Counting rational points on ruled varieties over function fields

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Abstract

Let K be the function field of an algebraic curve C defined over a finite field $\mathbb{F}_q$. Let $V \subset \mathbb{P}^n_K$ be a projective variety which is a union of lines. We prove a general result computing the number of rational points of bounded height on V/K. We first compute the number of rational points on a general line defined over K, and then sum over the lines covering V.

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1 Introduction

There are a multitude of fascinating and deep results which count the number of rational points of bounded height on an algebraic variety ... if that variety is defined over a number field. If that variety is instead defined over the Other Global Field — namely, the function field of a curve defined over a finite field — then results are thinner on the ground. In particular, there are papers of Bourqui ([3] and [4]) and Lai and Yeung ([9]) that compute counting functions for toric varieties (anticanonically embedded) and flag varieties, respectively. For ruled varieties, there seems to be very little known beyond this.

The purpose of this paper is to prove a general result on the counting functions of ruled varieties defined over function fields (Theorem 3.1). The main technical tool in this is Theorem 2.1, which computes a very precise and uniform upper bound for the counting function of a line over a function field. This bound depends inversely on the height of the line, and therefore is amenable to the following naive approach for computing an upper bound of the counting function of a ruled variety:

1. For each line of the ruling, use Theorem 2.1 to compute an upper bound for the counting function in terms of the height of the line.

2. Sum these upper bounds over all lines of the ruling.

The key point is, of course, the convergence of the sum. Somewhat surprisingly, the sum very often does converge, and even more surprising, often gives a fairly sharp upper bound on the counting function of the ruled variety, at least in terms of getting the right exponent on the main term.

This paper is parallel to [10], which proves the analogues of Theorem 2.1 and Theorem 3.1 in the number field case. The chief difference between the two papers — besides the obvious one — is that the details of the proof of Theorem 2.1 are quite different from their number field analogues. In particular, the results of section 3 follow from Theorem 2.1 in a completely formal way, and so the proofs are essentially identical to those of [10].

2 Rational Points on Lines

Let K be the function field of a smooth projective curve C, defined over a finite field $\mathbb{F}_q$ with q elements. Let L be a line in $\mathbb{P}^n_K$. We wish to compute an upper bound for the counting function:

$$N_L(B) = \operatorname{card}\{P \in L(K) \mid h(P) \le B\}$$

where $h(P)$ denotes the standard logarithmic height in projective space:

$$h([x_0 : \ldots : x_n]) = \sum_v \max_i \{|x_i|_v\})$$

where v ranges over all (isomorphism classes of) valuations on K. Note that we do not normalize the height to be independent of the field K, but only so that the product formula holds:

$$\sum_v |x|_v = 0$$

If L is $\mathbb{P}^1_K$, it is well known that asymptotically, the counting function $N_L(B)$ is given by:

$$N_L(B) \ll cq^{2B}$$

where c is a constant depending only on the field K. However, this estimate will not suffice for our present purpose, since we wish to control the set of points of small height on our lines as well. Furthermore, since our lines will not generally be identical to $\mathbb{P}^1$, we wish to explore the dependence of the constant c on the height $h(L)$ of the line L, which we define to be the height of the corresponding Plücker point in the Grassmannian $G(1,n)$.

More precisely, say L corresponds to a 2-dimensional subspace of K^{n+1}, spanned by the vectors $(a_0,\ldots,a_n)$ and $(b_0,\ldots,b_n)$. We define:

$$h(L) = h((a_0 dx_0 + \ldots + a_n dx_n) \wedge (b_0 dx_0 + \ldots + b_n dx_n))$$

where the result of the wedge product is interpreted as a point in $\mathbb{P}_K^{(n^2+n)/2}$ with homogeneous coordinates $\{dx_i \wedge dx_j\}$ for $i \neq j$.

We can now state the main result of this section:

Theorem 2.1 *The counting function for L satisfies the following inequalities:*

$$N_L(B) \leq c_K q^{2B-h(L)} + 1$$

where c_K is a positive real constant depending only on the field K.

Remark: Note that Theorem 2.1 is a direct analogue of Theorem 2.1 from [10]. If one rewrites Theorem 2.1 in terms of a multiplicative height, say, $H(P) = q^{h(P)}$ and $H(L) = q^{h(L)}$, then the main inequality becomes:

$$N_L(B) \leq \frac{c_K}{H(L)} B^2 + 1$$

which is precisely the inequality that appears in [10].

