Lecture notes for PM 464/764 – Week Four

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We can define a whole bunch of other stuff, too. Definitions ahoy!

Definition 0.1. *Let V be an algebraic variety. A* **Zariski closed** *subset of V is an algebraic subset of V. An subset $U \subset V$ is* **Zariski open** *if $V - U$ is Zariski closed.*

"Zariski" refers to Oscar Zariski, the great Russian-American mathematician of the twentieth century who was one of the founding minds behind algebraic geometry as a rigorous subject.

If you know something about general topology, then you'll recognize "closed" and "open", and that the Zariski open sets form a topology. BEWARE! It is a truly terrible topology, and doesn't work at all the way you think it should. It's very, very far from being Hausdorff, for example: if two Zariski open sets are disjoint, then at least one of them is empty!

If you don't know anything about general topology, then don't worry about that last paragraph at all.

There is a version of polynomial maps for rational functions, too.

Definition 0.2. *Let V and W be varieties. A* **rational map** *from V to W is a function $f\colon U \to W$ for some nonempty Zariski open subset $U \subset V$, such that for every point $P \in U$, there are rational functions $f_1, \ldots, f_r$ on V, all defined at P, such that*

$$f(Q) = (f_1(Q), \ldots, f_r(Q))$$

for all Q such that $f_1(Q), \ldots, f_r(Q)$ are all defined.

A rational map is said to be defined at $P \in V$ if there are rational functions $f_1, \ldots, f_r$ on V, all defined at P, such that

$$f(Q) = (f_1(Q), \ldots, f_r(Q))$$

for all Q such that $f(Q), f_1(Q), \ldots, f_r(Q)$ are all defined.

A rational map is a **morphism** *on a subset $V' \subset V$ if it is defined at every point of V'.*

Note that a rational map can be defined at a point outside U! The reason for this is a little silly – the definition of rational map lets you pick any old U, and you might pick one that's smaller than necessary – but the idea is that to build the rational map, you just need one nonempty Zariski open subset for a domain, but you're then later allowed to figure out cleverer representations that extend the definition of the map to more places.

But that morphism thing is gold. So much so that we will never use the phrase "polynomial map" again, using "morphism" instead, because every polynomial map is *a fortiori* also a morphism. On the other hand, there are lots of morphisms that aren't polynomial maps.

Rational functions let us define coordinate rings for arbitrary Zariski open sets.

Definition 0.3. *Let $U \subset V$ be a Zariski open subset. Then the* **ring of functions** *on U is the ring*

$$\Gamma(U) = \{f \in K(V) \mid f \text{ has no poles in } U\}$$

We have a corresponding algebra type theorem for these rings, although it's not quite as awesome as the one for affine varieties.

Theorem 0.4. *Let $\phi\colon V_1 \to V_2$ be a morphism of varieties, and let $U_i \subset V_i$ be Zariski open subsets such that $\phi(U_1) \subset U_2$. Then there is a $\mathbb{C}$-algebra homomorphism $\phi^*\colon \Gamma(U_2) \to \Gamma(U_1)$ defined by $\phi^*(f) = f \circ \phi$.*

Proof: All we have to do is check that ϕ^* is a $\mathbb{C}$-algebra homomorphism. Which is pretty easy. ♣

Let's do an example. If $U \subset \mathbb{A}^2$ is the set $\mathbb{A}^2 - \{x = 0\}$, then the corresponding ring of functions is

$$\Gamma(U) = \left\{ \frac{f}{x^n} \mid f \in \mathbb{C}[x,y],\, n \in \mathbb{Z} \right\} = \mathbb{C}[x,y,1/x]$$

because the only denominator that doesn't generate a pole with $x \neq 0$ is a power of x.

In fact, this example is representative of the general trend.

Theorem 0.5. *Let V be an affine variety, $U \subset V$ a nonempty Zariski open subset.*

If $U = V - V(f)$ for some $f \in \Gamma(V)$, then

$$\Gamma(U) = \Gamma(V)[1/f] = \{p/f^r \mid r \in \mathbb{Z},\, p \in \Gamma(V)\}$$

Proof: Say $V = V(f_1, \ldots, f_r) \subset \mathbb{A}^n$. Define another affine variety $W \subset \mathbb{A}^{n+1}$ by $W = v(f_1, \ldots, f_r, fx_{n+1} - 1)$, where we reinterpret the f_i as functions in $n+1$ variables.

Let ϕ: $W \to V$ be defined by $\phi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$. It's clear that the image of ϕ lies in V, and in fact it lies inside U as well, since if $f(x_1, \ldots, x_n) = 0$, then there is no value of x_{n+1} that satisfies the last defining equation for W, so ϕ maps no point of W to $V(f)$.

But ϕ admits an inverse morphism, at least on U:

$$\psi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 1/f(x_1, \ldots, x_n))$$

and so ϕ^* is an isomorphism between $\Gamma(W)$ and $\Gamma(U)$.

