DENSITY OF RATIONAL POINTS ON DIAGONAL QUARTIC SURFACES

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Abstract. Let a, b, c, d be nonzero rational numbers whose product is a square, and let V be the diagonal quartic surface in $\mathbb{P}^3$ defined by $ax^4 + by^4 + cz^4 + dw^4 = 0$. We prove that if V contains a rational point that does not lie on any of the 48 lines on V or on any of the coordinate planes, then the set of rational points on V is dense in both the Zariski topology and the real analytic topology.

1. Introduction

This paper is about the arithmetic of diagonal quartic surfaces, which are the surfaces $V_{a,b,c,d} \subset \mathbb{P}^3$ defined by the equation $ax^4+by^4+cz^4+dw^4 = 0$ for nonzero $a, b, c, d \in \mathbb{Q}$. We will prove the following theorem.

Theorem 1.1. *Let $a, b, c, d \in \mathbb{Q}^*$ be nonzero rational numbers with abcd square. Let $P = (x_0 : y_0 : z_0 : w_0)$ be a rational point on $V_{a,b,c,d}$, and suppose that $x_0y_0z_0w_0 \neq 0$ and that P does not lie on any of the 48 lines of the surface. Then the set of rational points of the surface is dense in both the Zariski and the real analytic topology.*

We will also prove a generalization to arbitrary number fields of a weaker version of Theorem 1.1. An easy consequence of Theorem 1.1 is the following.

Theorem 1.2. *Let $a, b, c, d \in \mathbb{Q}^*$ be nonzero rational numbers with abcd square and $a + b + c + d = 0$. Assume that no two of a, b, c, d sum to 0. Then the set of rational points of the surface $V_{a,b,c,d}$ is dense in both the Zariski and the real analytic topology.*

The surfaces $V_{a,b,c,d}$ are smooth quartic surfaces, which means that they are K3 surfaces. One of the most important open problems in the arithmetic of K3 surfaces is to determine whether there is a K3 surface over a number field on which the set of rational points is neither empty nor Zariski dense. Theorem 1.1 shows that a diagonal quartic surface over $\mathbb{Q}$ for which the product of the coefficients is a square does not have this property, unless all its rational points lie on the union of its 48 lines and the coordinate planes. However, no such diagonal surface is known and the authors believe that the condition in Theorem 1.1 that P not lie on one of the 48 lines or on one of the coordinate planes may not be necessary.

Noam Elkies [6] proved that the set of $\mathbb{Q}$-rational points on $V_{1,1,1,-1}$ is dense in both the Zariski topology and the real analytic topology. Martin Bright [3] has exhibited a Brauer-Manin obstruction to the existence of rational points on many examples. Sir Peter Swinnerton-Dyer in his paper [15] assumes like us that *abcd* is a square. He uses one of the two elliptic fibrations that exist in this case to show that under certain specific conditions on the coefficients, $V_{a,b,c,d}$ does not satisfy the Hasse principle, while under some other hypotheses, including Schinzel's hypothesis and the assumption that Tate-Schafarevich groups of elliptic curves are finite, the Hasse principle is satisfied. In particular, assuming these two big conjectures, it follows immediately from his work that if $abcd$ is a square but not a fourth power and no product of two coefficients or their negatives is a square and there is no Brauer-Manin obstruction to the Hasse principle, then the set of rational points is Zariski dense; the last hypothesis is obviously satisfied when $V_{a,b,c,d}(\mathbb{Q})$ is nonempty. By Theorem 1.1 the fact that the set $V_{a,b,c,d}(\mathbb{Q})$ is nonempty indeed implies that it is Zariski dense, independent from Schinzel's hypothesis and the assumption that Tate-Shafarevich groups of elliptic curves are finite, provided that we assume instead the existence of a rational point that does not lie on any of the 48 lines or any of the coordinate planes.

