pain.

Definition 1.2. *Let D be a domain. Let*

$$P_0 \supsetneq P_1 \supsetneq \ldots \supsetneq P_n = (0)$$

be a chain of maximal length such that each P_i is a prime ideal of D. Then the **Krull dimension** *of D is n. If no such maximal chain exists, then the Krull dimension of D is infinite.*

It's pretty easy to see from this (and the Nullstellensatz) that the dimension of V is the same as the Krull dimension of its coordinate ring $\Gamma(V)$.

(Incidentally, poor Krull often gets his name left out of this definition, so that people often just talk about the dimension of a ring.)

Neck still hurts. Here's the beginning of an analgesic:

Theorem 1.3. *The dimension of* $\mathbb{A}^n$ *is* n*.*

Ok, so it's not a big shock. But it turns out to be surprisingly annoying to prove, and we will skip the proof.

It does, however, give us a way to compute the dimension of an arbitrary variety. Here's an example.

Example: Let's compute the dimension of the twisted cubic C, defined as $C = V(y-x^2, z-x^3) \subset \mathbb{A}^3$. (We're going to assume that C is irreducible for the purposes of this calculation.

First off, C contains the point $(1,1,1)$, so it's not empty, which means it has a dimension.

Next, notice that C also contains the point $(2,4,8)$, so C isn't just one point. So if we set $V_0 = (1,1,1)$, we have $V_0 \subsetneq C$, and thus C has dimension at least one.

Next consider the set $V_2 = V(y - x^2)$. It contains C, and it's irreducible (because $\mathbb{C}[x,y,z]$ is a UFD and $y-x^2$ is irreducible). Better yet, $C \neq V_2$, because $(2,4,12398) \in V_2$, but $(2,4,12398 \notin C$. So we have $C \subsetneq V_2$.

And obviously $V_2 \subsetneq \mathbb{A}^3$. So we have the following chain of varieties:

$$V_0 \subsetneq C \subsetneq V_2 \subsetneq \mathbb{A}^3$$

which has to be maximal because $\mathbb{A}^3$ is three-dimensional! So C can't have dimension more than 1, because otherwise $\mathbb{A}^3$ would have dimension more than 3. And the very existence of the chain $V_0 \subsetneq C$ shows that the dimension of C is at least 1.

So we've shown that $\dim C = 1$. Woo hoo!

Here's the general strategy suggested by this example:

1. Check if $V \subset \mathbb{A}^n$ is empty. If it is, we're done.
2. Find a maximal chain of varieties $V_0 \subsetneq \ldots \subsetneq \mathbb{A}^n$ that has V in it.
3. This involves showing that each inclusion $V_i \subsetneq V_{i+1}$ is strict (find a point of V_{i+1} that isn't in V_i).
4. This also involves showing that each V_i is irreducible.
5. Once you've done all that, find the m for which $V = V_m$, and conclude that $\dim V = m$.

There are other ways to compute the dimension of a variety, but this will be our go-to for this term. And on a homework assignment, you don't need to reprove this strategy – you can take it for granted that it works.

2 Smoothness, singularity, and local coordinates

Now that we've got a notion of dimension under our belts, we can use this to sort varieties into the good ones and the bad ones, just like in the rest of real life.

Ideally, when you do geometry, you'd like to have a set of local coordinates. Like when you're geocaching, and you use longitude and latitude to determine where you are and where you're going.

This crucially depends on the *existence* of local coordinates. Because duh.

So when does a variety admit a set of local coordinates near a point P?

Our variety V is defined by polynomials $f_1, \ldots, f_r$. The gradient vectors of these polynomials at P:

$$\nabla f_1(P), \ldots, \nabla f_r(P)$$

are all normal to V at P, by which I mean that they are perpendicular to all the tangent vectors to V at P.

If there are good local coordinates near P, then the derivative vectors of those good local coordinates will be tangent to V. And there will be exactly $d = \dim V$ of them, and they will span the tangent space of V at P (whatever that turns out to be).

So if there are good local coordinates near P, then they will span a d-dimensional space, and the vectors $\nabla f_1(P), \ldots, \nabla f_r(P)$ will span an $(n-d)$-dimensional perpendicular space, to entirely fill up the n-dimensional ambient space in which V lives.

In other words, we expect that there will be good local coordinates near P if and only if the span of $\nabla f_1, \ldots, \nabla f_r$ has dimension $n-d$, where $V \subset \mathbb{A}^n$ has dimension d.