All we need to do now is show that $\Gamma(W) \cong \Gamma(V)[1/f]$. But this is easy:

$$\begin{aligned}\Gamma(W) &= \mathbb{C}[x_1, \ldots, x_{n+1}]/(f_1, \ldots, f_r, fx_{n+1} - 1) \\ &\cong \Gamma(V)[x_{n+1}]/(fx_{n+1} - 1) \\ &\cong \Gamma(V)[1/f]\end{aligned}$$

So we're done. ♣

Now let's say that $U = \mathbb{A}^2 - \{(0,0)\}$. What's $\Gamma(U)$ now?

Well, certainly $\Gamma(\mathbb{A}^2) \subset \Gamma(U)$. What else is in there? Well, let's say $f/g \in \Gamma(U)$. Since $\Gamma(\mathbb{A}^2) = \mathbb{C}[x,y]$ is a UFD, we can assume that f/g is in lowest terms, so g can only have a pole at $(0,0)$.

But this is a problem.

Theorem 0.6 (Krull's Hauptidealsatz)**.** *Let X be a variety of dimension n, and $f \in \Gamma(X)$ a non-constant function. Then* *every irreducible component of the algebraic set $V(f) \subset X$ has dimension $n-1$.*

(This isn't exactly how Krull stated it, but it's equivalent. We will, um, not be proving this.)

So the zero set of g has dimension 1. Which means that if f/g has a pole at $(0,0)$, then its pole set must be a curve. Which it isn't.

(Lots of words I haven't defined there, I know. Just go with it for now – I promise to make good later.)

So that means g does not *actually* have a zero at $(0,0)$ at all! Or anywhere else, for that matter. So g must be constant, and f/g is actually in $\Gamma(\mathbb{A}^2)$. So $\Gamma(U) = \Gamma(\mathbb{A}^2) = \mathbb{C}[x,y]$!

Remember that awesome theorem we had, that two algebraic sets are isomorphic if and only if their coordinate rings are isomorphic? We've got something similar with local rings. I mean, you can't really expect two varieties to be isomorphic if their local rings are isomorphic, but you can get the other way around.

Theorem 0.7. *Let V and W be two varieties, $\phi\colon V \dashrightarrow W$ a rational map, $P \in V$ a point where ϕ is defined. Then ϕ^* induces an morphism from $\mathcal{O}_{\phi(P)}(W)$ to $\mathcal{O}_P(V)$, given (as usual) by $\phi^*(p) = p \circ \phi$.*

Moreover, if ϕ is birational, and if ϕ^{-1} is defined at $\phi(P)$, then ϕ^* induces an isomorphism from $\mathcal{O}_{\phi(P)}(W)$ to $\mathcal{O}_P(V)$.

Proof: Define $\psi\colon \mathcal{O}_{\phi(P)}(W) \to \mathcal{O}_P(V)$ be defined by

$$\psi\left(\frac{f}{g}\right) = \frac{\phi^* f}{\phi^* g}$$

where f and g are elements of $\Gamma(W)$ satisfying $g(\phi(P)) \neq 0$. It's well defined because if $f/g = p/q$, then $\phi^* f/\phi^* g = \phi^* p/\phi^* q$. And if $g(\phi(P)) \neq 0$, then $(\phi^* g)(P) = g(\phi(P)) \neq 0$, so $\phi^* f/\phi^* g \in \mathcal{O}_P(V)$. So we have a homomorphism ψ.

In the moreover case, it's got an inverse, too! So let's do the same thing with ϕ^{-1}:

$$\psi^{-1}\left(\frac{f}{g}\right) = \frac{(\phi^{-1})^* f}{(\phi^{-1})^* g}$$

It's pretty easy to see that this is indeed a well defined inverse to ψ. So we have an isomorphism on our hands. ♣

In fact, we can generalize this a bit, and get a stronger theorem. To do this, we will – *of course* – require a definition.

Definition 0.8. *A rational map $f \colon V \dashrightarrow W$ is* **birational** *if and only if there is a rational map $g \colon W \dashrightarrow V$ such that $f \circ g$ and $g \circ f$ are both defined, and both equal to the identity function wherever they are defined.*

The map f is said to be **dominant** *if and only if there is no proper closed subset $Y \subset W$ such that $f(V) \subset Y$.*

Rational maps come with their algebra partners too.

Theorem 0.9. *Let $f \colon V \dashrightarrow W$ be a dominant rational map of varieties. Then there is a $\mathbb{C}$-algebra homomorphism $f^* \colon K(W) \to K(V)$ of function fields defined by*

$$f^*(p) = p \circ f$$

for all $p \in K(W)$. Moreover, f is birational if and only if f^* is an isomorphism.

Proof: Well, we're told what f^* has to be. All we gotta check is that it's a $\mathbb{C}$-algebra homomorphism.

Honestly, that's all pretty clear, except for the possibility that $p \circ f$ might not be defined. You've got to remember: f and p are NOT FUNCTIONS. So it's quite possible that their composition is simply junk:

$$f(x, y) = (x, x),\ p = \frac{1}{x - y}$$

Oof. But see, the problem here is that the image of f is entirely contained in the pole set of p.