Jean-Louis Colliot-Thélène, Alexei Skorobogatov, and Sir Peter Swinnerton-Dyer [5] also use Schinzel's hypothesis and finiteness of Tate-Schafarevich groups to show that over arbitrary number fields, on semistable elliptic fibrations satisfying certain technical conditions, the Brauer-Manin obstruction coming from the vertical Brauer group is the only obstruction to the Hasse principle; furthermore, that if such a fibration contains a rational point, then its set of rational points is Zariski dense. Olivier Wittenberg [16] generalizes their theory to the extent that Sir Peter Swinnerton-Dyer's aforementioned result over the rational numbers becomes a special case of this more general setting, thus extending the result to arbitrary number fields.

Jean-Louis Colliot-Thélène pointed out Richmond's method [14] to the authors that takes a rational point P on $V = V_{a,b,c,d}$ to construct two new points over $\mathbb{Q}(\sqrt{abcd})$. Each of these two points is the unique last point of intersection between V and one of the two tangent lines to the singular node in the intersection between V and the tangent plane to V at P. In this paper we reinterpret this construction to study the arithmetic of the surface V.

In the next section, we exhibit two endomorphisms e_1 and e_2 of $V_{a,b,c,d}$ such that $e_1(P)$ and $e_2(P)$ are the two points given by Richmond's construction. The diagonal surfaces have two elliptic fibrations and each fibration is fixed by one of the two endomorphisms. Thus, if e_i is one of the endomorphisms and P is a rational point on the surface, we will consider the fibre C_i of the fibration fixed by the endomorphism e_i that passes through P. This is a curve, so we can study the divisor $(e_i(P)) - (P)$ on it. Subject to the hypotheses, we will see that it is almost never a torsion divisor, and hence that fibres with rational points tend to have positive rank.

Our results are very much in the spirit of the potential density results of Bogomolov and Tschinkel [1, 2], and Harris and Tschinkel [8]. These papers describe a variety of techniques for proving density and potential density of rational points on a variety of surfaces, including in particular the diagonal quartic surfaces we consider in this paper. Our results improve on these only in that we strengthen the conclusion of the potential density results to actual density, and that we weaken the hypotheses on the density results to demanding only a single rational point satisfying a weak genericity condition. For an excellent overview of techniques used to prove density and potential density of rational points on algebraic varieties, please see Brendan Hassett's survey [9].

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2. The elliptic fibrations and endomorphisms

We begin by introducing some notation.

Definition 2.1. For $a, b, c, d \in \mathbb{Q}^*$ we let $V_{a,b,c,d}$ be the surface in $\mathbb{P}^3$ given by $ax^4 + by^4 + cz^4 + dw^4 = 0$. Set $V_0 = V_{1,1,1,1}$ and $V_0' = V_{1,1,-1,-1}$. Let $\tau \colon \mathbb{P}^3 \to \mathbb{P}^3$ be the map that squares all four coordinates. Set $Q_{a,b,c,d} = \tau(V_{a,b,c,d})$.

Suppose $a, b, c, d \in \mathbb{Q}^*$ with $abcd \in (\mathbb{Q}^*)^2$ and write V and Q for $V_{a,b,c,d}$ and $Q_{a,b,c,d}$ respectively. Suppose that V has a rational point. Then Q, which is a nonsingular quadric surface defined by $ax^2 + by^2 + cz^2 + dw^2 = 0$, also contains a rational point. Since $abcd$ is a square, the two rulings on Q are defined over $\mathbb{Q}$, see [4, Lemma 2.5].

Definition 2.2. Fix a rational point R on Q, and decompose the intersection of Q with the tangent plane to Q at R into two lines l_1, l_2. Let $\pi_1, \pi_2 \colon Q \to \mathbb{P}^1$ be two rulings on Q such that l_i is a fibre of π_i. For $i = 1, 2$, set $f_i = \pi_i \circ \tau \colon V \to \mathbb{P}^1$.

Our description of the two rulings shows that they can be defined over $\mathbb{Q}$. However, the two rulings do not depend on R in the following sense. Let R' be another rational point on Q. Then by the same construction we obtain two rulings, which we can number $\pi_1', \pi_2' \colon Q \to \mathbb{P}^1$, such that for each i the maps π_i and π_i' coincide up to a linear automorphism of $\mathbb{P}^1$.