Definition 2.1. *Let P be a point on an algebraic set $V \subset \mathbb{A}^n$ with $I(V) = (f_1, \ldots, f_m)$. The* **Jacobian matrix** *of V at P with respect to $\{f_1, \ldots, f_m\}$ is the m by n matrix*

$$J_V(P) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(P) & \frac{\partial f_1}{\partial x_2}(P) & \ldots & \frac{\partial f_1}{\partial x_n}(P) \\ \frac{\partial f_2}{\partial x_1}(P) & \frac{\partial f_2}{\partial x_2}(P) & \ldots & \frac{\partial f_2}{\partial x_n}(P) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(P) & \frac{\partial f_m}{\partial x_2}(P) & \ldots & \frac{\partial f_m}{\partial x_n}(P) \end{pmatrix}$$

Notice that the Jacobian matrix of V at P is just the matrix whose rows are $\nabla f_1(P), \ldots, \nabla f_r(P)$.

Definition 2.2. *Let P be a point on a variety $V \subset \mathbb{A}^n$ with $I(V) = (f_1, \ldots, f_m)$. Then P is a* **smooth point** *of V if and only if the rank of $J_V(P)$ equals $n - \dim V$. If P is not a smooth point of V, then we say it is a* **singular point** *of V.*

Let's do some examples to illustrate this.

The point $(1,0)$ on the unit circle $x^2+y^2 = 1$ is smooth. The unit circle is defined by the single polynomial x^2+y^2-1, whose gradient vector is

$$\nabla(x^2 + y^2 - 1) = (2x, 2y)$$

At the point $(1,0)$, this is the vector $(2,2)$, which is also the Jacobian matrix. Our ambient space has dimension 2, our variety has dimension 1 (easy check), and so smoothness at $(1,0)$ is equivalent to the rank of the Jacobian being 1. Which it is.

In fact, for a point on a variety defined by a single equation, the definition of smoothness boils down to the following simple principle:

Theorem 2.3. *Let V be a subvariety of $\mathbb{A}^n$ with $I(V) = (f)$ for a nonzero polynomial f. Let $P \in V$ be a point. Then* $\dim V = n-1$*, and V is smooth at P if and only if $\nabla(f)(P) \neq \mathbf{0}$.*

Proof: First, the dimension of V is $n-1$. This is not at all obvious, but is a consequence of Krull's Hauptidealsatz. In practice, for any given example, it's easy to verify that the dimension of V is $n-1$, but in general it's a pain in the neck.

Once we know the dimension of V, the rest is easy. The Jacobian matrix has only one row, so its rank is either one or zero. Therefore, $\nabla(f)(P) = \mathbf{0}$ if and only if the rank of the Jacobian is zero, if and only if its sum with $\dim V$ is n, so V is not smooth at P. ♣

Let $P = (1,1,1)$ on the twisted cubic $C = V(y-x^2, z-x^3) \subset \mathbb{A}^3$. Then P is again smooth:

$$J_C = \begin{pmatrix} -2x & 1 & 0 \\ -3x^2 & 0 & 1 \end{pmatrix}$$

which, when evaluated at $x = y = z = 1$, gives the matrix:

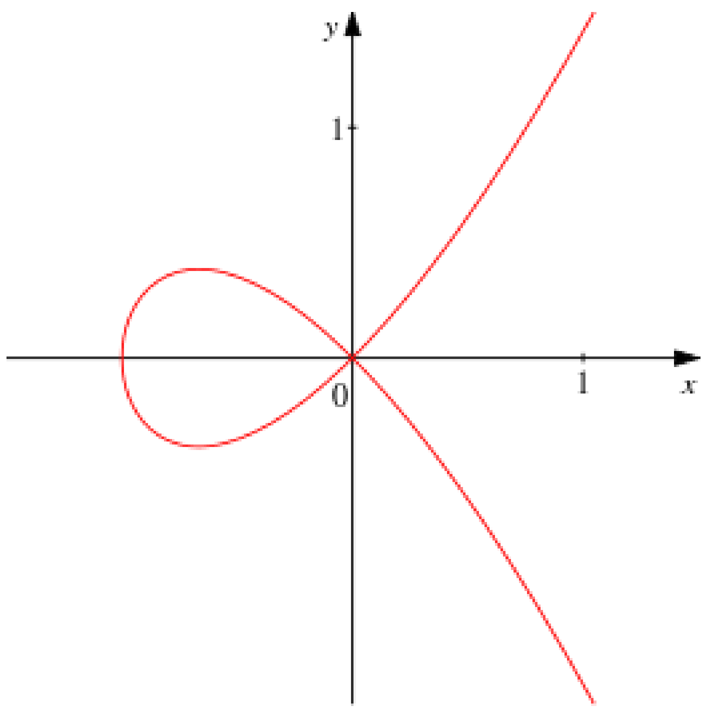
$$J_C(P) = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

which has rank two. Since we already know that the dimension of C is 1 (see the dimension section), we conclude that C is smooth at P.

