Lecture notes for PM 464/764 – Week Eleven

David McKinnon
Department of Pure Mathematics
University of Waterloo

Spring 2021

1 More about divisors and projective embeddings

We can now see that if H and H' are hyperplanes in $\mathbb{P}^n$, then $\mathrm{div}(H) - \mathrm{div}(H') = \mathrm{div}(H/H')$ is the divisor of a rational function. We would like to be able to say conversely that if $\zeta \in K$ is any nonconstant rational function and H any hyperplane in $\mathbb{P}^n$, then $\mathrm{div}(H) + \mathrm{div}(\zeta) = \mathrm{div}(H')$ for some hyperplane H', but sadly, this is not true.

Example 1.1. Let C be $\mathbb{P}^1$, and consider the embedding $\phi\colon C \to \mathbb{P}^3$ defined by $\psi([a:b]) = ([a^4 : a^3b : ab^3 : b^4])$. It's clear that ψ is defined everywhere, and the image is defined by the relations $xz^2 = y^3$, $wz^3 = y^4$, $yw^2 = x^3$, and $w^3z = x^4$ (variables listed in alphabetical order). A quick check of the Jacobian matrix shows that $\psi(C)$ is a smooth curve.

Now consider the hyperplane $w = 0$, and the rational function y^2/wz. The divisor $\mathrm{div}(w)$ is supported entirely on the point

$P = [0 : 0 : 0 : 1]$, since this is the only point of C on which w vanishes. Specialize to the affine piece $z = 1$ to do calculations. The tangent line to C at P is the line $w = x = 0$, so y/z is a uniformizer at P. Thus, the relation $wz^3 = y^4$ shows that $\text{ord}_P(w) = 4$, so $\text{div}(w) = 4P$.

Next, notice that $\text{div}(y^2/wz)$ is supported on the two points $P = [0 : 0 : 0 : 1]$ and $Q = [1 : 0 : 0 : 0]$. At P, we have (setting $z = 1$):

$$\text{ord}_P(y^2/wz) = \text{ord}_P(y^2/w) = 2 - 4 = -2$$

and so to make sure the degree is zero, we must have

$$\text{div}(y^2/wz) = 2Q - 2P$$

Therefore, we get $\text{div}(w) + \text{div}(y^2/wz) = 2P + 2Q$.

Let's assume for the moment that $2P+2Q = \text{div}(H)$ for some hyperplane H, and try to find H. H would have to contain the tangent line to P and the tangent line to Q, since that's the only way H can possibly vanish to order greater than one at P and Q. But these tangent lines are $w = x = 0$ and $y = z = 0$, respectively, and these two lines are skew! That is, they don't intersect, and they're not parallel, so there's no way they can be contained in a single plane. Thus, H cannot exist.

Darn. However, all is not lost. Soon enough, we'll show that if $\text{div}(H)$ is a hyperplane section of C in some embedding, and ζ is a rational function on C, then we can find some embedding of C in projective space such that $\text{div}(H) + \text{div}(\zeta)$ is a hyperplane section of C. For instance, in Example 1.1, if we change the embedding ψ to $\phi([a : b]) = [a^4 : a^3b : a^2b^2 : ab^3 : b^4] \in \mathbb{P}^4$, with homogeneous coordinates v through z, then the hyperplane $x = 0$ will satisfy $\mathrm{div}(H) = 2P + 2Q$.

So here's the plan. We want to take the set of divisors on C, and define an equivalence relation on them such that two divisors D and D' are equivalent if and only if there's some embedding of C in some $\mathbb{P}^n$ such that $D = \mathrm{div}(H)$ and $D' = \mathrm{div}(H')$ for hyperplanes H and H' in $\mathbb{P}^n$. We make the following definition:

Definition 1.2. *Let D and D' be two divisors on C. We say that D is linearly equivalent to D' (written $D \equiv D'$) if and only if there is some rational function $\zeta \in K$ such that $D - D' = div(\zeta)$.*

For example, let C be the curve $x = 0$ in $\mathbb{P}^2$, and define divisors $D_1 = P_1$ and $D_2 = P_2$, where $P_1 = [0 : 1 : 1]$ and $P_2 = [0 : 1 : 0]$. Then we have $D_1 = D_2 + \mathrm{div}(\zeta)$, where $\zeta = y - 1 \in K$. (To see this, recall that in the previous example, we computed $\mathrm{div}(\zeta) = P_1 - P_2$.) Thus, we deduce that $D_1 \equiv D_2$.