Proof: Let v be a valuation of K, corresponding to an effective K-rational divisor D on C. Since D is effective, it is ample, so some multiple of D is very ample, and so we can find an affine subset T of C such that $C - T$ is precisely the support of D. (In the case that $C = \mathbb{P}^1_K$, we may choose D to be the point at infinity, and T to be the affine line $\mathbb{A}^1_K$.) Let $\mathcal{O}$ be the coordinate ring of T.

Our first step will be to identify the K-rational points of L with a certain subset of a rank two $\mathcal{O}$-module in a finite-dimensional K-vector space. Let $[x_0 : \ldots : x_n]$ be a K-rational point on L. By clearing denominators, we can ensure that $x_i \in \mathcal{O}$ for all i. In fact, by choosing a fixed set $\mathcal{J}$ of representatives for the ideal class group of $\mathcal{O}$, we can ensure that the coordinates x_i

generate an ideal in $\mathcal{J}$. This representation for $[x_0 \colon \ \dots : x_n]$ is unique up to multiplication by a unit of $\mathcal{O}$. Note that the ideal class group of $\mathcal{O}$ is finite by [11], Lemma 5.6, Corollary 14.2, and Theorem 14.5, and so $\mathcal{J}$ is finite as well. Note further that the units of $\mathcal{O}$ are precisely the constants (elements of $\mathbb{F}_q$), since a unit of $\mathcal{O}$ must be a rational function which has no poles or zeroes on T, and therefore no zeroes or poles on C, since there is only one place at infinity. The only such functions are the constants. (Geometrically, there may be several points at infinity, but by our construction they are all conjugate over $\mathbb{F}_q$, and so a rational function over $\mathbb{F}_q$ has the same order of vanishing at every point of $C - T$.)

Let M be the $\mathcal{O}$-submodule of K^{n+1} generated by the vectors $(x_0, \dots, x_n)$ such that $[x_0 : \dots : x_n]$ is a K-rational point on L satisfying $x_i \in \mathcal{O}$ and $(x_0, \dots, x_n) \in \mathcal{J}$. It is clear that there is a natural correspondence between $L(K)$ and a subset of M, so that if we wish to bound from above the number of points of $L(K)$ of bounded height, then it suffices to bound the number of nonzero vectors in M of bounded $|\cdot|_v$-size, modulo K-linear equivalence. (Note that K^* does not act on M, but by an abuse of terminology we use the term K-linear equivalence to refer to the equivalence relation induced on $M - \{0\}$ by the K^*-action on $K^{n+1} - \{0\}$. Note further that we measure the $|\cdot|_v$-size of an element $\mathbf{w}$ of any $\mathcal{O}$-submodule W of K^m by $|\mathbf{w}|_v = \max_i\{|w_i|_v\}$.)

It is easy to see that M has rank two, and so since $\mathcal{O}$ is a Dedekind domain, M is isomorphic to $\mathcal{O} \oplus I$, where I is an ideal of $\mathcal{O}$ that can be chosen from the set $\mathcal{J}$. Since $\mathcal{J}$ is finite, we can find a free submodule N of M such that the index of N in M is bounded by a constant depending only on $\mathcal{O}$, and thus only on the field K. In particular, if we wish to count the number of K-linear equivalence classes of nonzero vectors of bounded $|\cdot|_v$-size in M, then up to a constant factor depending only on the field K, we may instead count the number of K-linear equivalence classes of nonzero vectors of bounded $|\cdot|_v$-size in N.

Since N is free, we may write $N = \mathcal{O}\mathbf{x} + \mathcal{O}\mathbf{y}$ for two vectors $\mathbf{x}$ and $\mathbf{y}$ in N. In particular, this means that we may assume that the coordinates of the vector $\mathbf{x} \wedge \mathbf{y}$ generate an ideal in $\mathcal{J}$. Thus, up to a constant that depends only on $\mathcal{O}$, we have:

$$h(L) = \max_{i,j} |x_i y_j - x_j y_i|_v \tag{1}$$

where v is, as defined above, the unique place of K at infinity.

Let (i, j) be a pair of indices for which the maximum in (1) is attained.

