Lecture notes for PM 464/764 – Week Three

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Spring 2021

1 Rational functions

So far, our “algebra” has been limited to the three operations of addition, subtraction, and multiplication. It’s time to add division to the list.

Division works better when there aren’t any zero divisors. I mean, you can’t divide by zero, right? But if $ab = 0$ for $a, b \neq 0$, that means it’s not ok to divide by a or b either: do one and then the other, and badness ensues.

Luckily, the geometry suggests that it’s ok to rule out zero divisors for us. I mean, remember from last week that an algebraic set X is irreducible if and only if its corresponding ideal $I(X)$ is prime, which is true if and only if the coordinate ring $\Gamma(X)$ is a domain. (For me, “domain” means the same thing as “integral domain”, but wastes fewer electrons.) And domains are, of course, rings with no zero divisors. (If you believe that zero is a zero divisor, then more power to you. But I don’t.)

But an algebraic set X is reducible if and only if it can be written as the union of two algebraic subsets: $X = Y \cup Z$. If you can understand Y and Z, then surely you understand X, no? And it's not too hard to show – using the Hilbert Basis Theorem – that every algebraic set X is the union of finitely many irreducible algebraic sets, and that this union is unique.

(For any reducible X, write it as the union of two proper subvarieties. If they're reducible, write *them* as unions of proper subvarieties. If that process never stops, you get an infinite descending chain of subvarieties $X \supsetneq X_1 \supsetneq \ldots$, which translates into an ascending chain of ideals $I(X) \subset I(X_1) \subset \ldots$, which is impossible because $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian.)

Definition 1.1. *Let X be an algebraic set, and write $X = X_1 \cup \ldots \cup X_r$ be an expression of X as the union of finitely many irreducible algebraic subsets. The sets X_i are called the* **irreducible components** *of X.*

This means we need a snappy name for these beasts, because "irreducible algebraic set" is completely environmentally unsustainable.

Definition 1.2. *A* **variety** *is an irreducible algebraic set.*

Warning: Some people define "variety" differently, sometimes to mean "algebraic set". These people are silly of course, but it's important to know that they're out there.

Another Warning: Some other people use the word "variety" to mean something totally different, like a kind of structure in universal algebra, or (if capitalized) a quaint thing from the last century called a "magazine". These people are not silly – or at least, not silly for using this terminology – but it's important not to confuse their varieties (or Variety) with ours.

Not Really a Warning: Actually quite a few people use the phrase "algebraic variety" to distinguish our kind of variety from universal algebra structures and magazines, etc, but I can't be bothered.

ANYWAY. We now have a shorthand for irreducible algebraic sets, so we will move on.

Definition 1.3. *Let X be a variety. The* **function field** $K(X)$ *of X is the fraction field of the coordinate ring* $\Gamma(X)$*. An element of $K(X)$ is called a* **rational function**.

The easiest example of this – and it's one you've seen before – is affine space $\mathbb{A}^n$. The function field of $\mathbb{A}^n$ is, of course, the field of rational functions $\mathbb{C}(x_1, \ldots, x_n)$. It's the field consisting of fractions of polynomials in $x_1, \ldots, x_n$. Which are called rational functions.

And in general, the function field of any variety X is the set of fractions whose numerator and denominator are elements of $\Gamma(X)$.

That might seem a little freaky. I mean, elements of $\Gamma(X)$ are equivalence classes of polynomials. How do you make a fraction out of them?

It's actually completely straightforward – you just have to get used to it. Honest. I'll show you in an example.

Let's say we're working with the variety $X = V(y^2 - x^2 - 1)$ – a hyperbola. ("But it's in two pieces!" I hear you cry. "Don't worry, little student," I softly reassure you. "We're working over $\mathbb{C}$, where the two real halves of this wonderful variety are lovingly brought together into an irreducible complex whole.")

The coordinate ring of X is

$$\Gamma(X) = \mathbb{C}[x, y]/(x^2 - y^2 - 1)$$

So an example of a rational function on X is

$$\frac{xy}{2x^2 + y^2 + y}$$

where we interpret x and y modulo $I(X)$. So this rational function is the same rational function as

$$\frac{xy}{2x^2 + y^2 + y} = \frac{xy}{3x^2 + y - 1}$$

because $y^2 \equiv x^2 - 1 \pmod{I(X)}$. The numerator and denominator here are the same functions – the same elements of $\Gamma(X)$ – it's just that the denominator has a different name – a different choice of representative – in the two representations of the fraction.