And if you think about it, this is the only way $p \circ f$ can be not well defined.

But the pole set of p is Zariski closed. And I said f was dominant in the statement of the theorem. So $p \circ f$ is always well defined.

Whew! Good thing I saw that coming.

So all that's left is the moreover. If f is birational, then f^* is a homomorphism from $K(W)$ to $K(V)$, with an inverse homomorphism. (Because $\mathrm{id}^* = (f \circ g)^* = g^* \circ f^*$.) Must be an isomorphism.

Conversely, let's say that $f^* \colon K(W) \to K(V)$ is an isomorphism, and let $g^* \colon K(V) \to K(W)$ be its inverse. ♣

"Alright, Mr. DeMille, I'm ready for my theorem ..."

Theorem 0.10. *Let $f \colon V \dashrightarrow W$ be a birational map with $f(P) = Q$ and $f^{-1}(Q) = P$. Then $f^* \colon K(W) \to K(V)$ induces an isomorphism from $\mathcal{O}_Q(W)$ to $\mathcal{O}_P(V)$.*

Proof: First, since $f(P) = Q$, if $p \in \mathcal{O}_Q(W)$, then $f^*(p) = p \circ f$ is indeed in $\mathcal{O}_P(V)$. (If p is defined at $Q = f(P)$, then $p \circ f$ is defined at P.) So we do have a homomorphism from $\mathcal{O}_Q(W)$ to $\mathcal{O}_P(V)$.

But we can do the same thing to the birational inverse g of f, so we have an inverse homomorphism from $\mathcal{O}_P(V)$ to $\mathcal{O}_Q(W)$! Two homomorphisms, inverse to one another. Sounds like an isomorphism to me. ♣

And now a little algebra digression. An example of how algebra has been influenced by geometry.

Definition 0.11. *Let A be a domain, P a prime ideal of A, K the fraction field of A. The localization of A at P is the set*

$$A_P = \left\{ \frac{a}{b} \mid a, b \in A,\, b \notin P \right\}$$

Notice that if $a/b \in A_P$, it's still possible that $b \in P$! There are many different fractions that represent the same element of K – all you need is for *one* of them to have a denominator outside P, and the fraction ends up in A_P. Otherwise, nothing would be in A_P: if $p \in P$, then $a/b = pa/pb$.

Theorem 0.12. *The localization of A at P is a local ring.*

Recall that a local ring is a ring with a unique maximal ideal; that is, the set of all non-units is an ideal.

Proof: Let a/b and c/d be in A_P, such that b and d are not in P. Then all of $a/b \pm c/d$ and $(a/b)(c/d)$ can be written with denominator bd. Since neither of them is in P and since P is prime, it follows that $bd \notin P$ and so A_P is closed under plus, minus, and times. Since A_P clearly contains 0 and 1, it's a subring of K.

To see the local part: the non-units are exactly the elements of K of the form a/b, where $a \in P$. (If $a/b = c/d$, then $bc = ad \in P$, so $b \notin P$ means $c \in P$. So you can't have one representation with numerator in P, but a different one where the numerator isn't in P.) This is an ideal, because if a/b and c/d satisfy $a, c \in P$ and $b, d \notin P$, then $a/b \pm c/d$ also satisfies that, as does $(a/b)(u/v)$ for any $u/v \in A_P$.

In other words, the unique maximal ideal of A_P is the ideal (of A_P) generated by P, often written P_P, or even just P if the context is clear. ♣

It's not too hard to see, that the ring $\mathcal{O}_P(X)$ is just the localization of $\Gamma(X)$ at the maximal ideal $I(P)$. And this is where the name "localization" comes from: elements of $\mathcal{O}_P(X)$ are elements that are locally defined near P.

And, incidentally, you can make local rings for any subvariety of X. A subvariety Y is, by definition, irreducible, and so its corresponding ideal $I(Y)$ is prime, so we can construct the local ring $\mathcal{O}_Y(X)$ as the localization of $\Gamma(X)$ at the prime ideal $I(Y)$. We won't really deal with that, but it's nice to know.

One last useful theorem about localization.

Theorem 0.13. *Let A be a noetherian domain, P a prime ideal of A. Then the localization A_P is noetherian.*

Proof: Let I be an ideal of A_P. We want to show that I is finitely generated.

Consider $J = I \cap A$. Since A is noetherian, J is finitely generated, say, by $\{x_1, \ldots, x_n\}$.

If $x \in I$, then we can write $x = a/b$ for some $a, b \in A$, $b \notin P$, and so $a = xb \in I$ as well. But $a \in A$, so $a \in J$, so we can write

$$a = a_1 x_1 + \ldots + a_n x_n$$

for some $a_1, \ldots, a_n \in A$. Dividing both sides by b gives

$$x = (a_1/b)x_1 + \ldots + (a_n/b)x_n$$

with $a_i/b \in A_P$ because $b \notin P$. So I is generated by $\{x_1, \ldots, x_n\}$ as well. ♣