Any two linear forms defining l_i define a map to $\mathbb{P}^1$ equal to π_j up to a linear transformation of $\mathbb{P}^1$, with $i \neq j$. Using additional lines in the same family, we can obtain alternative equations for the rulings and remove the base locus. The f_i are elliptic fibrations on V, also well-defined up to an automorphism of $\mathbb{P}^1$.

Definition 2.3. Fix fourth roots of a, b, c, d. Let $\iota_{a,b,c,d}$ be the $\overline{\mathbb{Q}}$-isomorphism $V_{a,b,c,d} \to V_0$ defined by $(\sqrt[4]{a}x : \sqrt[4]{b}y : \sqrt[4]{c}z : \sqrt[4]{d}w)$. Fix fourth roots of 1 and -1, and let $\iota_{a,b,c,d}'$ be the $\overline{\mathbb{Q}}$-isomorphism $V_{a,b,c,d} \to V_0'$ defined by ${\iota_{1,1,-1,-1}}^{-1} \circ \iota_{a,b,c,d}$.

Note that the fibrations f_i of $V = V_{a,b,c,d}$ were constructed geometrically. Therefore, the fibrations of $V_{a,b,c,d}$ coincide with those of $V_{1,1,1,1}$ up to composition with $\iota_{a,b,c,d}$ and a linear automorphism of $\mathbb{P}^1$.

When we study the geometric properties of diagonal quartics, it suffices to consider V_0 or V_0'. It is only when we consider the arithmetic properties that we need to allow the coefficients to vary. While some formulas are more symmetrical on V_0, some things are defined over $\mathbb{Q}$ for V_0' that are not for V_0. For example, the two elliptic fibrations on V_0' are defined by $(x^2 - z^2 : y^2 - w^2)$ and $(x^2 - w^2 : y^2 - z^2)$, whereas on V_0 they can only be described over $\mathbb{Q}$ as maps to a curve isomorphic to the conic $x^2 + y^2 + z^2 = 0$. To give a fibration of V_0 over $\mathbb{P}^1$ requires changing base to a field over which this conic has a point.

Definition 2.4. Let μ denote the group of automorphisms of $\mathbb{P}^3$ that multiply each coordinate by a fourth root of unity, and let S_4 act on the coordinates of $\mathbb{P}^3$. We will regard μ as inducing a subgroup of $\operatorname{Aut} V$. Any permutation $\pi \in S_4$ induces an isomorphism from $V_{a,b,c,d}$ to $V_{a',b',c',d'}$, where (a', b', c', d') is the appropriate permutation of (a, b, c, d).

Definition 2.5. Let G be the semidirect product $\mu \rtimes S_4$, with the obvious action of S_4 on μ. We will view G as a subgroup of $\operatorname{Aut} V_0$, and through conjugation with $\iota_{a,b,c,d}$ also as a subgroup of $\operatorname{Aut} V_{a,b,c,d}$.

Note that when G is viewed as acting on V_0, the elements of S_4 correspond to $\mathbb{Q}$-automorphisms of V_0; this is not the case when G is considered as acting on a general $V_{a,b,c,d}$.

The surface V contains exactly 48 lines, on which G acts transitively. On V_0' one of these lines is given by $x = z$ and $y = w$. (For facts regarding the set of lines on $V_{a,b,c,d}$, see for example [13].)

Definition 2.6. Let G_0 denote the index-2 subgroup of G that fixes the fibrations f_i (up to an automorphism of $\mathbb{P}^1$).

The group G_0 partitions the 48 lines into two orbits Λ_i of size 24 (with $i = 1, 2$), where Λ_i consists of the irreducible components of the 6 singular fibres of f_i, each being of type I_4 [15, page 517]. The singular points of the fibres are exactly the 24 points with two coordinates zero, and each of these points is singular on its fibre in both fibrations.

Definition 2.7. Let Ω denote the set of these 24 points, and let U be the complement of Ω in V.