Define a map $\psi\colon N \to \mathcal{O}^2$ by $\psi(\mathbf{w}) = (w_i, w_j)$ — that is, ψ is the projection onto the ith and jth coordinates. Note further that ψ is injective. We have:

$$h([w_0 : \ldots : w_n]) \leq \max\{|w_i|_v, |w_j|_v\}$$

and therefore, up to a constant depending only on $\mathcal{O}$, the number of elements $\mathbf{w}$ of N with $|\mathbf{w}|_v \leq B$ is equal to the number of elements $\mathbf{w}$ of $\psi(N) \subset \mathcal{O}^2$ with $|\mathbf{w}|_v \leq B$, in both cases modulo K-linear equivalence.

Moreover, it is clear that $|\det(\psi(N))|_v = h(L)$, by construction. Thus, we have reduced to showing that the number of vectors $\mathbf{w}$ in $\psi(N)$ of $|\cdot|_v$-size at most B, up to K-linear equivalence, is at most:

$$c_K q^{2B - |det(\psi(N))|_v} + 1$$

where, as before, c_K is a nonzero constant depending only on the field K.

Let $f\colon \mathcal{O}^2 \to \mathcal{O}^2/\psi(N)$ be the quotient map. The quotient $\mathcal{O}^2/\psi(N)$ is a finite set, isomorphic to $\mathcal{O}/\det(\psi(N))\mathcal{O}$, so its cardinality is q^α, where $\alpha = |\det(\psi(N))|_v$. Fix a positive number B — without loss of generality, we may assume that B is an integer. If up to K-linear equivalence there is at most one nonzero vector $\mathbf{w} \in \psi(N)$ with $|\mathbf{w}|_v \leq B$, then we are done. Thus, assume that there are at least two, denoted by $\mathbf{w}_1$ and $\mathbf{w}_2$. If we choose $\mathbf{w}_1$ and $\mathbf{w}_2$ to be of minimal $|\cdot|_v$-size in $\psi(N)$, then the $\mathcal{O}$-module generated by $\mathbf{w}_1$ and $\mathbf{w}_2$ is a submodule of $\psi(N)$ whose index is bounded by a constant depending only on the field K. (To see this, note that $\psi(N)/\mathcal{O}\mathbf{w}_1$ is a torsion-free $\mathcal{O}$-module of rank one, and so $\psi(N)/(\mathcal{O}\mathbf{w}_1 + \mathcal{O}\mathbf{w}_2)$ is a finite set whose cardinality is bounded by a constant depending only on the class number of $\mathcal{O}$.)

Thus, we may assume that $\psi(N)$ is generated as an $\mathcal{O}$-module by $\mathbf{w}_1$ and $\mathbf{w}_2$. Let $\gamma_i = |\mathbf{w}_i|_v$, and assume without loss of generality that $\gamma_2 \geq \gamma_1$. We have:

$$\mathrm{card}(\mathcal{O}^2/\psi(N)) = q^{|\det(\psi(N))|_v}$$

and

$$\mathrm{card}\{\mathbf{w} \in \mathcal{O}^2 \mid |\mathbf{w}|_v < A\} = cq^{2A}$$

where A is any positive integer and c is a positive constant depending only on the field K. We have $|\det(\psi(N))|_v \ll q^{\gamma_1\gamma_2}$, where the implied constant depends only on the field K. Consider the set:

$$Y = \{\mathbf{w} \in \mathcal{O}^2 \mid |\mathbf{w}|_v < \gamma_2$$

Up to multiplication by a constant depending only on the field K, the cardinality of Y is $q^{2\gamma_2}$. Furthermore, Y is an $\mathbb{F}_q$-vector space, and the only elements of $\psi(N) \cap Y$ are $\mathcal{O}$-linear multiples of $\mathbf{w}_1$, of which there are $q^{\gamma_2-\gamma_1}$, up to multiplication by a constant depending only on K. Thus, Y is the union of $q^{\gamma_2-\gamma_1}$ sets of representatives for $\mathcal{O}^2/\psi(N)$ (up to multiplication by a constant depending only on the field K). Thus, reduction modulo $\psi(N)$ induces a surjective map of $\mathbb{F}_q$-vector spaces from the space of vectors in $\mathcal{O}^2$ of $|\cdot|_v$-size at most B to the space $\mathcal{O}^2/\psi(N)$, of which the kernel is precisely the set of elements of $\psi(N)$ of $|\cdot|_v$-size at most B. A quick division shows that this kernel has $c_K q^{2B-|\det(\psi(N))|_v}$ elements, where c_K is a nonzero constant depending only on the field K. □