Notice that in our definition of linear equivalence, we allow D to be any divisor on C – we don't require it to be a hyperplane section. This is mostly because we don't know in advance which divisors are going to be hyperplane sections and which ones aren't. However, it's also going to turn out to be whoppingly useful to carry around all those other divisors with us, without worrying about the fact that they can't possibly be hyperplane sections.

Which brings us to another matter. At this point, we've stopped talking about C as a subset of projective space, really, and we've started talking about all possible ways to embed C in projective space. In this context, it's not really fair to say that some divisor D "is a hyperplane section", because in some embeddings it is a hyperplane section, and in others it's not. So we'll change the terminology:

Definition 1.3. *A divisor D on a smooth curve C is very ample if and only if there is some embedding $\phi\colon C \to \mathbb{P}^n$ such that $D = div(H)$ on $\phi(C)$ for some hyperplane $H \subset \mathbb{P}^n$.*

This terminology is not quite the same as the modern standard usage, since being very ample is usually thought of as a property of a linear equivalence class rather than a divisor, so that a very ample divisor is really any divisor which is linearly equivalent to a hyperplane section in some embedding. And in case you're wondering, there is indeed such a thing as an "ample divisor", but for us it will mean exactly the same thing as "effective divisor", so we won't bother with it.

In any case, let's prove some elementary facts about divisors and linear equivalence.

Theorem 1.4. *Let $\zeta \in K$ be a rational function. Then $div(\zeta)$ is effective if and only if $\zeta \in \mathbb{C}$ is constant.*

Proof: Assume that $\text{div}(\zeta)$ is effective. Then clearly ζ has no poles. Let $\zeta(P) = \alpha$ for some element $\alpha \in \mathbb{C}$ and some point $P \in C$. Then $\zeta - \alpha$ also has no poles ... but $(\zeta - \alpha)(P) = 0$, so it has at least one zero! This is impossible by the previous theorem unless $\zeta - \alpha$ is identically zero, so we deduce that $\zeta = \alpha \in \mathbb{C}$.

Conversely, if $\zeta \in \mathbb{C}$, then it's clear that $\text{div}(\zeta) = 0$ is effective. ♣

Theorem 1.5. *Let ζ_1 and ζ_2 be any two nonzero rational functions in K. Then $div(\zeta_1\zeta_2) = div(\zeta_1) + div(\zeta_2)$, and $div(\zeta_1) = div(\zeta_2)$ if and only if $\zeta_1/\zeta_2 \in \mathbb{C}$.*

Proof: For the first claim, it suffices to show that for each $P \in C$, we have $\mathrm{ord}_P(\zeta_1\zeta_2) = \mathrm{ord}_P(\zeta_1) + \mathrm{ord}_P(\zeta_2)$. This is immediate if we write $\zeta_i = u_i t^{\mathrm{ord}_P(\zeta_i)}$ for units u_i and uniformizer t of $\mathcal{O}_P$. The second claim follows immediately from the first. ♣

Theorem 1.6. *Linear equivalence is an equivalence relation on divisors.*

Proof: Since $\mathrm{div}(0) = 0$, it follows that linear equivalence is reflexive. Symmetry similarly follows immediately from the fact that $\mathrm{div}(1/\zeta) = \mathrm{div}(1) - \mathrm{div}(\zeta) = -\mathrm{div}(\zeta)$, so if $D_1 = D_2 + \mathrm{div}(\zeta)$, then $D_2 = D_1 + \mathrm{div}(1/\zeta)$. Transitivity follows immediately from $\mathrm{div}(\zeta_1\zeta_2) = \mathrm{div}(\zeta_1) + \mathrm{div}(\zeta_2)$: if $D_1 = D_2 + \mathrm{div}(\zeta_1)$ and $D_2 = D_3 + \mathrm{div}(\zeta_2)$, then $D_1 = D_3 + \mathrm{div}(\zeta_1\zeta_2)$. ♣

Theorem 1.7. *Let D_1 and D_2 be linearly equivalent divisors on a smooth plane curve C. Then $\deg(D_1) = \deg(D_2)$, and for any linearly equivalent divisors $D_3 \equiv D_4$ on C, we have $D_3 + D_1 \equiv D_4 + D_2$.*

Proof: Let $D_1 = D_2 + \mathrm{div}(\zeta)$. Then $\deg(D_1) = \deg(D_2 + \mathrm{div}(\zeta)) = \deg(D_2) + \deg(\mathrm{div}(\zeta)) = \deg(D_2)$. For the second claim, if $D_3 = D_4 + \mathrm{div}(\xi)$, then $D_3 + D_1 = D_4 + D_2 + \mathrm{div}(\zeta\xi)$, so $D_3 + D_1 \equiv D_4 + D_2$. ♣