We will see that the "tangents" to the node at P that we described in the introduction can be characterized in a different manner as well, namely as the tangents to the fibres of the f_i through P. We first show that these tangents do not interfere too much with the singular fibres.

Lemma 2.8. *Fix a point $P \in U(\overline{\mathbb{Q}})$, and i, j such that $\{i, j\} = \{1, 2\}$. For $k = 1, 2$, let C_k be the fibre of f_k through P, and let M be the tangent line to C_j at P. Then M is not contained in C_i.*

Proof. Since this statement is completely geometric, we assume $V = V_0'$ without loss of generality. Note that M is well-defined because C_j is smooth at P. Suppose M is contained in C_i. Then M is one of the 48 lines. After acting on V by an appropriate element of G, the line M is given by $x = z$ and $y = w$, so there are $s, t \in \overline{\mathbb{Q}}$ such that $P = (s : t : s : t)$. Since M is contained in the fibre above $(0 : 1)$ of the fibration that sends $(x : y : z : w)$ to $(x^2 - z^2 : w^2 + y^2)$, the curve C_j is a fibre of the other fibration, so f_j can be given by $(x^2 + z^2 : w^2 + y^2)$, or equivalently $(w^2 - y^2 : x^2 - z^2)$. Since $f_j(P) = (s^2 : t^2)$, we conclude that C_j is given by $s^2(w^2 + y^2) = t^2(x^2 + z^2)$ and $t^2(w^2 - y^2) = s^2(x^2 - z^2)$. The tangent line to C_j at P is therefore also given by $s^2t(w+y) = t^2s(x+z)$ and $t^3(w-y) = s^3(x-z)$. Simple linear algebra shows that this does not contain M unless $st = 0$. This contradicts the assumption that $P \in U$, which shows that M is not contained in C_i. $\square$

The following proposition is fundamental to our work and shows how the case of diagonal V is special.

Proposition 2.9. *Fix a point $P \in U(\overline{\mathbb{Q}})$ and set $R = \tau(P)$. Let T_R denote the tangent space to Q at R, and set $A = \tau^{-1}(T_R)$. Fix $i \in \{1, 2\}$, and let C_i be the fibre of f_i through P. Let M denote the tangent line to C_i at P. Then M is contained in A. Furthermore, let T_P denote the tangent plane to V at P. If C_i is irreducible, then the intersection multiplicities $(M \cdot (T_P \cap V))_P$ and $(T_P \cdot C_i)_P$ are at least* 3.

Proof. Note that $C_i = \tau^{-1}(L_i)$, where L_1 and L_2 are the lines in $T_R \cap Q$, so we have $C_1 \cup C_2 = \tau^{-1}(T_R \cap Q) = \tau^{-1}(T_R) \cap \tau^{-1}(Q) = A \cap V$. By the assumption $P \in U$, the curve C_i is smooth at P, so M is well-defined. Without loss of generality, we assume that P is contained in the affine part $w = 1$, given by $P = (x_0, y_0, z_0)$. Since the statement of the lemma is completely geometric, we may assume that Q is given by $x^2 + y^2 + z^2 + 1 = 0$, so that V is given by $q_V = 0$ with $q_V = x^4 + y^4 + z^4 + 1$, and A by $q_A = 0$ with $q_A = x_0^2 x^2 + y_0^2 y^2 + z_0^2 z^2 + 1$. Note that at most one of the coefficients of the equation defining A is 0, so A is irreducible and smooth at P. The common tangent space T_P to V and A at P is given by $l = 0$, where

$$l = x_0^3(x - x_0) + y_0^3(y - y_0) + z_0^3(z - z_0) = x_0^3 x + y_0^3 y + z_0^3 z + 1.$$