3 Ruled Varieties

In the spirit of Heath-Brown's remark in [7], Theorem 2.1 enables us to give easy upper bounds for the counting functions for rational points on ruled varieties. Indeed, in [10], a direct analogue of Theorem 2.1 is proven for lines over number fields, and number field analogues of all the results in this section follow in that paper in a formal algebro-geometric way. Thus, the results in this section are proved in a nearly identical manner to those in the last section of [10].

Let $V \subset \mathbb{P}^n$ be a projective variety defined over a number field K. Assume that V admits a K-rational morphism $\phi\colon V \to X$ to a projective variety X over K such that the fibres of ϕ are lines. Then we can define a morphism $\psi\colon X \to G(1,n)$ by $\psi(P) = [\phi^{-1}(P)]$, where $G(1,n)$ denotes the Grassmannian of lines in $\mathbb{P}^n$. Let D be the Plücker divisor on $G(1,n)$ – that is, the pullback of $\mathcal{O}(1)$ via the Plücker embedding of $G(1,n)$. Note that ψ is injective, since ϕ is a morphism. We now have the following result:

Theorem 3.1 *Using the notation of the previous paragraph, assume that the counting function of X with respect to the divisor $A = \psi^*(D)$ satisfies:*

$$N_X(B) = \mathrm{card}\{P \in X(K) \mid h_A(P) \leq B\} = O(q^{\epsilon B})$$

for some $\epsilon < 1/m$, where m is the least integer such that $\mathcal{O}(m) - \phi^* A$ is effective. Then we have:

$$(1/c)q^{2B} \leq N_V(B) \leq cq^{2B}$$

for some positive constant c.

Proof: The first inequality is clear, since V contains at least one K-rational line. Thus, we turn our attention to the second inequality. Write h for the usual height function in $\mathbb{P}^n$, and let $F = \phi^* A$. Via the height machine, we obtain a constant α such that for all K-rational points P of V:

$$(1/m)h_F(P) \leq \alpha + h(P) \tag{2}$$

We can now calculate as follows:

$$\begin{aligned} N_V(B) &\leq \sum_{P \in X(K),\, h_A(P) \leq mB + m\alpha} N_{\phi^{-1}(P)}(B) \\ &\leq \sum_{P \in X(K),\, h_A(P) \leq mB + m\alpha} \left(c_K q^{2B - h_A(P)} + 1 \right) \end{aligned}$$

where this last inequality is by Theorem 2.1 and the fact that $h(\phi^{-1}(P)) = h_A(P)$. The hypothesis of the theorem now easily implies that this sum is asymptotically less than cq^{2B} for a positive constant c, and the theorem is proven. □

Remarks: In particular, Theorem 3.1 applies to all (relatively) minimal ruled surfaces (see section V.2 of [6] for a discussion of such surfaces). (This is not quite true, since the two cases of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown up at a single point do not satisfy the hypotheses of Theorem 3.1, but they can be handled in a similar manner, or indeed by any number of elementary approaches as well.)

In the number field case, the arithmetic of relatively minimal ruled surfaces over a rational base curve has been dealt with admirably in several places. In the case of function fields over a finite field, however, results are more limited: Bourqui has two papers ([3] and [4]) which compute counting functions for surfaces which can be ruled, but he computes them only for the anticanonical divisor, with respect to which the surfaces are not ruled, in general. (He nevertheless manages to compute the precise main term and error terms, which we do not do here.) Lai and Yeung (in [9]) make a similar computation for flag varieties.

Theorem 2.1 can in principle be applied to any variety which is a union of lines in $\mathbb{P}^n$, by the simple expedient of summing the counting functions of the individual lines, and controlling the point of smallest height on each line. Such an analysis proceeds trivially for $\mathbb{P}^n$, for example, which is the union of a pencil of lines through a fixed point, and the main term on the upper bound thereby obtained is sharp ($q^{(n+1)B}$). Similar analyses can be done for cones. In both cases, the point of smallest height on (almost all) lines is the basepoint of the linear system which sweeps out the variety.

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