For any divisor D on C, denote by $|D|$ its linear equivalence class. The preceding discussion shows that the degree of $|D|$ is well defined, so we have a nice definition of the degree of a smooth curve:

Definition 1.8. *Let $C \subset \mathbb{P}^n$ be a smooth curve. The degree of C is defined to be* $\deg$ *div*(H)*, where H is any hyperplane in $\mathbb{P}^n$.*

Better yet, for any pair of divisors D_1 and D_2 on C, we have a well defined sum of linear equivalence classes $|D_1| + |D_2| = |D_1 + D_2|$. Thus, we can make the following definition:

Definition 1.9. *Let C be a smooth plane curve. The Picard group $Pic(C)$ of C is defined to be the set of linear equivalence classes of divisors on C, with the addition as defined above.*

As the terminology suggests, $\mathrm{Pic}(C)$ is a group, which isn't too hard to see. I mean, it's the quotient of the group of divisors by the subgroup of divisors of rational functions.

There's a subset of $\mathrm{Pic}(C)$ which is also quite interesting for our purposes. The set of linear equivalence classes of degree 0 is called $\mathrm{Pic}^0(C)$ (read: "Pick-zero of C"). This is well defined, because degree is well defined for linear equivalence classes. More on $\mathrm{Pic}^0(C)$ later.

Before that, though, let's justify the claim made above about linearly equivalent very ample divisors. Specifically, we claimed that if D and D' are linearly equivalent very ample divisors, then there is an embedding $\phi\colon C \to \mathbb{P}^n$ such that $D = \mathrm{div}(H)$ and $D' = \mathrm{div}(H')$ on $\phi(C)$. The key will be to use the linear equivalence class $|D| = |D'|$, but first, we'll need to identify which elements of $|D|$ have any shot at being hyperplane sections.

Definition 1.10. *Let D be a divisor on a smooth curve C. Define the set:*

$$L(D) = \{f \in K(C) \mid D + div(f) \text{ is effective}\} \cup \{0\}$$

Note that $L(D)$ is a vector space, because $\mathrm{div}(f)$ is invariant under scalar multiplication, and if $D+\mathrm{div}(f)$ and $D+\mathrm{div}(g)$ are both effective, then $D + \mathrm{div}(f+g)$ is effective because $\mathrm{ord}_P(f + g) \geq \min\{\mathrm{ord}_P(f), \mathrm{ord}_P(g)\}$.

Our first step will be to prove that $L(D)$ is finite dimensional. Note first of all that if $\deg(D) < 0$, then $L(D) = \{0\}$, and that $L(0) = k$, since 0 is the only effective divisor of degree zero. Moreover, if $\deg(D) = 0$, then $L(D) = \{0\}$ unless $D \equiv 0$.

Theorem 1.11. *Let D be an effective divisor on a smooth curve C. Then* $\dim(L(D)) \leq \deg(D) + 1$.

Proof: First, notice that if $D = 0$, then $\dim L(D) = 1$ and $\deg D = 0$, so the claim is true. Then, notice that if $D - D'$ is effective, then $L(D') \subset L(D)$, because all the coefficients of D are no smaller than those of D'. Thus, by induction on the degree of D, it suffices to show that if $D = D' + P$, then $\dim L(D) - \dim L(D') \leq 1$.

To prove this, let t be a uniformizer for $\mathcal{O}_P(C)$, and let n be the coefficient of P in D. Then for all $f \in L(D)$, we know that $\mathrm{ord}_P(f) \geq -n$, with $\mathrm{ord}_P(f) > -n$ if and only if $f \in L(D')$. If we define a linear transformation $\phi \colon L(D) \to k$ by $\phi(f) = (t^n f)(P)$, then it's clear that the kernel of this map is precisely $L(D')$, and hence $\dim L(D) - \dim L(D') \leq 1$, as desired. ♣

We are now in a position to define our projective embedding ϕ. Let $\psi \colon C \to \mathbb{P}^n$ be an embedding such that $D = \mathrm{div}(H)$ on $\psi(C)$ for some hyperplane H. After a change of coordinates, we can assume that H is the hyperplane $x_0 = 0$, where $\{x_0, \ldots, x_n\}$ are homogeneous coordinates on $\mathbb{P}^n$. The rational functions x_i/x_0 are all elements of $L(D)$. Thus, if we write $f_i = x_i/x_0$, we can extend the set of f_i's to a spanning set

$\{f_0, \ldots, f_m\}$ of $L(D)$. (Note that $f_0 = 1$.)