A less illustrative but quite important example is that every point of $\mathbb{A}^n$ is smooth, because $\mathbb{A}^n$ is generated by no polynomials at all. This means that the rank of the Jacobian matrix is zero, which is equal to the ambient dimension (n) minus the dimension of our variety (also n).

Now let's do an example of a point that is *not* smooth.

Let $P = (0,0) \in V = V(y^2 - x^3 - x^2)$. We have $\nabla(y^2 - x^3 - x^2) = (-3x^2 - 2x, 2y)$, which equals $(0,0)$ when you plug in $x = y = 0$. So by our handy Theorem, we see that P is not a smooth point of V. You can see a picture of this curve – from the Wikipedia page for "Cubic Plane Curve" on May 6, 2021 – included below:



You can see in the picture that the point $(0,0)$ really doesn't have a good tangent line. In fact, it has *two* tangent lines ... but we won't talk about that right now. Suffice it to say that there's no good set of local coordinates near $(0,0)$ because there are two competing sets of local coordinates that get in each other's way.

The Jacobian matrix depends on the choice of generators $\{f_1, \ldots, f_m\}$ of the ideal of V. I mean, it really does – it's terrible. But for smoothness, we only deal with its *rank*, which doesn't change:

Theorem 2.4. *Let P be a point on a variety $V \subset \mathbb{A}^n$. Let $M = M(P) \subset \Gamma(V)$ be the maximal ideal corresponding to P. Then*

$$\dim M/M^2 + rank(J_V(P)) = n$$

where the dimension on the left hand side is the dimension as a vector space over $\mathbb{C}$.

In particular, the rank of the Jacobian matrix does not depend on the choice of generators for the ideal of V.

Proof: To be honest, the statement of the theorem reminds me of the Rank-Nullity Theorem. Maybe there's a way to turn M/M^2 into the kernel of a linear transformation.

Yeah. Like I would write that in if it didn't go anywhere.

The set M/M^2 is a vector space spanned by the linear polynomials $\{x_1 - a_1, \ldots, x_n - a_n\}$. (When you mod out by M^2, all the higher degree stuff goes away.) We want to work with M ... but it's kind of annoying, because $\Gamma(V)$ is a quotient ring, which means that it's terrible. So we'll try to shift this calculation to the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$.

Let N be the ideal $(x_1 - a_1, \ldots, x_n - a_n)$ of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Define a linear transformation $T\colon \mathbb{C}[x_1, \ldots, x_n] \to$

$\mathbb{C}^n$ by

$$T(f) = \left(\frac{\partial f}{\partial x - 1}(P), \ldots, \frac{\partial f}{\partial x_n}(P) \right)$$

Any $g \in N$ can be written in terms of the generators:

$$g = (x_1 - a_1)g_1 + \ldots + (x_n - a_n)g_n$$

for some polynomials $g_1, \ldots, g_n$. Therefore, the ith partial derivative of g equals

$$\frac{\partial g}{\partial x_i} = (x_1 - a_1)\frac{\partial g_1}{\partial x_i} + \ldots + (x_n - a_n)\frac{\partial g_n}{\partial x_i} + g_i$$

by the Chain Rule. In other words, if $T(g) = 0$, then we have $g_i(P) = 0$ for all i. Which means $g_i \in N$ for all i. Which means $g \in N^2$. Conversely, it's easy to check that if $g \in N^2$, then $T(g) = 0$.

Great! So T induces a well defined – and injective! – linear transformation of vector spaces from N/N^2 to $\mathbb{C}^n$. And this linear transformation is an isomorphism, because the polynomials $(x_i - a_i)$ map to the standard basis of $\mathbb{C}^n$.

Next, notice that $T(f_i)$ is just the ith column of $J_V(P)$. Therefore, the span U of $\{T(f_1), \ldots, T(f_n)\}$ is just the column space of $J_V(P)$. Its dimension is the rank of $J_V(P)$. So $\dim U = \text{rank} J_V(P)$.

But U is also just the image of $I = I(V)$ under T. Since T is an isomorphism when regarded as a linear transformation from N/N^2 to $\mathbb{C}^n$, this means that the rank of $J_V(P)$ is equal to the dimension of $I \subset N$ modulo N^2. In other words:

$$\text{rank} J_V(P) = \text{nullity}(q)$$

where $q\colon N/N^2 \to N/(N^2+I)$ is the quotient map.

But this q is surjective, because it's a quotient map. And we have

$$M/M^2 \cong (N+I)/(N^2+I) \cong N/(N^2+I)$$

and so by the Rank-Nullity Theorem, we get

$$\dim M/M^2 + \operatorname{rank}(J_V(P)) = n$$

as desired. ♣