And we're calling these freaky things "rational functions". So, are they functions?

Wellllllllllllll. Um.

No.

But they almost are! I mean, the numerator and denominator are both functions, right? So we just divide them, and get another function ... except that maybe the denominator is zero and all is lost.

So, of course, we make a definition to pigeonhole all our problems.

Definition 1.4. *Let f be a rational function on a variety X, and let $P \in X$ be a point. Then f is* **defined at** *P if and only if there is some expression*

$$f = \frac{p}{q}$$

for $p, q \in \Gamma(X)$ with $q(P) \neq 0$. If f is not defined at P, we say that P is a **pole** *of f, or that f has a pole at P.*

So, for example, x/y has a pole at $(3, 0)$ (and, in fact, at $(a, 0)$ for all complex numbers a) on $\mathbb{A}^2$. Technically, I have to show that there is no way of rewriting x/y to make the zero in the denominator go away at $(3, 0)$, but since $\mathbb{C}[x, y]$ is a UFD, and since x/y is in lowest terms, this is pretty obvious.

You might be wondering "does that happen every time?" I mean, maybe you just write down a non-stupid representation of the rational function, and if the denominator is zero, then there's a pole?

Alas, life is not so good. Let X be the variety $V(y^2 - x^3 + x)$, $P = (0, 0)$, and $f = x/y$. Sure looks like f is written nicely, and for sure the denominator is zero at P. But it's not a pole:

$$\frac{x}{y} = \frac{y}{x^2 - 1}$$

and that second fraction has a nonzero denominator at P! (If you're wondering why those two fractions are the same, cross-multiply and work modulo $I(X)$.)

Worse yet, the second representation of f is also imperfect, because it appears to have poles when $x = \pm 1$ that are obviously not poles because of the x/y version of f. So there is no "best" fraction that represents the rational function f – you have to use different versions for different points.

There are even more terrible things that can happen if you use other curves. But I'll leave those examples for the homework. (Mwhahaha.)

There is the following shortcut that usually works, though:

Theorem 1.5. *Let f be a rational function on a variety X, $P \in X$ a point. If there is a representation*

$$f = \frac{p}{q}$$

for which $q(P) = 0$ and $p(P) \neq 0$, then P is a pole of f.

Proof: Choose any representation $f = a/b$ for elements $a, b \in \Gamma(X)$. Then since $a/b = p/q$, we have $aq = bp$. Since $q(P) = 0$, we get $(bp)(P) = 0$, and since $p(P) \neq 0$, this means $b(P) = 0$. So the denominator of f is always zero at P, and P is a pole. ♣

So the only way you can have a stealth non-pole is if *both* the numerator and denominator are zero.

We can use all this to study the structure of a variety near a point P.

Definition 1.6. *Let X be a variety, $P \in X$. The* **local ring** *at P is the ring*

$$\mathcal{O}_P(X) = \{f \in K(X) \mid f \text{ is defined at } P\}$$

It's pretty easy to show that $\mathcal{O}_P(X)$ is a subring of $K(X)$. And if you remember that a local ring is a ring with a unique maximal ideal – that is, the set of nonunits is an ideal – then you'll notice that $\mathcal{O}_P(X)$ is also a local ring, whose maximal ideal $\mathcal{M}_P(X)$ is

$$\mathcal{M}_P(X) = \left\{ f \in K(X) \mid f = \frac{a}{b} \text{ for } a(P) = 0, b(P) \neq 0 \right\}$$

(A unit of $\mathcal{O}_P(X)$ is a rational function whose reciprocal is also in $\mathcal{O}_P(X)$. So the units are the ones where the denominator doesn't vanish at P, and neither does the numerator – if the numerator vanished, then by the shortcut theorem we proved above, the reciprocal would have a pole at P.)

Recall that the ideal $I(P)$ in $\Gamma(X)$ is the ideal of all the functions that vanish at P. So in fact, we have

$$\mathcal{M}_P(X) = I(P)\mathcal{O}_P(X)$$

which is to say, $\mathcal{M}_P(X)$ is just the ideal of $\mathcal{O}_P(X)$ generated by $I(P)$.

More on local rings next week.