Define $\phi\colon C \to \mathbb{P}^m$ by $\phi(P) = [f_0(P) : \ldots : f_m(P)]$. By choice of f_0 through f_n, it's clear that ϕ is an embedding, since composition with the linear projection onto the first n coordinates results in the embedding ψ. And the great thing about ϕ is that for any effective divisor D' with $D \equiv D'$, we will show that D' is a hyperplane section of $\phi(C)$.

Note that D is certainly a hyperplane section of $\phi(C)$; in fact, it's the hyperplane section $x_0 = 0$! Now let D' be any effective divisor which is linearly equivalent to D. Then $D - D' = \mathrm{div}(f)$ for some rational function $f \in L(D)$, so we can write (possibly in many different ways) $f = a_0 f_0 + \ldots + a_m f_m$ for some constants $a_m \in k$. We claim that $D' = \mathrm{div}(H)$, where H is the hyperplane $a_0 x_0 + \ldots + a_m x_m = 0$.

We already know that $D = \mathrm{div}(x_0)$. In fact, for each i, we have $x_i = x_0 f_i$ (as elements of the homogeneous coordinate ring of $\phi(C)$). To see this, note that if $i \leq n$, then $x_i = x_0 f_i$ as elements of the homogeneous coordinate ring of $\psi(C)$. Therefore, multiplying through by x_0 is precisely what is needed to clear the denominators of f_0 through f_n. But by definition of $L(D)$, the divisor $\mathrm{div}(f_i) + \mathrm{div}(x_0)$ is effective for all i, so multiplying through by x_0 will clear the denominators of *all* the f_i, and so we conclude that $x_i = x_0 f_i$. We then notice that $\sum a_i x_i = x_0 \sum a_i f_i$, so that $\mathrm{div}(H) = \mathrm{div}(D) + \mathrm{div}(f) = \mathrm{div}(D')$, as desired.

Example 1.12. Let's work this out for Example 1.1. Our original curve C is $\mathbb{P}^1$, and our original divisor is $D = 4P$, where P is the point $[0 : 1]$ on $\mathbb{P}^1$. We constructed above an embedding of C in $\mathbb{P}^3$ such that D was $\mathrm{div}(w)$: $\psi([a : b]) = [a^4 : a^3 b : a b^3 : b^4]$.

To construct the embedding ϕ described above, we need to find a basis of $L(D)$.

By Theorem 1.11, we know that $\dim L(D) \leq \deg(D) + 1 = 5$. If we regard C as being embedded (trivially!) in $\mathbb{P}^1$ with homogeneous coordinates a and b, then $\mathrm{div}(b/a) = Q - P$, so $\{1, u, u^2, u^3, u^4\}$ is a linearly independent subset of $L(D)$, where $u = b/a$. (They're all in $L(D)$ because their divisors are effective when you add $4P$ to them, and they're linearly independent because any linear relation would become a polynomial in u that's identically zero.)

We can now write down our embedding:

$$\phi([a : b]) = [1 : b/a : b^2/a^2 : b^3/a^3 : b^4/a^4] = [a^4 : a^3b : a^2b^2 : ab^3 : b^4]$$

which is the same as the ϕ we described at the end of Example 1.1, up to a projective change of coordinates.

A neat thing to notice is that we never really used the fact that D is very ample in all this discussion, except to show that the map ϕ was an embedding. In fact, as long as $L(D)$ has positive dimension, you can always construct a rational map $\phi\colon C \to \mathbb{P}^m$ as above. Even more, there's no reason that the set $\{f_0, \ldots, f_m\}$ needs to span $L(D)$ – it just needs to contain some nonzero function.

Generally, then, let D be a divisor on C, and let $\{f_0, \ldots, f_m\}$ be a subset of $L(D)$ such that at least one of the f_i is not identically zero. Define a rational map $\phi\colon C \to \mathbb{P}^m$ as follows:

$$\phi(P) = [f_0(P) : \ldots : f_m(P)]$$

The first thing to observe is that since C is smooth, this rational map is always defined everywhere.

So to any finite set of elements of $L(D)$, not all zero, we can associate a morphism ϕ from C to $\mathbb{P}^m$. If our finite set generates $L(D)$ as a vector space, and if D is very ample, then ϕ is an embedding. We've already seen in Example 1.1 that if D is very ample, you can sometimes get an embedding even if the finite set does not span $L(D)$, but by definition there's no way it's going to be an embedding if D is not very ample.

There are a number of elementary properties of this construction:

1. The subset $\{f_0, \ldots, f_m\}$ results in a map to $\mathbb{P}^m$ – the dimension of the target space of the embedding is one less than the number of elements of $L(D)$ you pick out.

2. Any linear relation $\sum a_i f_i = 0$ corresponds to a hyperplane $\sum a_i x_i = 0$ which contains $\phi(C)$.

3. If ϕ is the map corresponding to $\{f_0, \ldots, f_m\}$ and ψ is the map corresponding to $\{f_0, \ldots, f_{m-1}\}$, then $\psi = \pi \circ \phi$, where $\pi \colon \mathbb{P}^m \to \mathbb{P}^{m-1}$ is the linear projection $\pi([x_0 : \ldots : x_m]) = [x_0 : \ldots : x_{m-1}]$.

4. If ϕ is an embedding, then the degree of $\phi(C)$ equals the degree of D.

5. If $\{f_0, \ldots, f_n\}$ and $\{g_0, \ldots, g_n\}$ span the same subspace of $L(D)$, then the corresponding maps to projective space differ only by a projective change of coordinates.

Moreover, any embedding of C into projective space can be constructed in this way. Let $\phi \colon C \to \mathbb{P}^n$ be an embedding, and let D_i be the divisor $\operatorname{div}(x_i)$. Then clearly $D_i \equiv D_0$ for all i, so choose rational functions $f_i \in K(C)$ such that $D_i - D_0 = \text{div}(f_i)$. Then since D_i is effective, it follows that $f_i \in L(D_0)$, and it's clear that ϕ is precisely the map associated to $\{f_0, \ldots, f_n\}$.

Putting all this together, we see that if $\phi \colon C \to \mathbb{P}^n$ is any embedding of C with $\text{div}(x_0) \equiv D$, then there is some subset $S_\phi = \{f_0, \ldots, f_n\} \subset L(D)$ such that ϕ is induced by S_ϕ. If $\psi \colon C \to \mathbb{P}^m$ is another such embedding, associated to $\{g_0, \ldots, g_m\}$, then both ϕ and ψ can be obtained by a sequence of linear projections from the embedding associated to $\{f_0, \ldots, f_n, g_0, \ldots, g_m\} \subset L(D)$. Thus, we have proven the following result:

Theorem 1.13. *Any embedding of C into projective space with div$(x_0) = D$ can be obtained by finding the embedding of C associated to some basis of $L(D)$, and then composing with a projective change of coordinates, an embedding of $\mathbb{P}^n$ into $\mathbb{P}^m$ as a linear subspace, and a sequence of linear projections.*

Proof: Pretty much as explained above. The embedding of $\mathbb{P}^n$ as a linear subspace of $\mathbb{P}^m$ comes from the fact that the set of rational functions you pick to create your embedding might not be linearly independent. ♣

A similar fact is true about general maps from C to projective space, but unfortunately we lack the technology to state it precisely.

Nevertheless, this tells us something about embeddings of $\mathbb{P}^1$ in projective space, for example. Say that $\phi \colon \mathbb{P}^1 \to \mathbb{P}^n$ is an embedding. Then the associated divisor – the hyperplane section associated to the embedding – is a divisor with positive degree d.

For any two points $P = [a : b]$ and $Q = [c : d]$ on $\mathbb{P}^1$, the divisor $P - Q$ is the divisor of a rational function, namely $(bx - ay)/(dx - cy)$. So any two divisors of the same degree are linearly equivalent!

So we just need to pick one divisor of degree d, compute the corresponding $L(D)$, and see what we get.

The easiest one is $d[0 : 1]$. So let's compute $L(d[0 : 1])$.

It's all the rational functions that have no poles except maybe at $[0 : 1]$, and those poles are only allowed to be of order up to d. It's pretty easy to see that the following is a basis for $L(d[0 : 1])$:

$$1, \frac{y}{x}, \frac{y^2}{x^2}, \ldots, \frac{y^d}{x^d}$$

(The denominator of anything in $L(d[0 : 1])$ – in lowest terms – is obviously x^a for some $a \leq d$. That means the numerator is some homogeneous polynomial in x and y of degree a. All the terms with x in them will cancel stuff with the x^a in the denominator, and you're left with a linear combination of the basis elements listed.)

So the following morphism is really important:

$$[x^d : yx^{d-1} : \ldots : y^d]$$

where d is a positive integer.

This map is called the dth Veronese embedding of $\mathbb{P}^1$, or the d-uple embedding of $\mathbb{P}^1$, and its image is called the rational normal curve of degree d. So every embedding of $\mathbb{P}^1$ in $\mathbb{P}^n$ is a composition of the dth Veronese embedding and a linear map.