It turns out that since Q is diagonal, the surfaces A and V are more similar locally at P than is implied by the fact that they share a tangent space. Let $\mathcal{O}_{\mathbb{P}^3,P}$ and $\mathfrak{m}$ be the local ring of P in $\mathbb{P}^3$ and its maximal ideal. Set $g = x_0^2(x - x_0)^2 + y_0^2(y - y_0)^2 + z_0^2(z - z_0)^2 \in \mathfrak{m}^2$. Then the quadratic approximations of q_A and q_V are $q_A \equiv 2l + g \mod \mathfrak{m}^3$ and $q_V \equiv 4l + 6g \mod \mathfrak{m}^3$. (Note that in fact $q_A = 2l + g$ as well.) Let $q_1, q_2 \in k[x, y, z]$ be quadrics such that C_i is given on A by $q_i = 0$. From $C_1 \cup C_2 = V \cap A$ we conclude that $q_V \equiv cq_1q_2 \mod q_A$ for some nonzero constant c. Replacing q_1 by cq_1, we find that there exists a quadric $r \in k[x, y, z]$ such that $q_V = q_1q_2 + q_Ar$. From $q_i \in \mathfrak{m}$ we find $4l \equiv q_V \equiv q_Ar \equiv 2lr \mod \mathfrak{m}^2$, and since $2l \neq 0$ in $\mathfrak{m}/\mathfrak{m}^2$, this implies that $r \equiv 2 \mod \mathfrak{m}$.

Let $\mathcal{O}_{M,P}$ and $\mathfrak{n}$ denote the local ring of P on M and its maximal ideal, and let the reduction map $\mathcal{O}_{\mathbb{P}^3,P} \to \mathcal{O}_{M,P}$ be given by $s \mapsto \overline{s}$. Since M is contained in T_P, we have $\bar{l} = 0$. Since C_i is tangent to M, we have $\overline{q_i} \in \mathfrak{n}^2$, so $\overline{q_1q_2} \in \mathfrak{n}^3$. Note that this also holds in case M is a component of C_i, because then we have $\overline{q_i} = 0$. Therefore, we find $6\overline{g} \equiv \overline{q_V} \equiv \overline{rq_A} \equiv 2\overline{g} \mod \mathfrak{n}^3$, and so $\overline{g} \in \mathfrak{n}^3$. This implies $\overline{q_A} = \overline{g} \in \mathfrak{n}^3$. If M were not contained in A, then this would imply that A intersects M at P with multiplicity at least 3, which is impossible because A is quadratic and M is linear. We conclude that M is indeed contained in A.

Now assume that C_i is irreducible. Then by Lemma 2.8 the line M is not contained in $C_1 \cup C_2 = A \cap V$, so $M \not\subset V$. This implies that the first intersection is between curves that have no common components. Because C_i is the intersection of two quadrics, it is not contained in a hyperplane. The same holds for the second intersection.

The first intersection takes place in T_P and the multiplicity equals the valuation of $\overline{q_V}$ in $\mathcal{O}_{M,P}$. From the congruence $\overline{q_V} \equiv 6\overline{g} \mod \mathfrak{n}^3$ we conclude that $\overline{q_V} \in \mathfrak{n}^3$, so the multiplicity is at least 3. Let $\mathcal{O}_{C,P}$ and $\mathfrak{p}$ denote the local ring of P on C_i and its maximal ideal, and let the reduction map $\mathcal{O}_{\mathbb{P}^3,P} \to \mathcal{O}_{C,P}$ be given by $s \mapsto \tilde{s}$. Since $\tilde{q_V}$ vanishes on C_i, we have $4\tilde{l} + 6\tilde{g} \equiv 0 \mod \mathfrak{p}^3$. Because $\tilde{q_A}$ vanishes on C_i, we also have $2\tilde{l} + \tilde{g} = 0$. Together this implies $l \in \mathfrak{p}^3$. As l defines T_P, this implies that T_P intersects C_i at P with multiplicity at least 3. □

Definition 2.10. For the rest of this section, fix a point $P \in U(\overline{\mathbb{Q}})$ and i, j such that $\{i, j\} = \{1, 2\}$. For $k = 1, 2$, let C_k be the fibre of f_k through P and M_k the tangent line to C_k at P.

Corollary 2.11. *The line M_j intersects C_i in a scheme of dimension 0 and degree 2 that contains the reduced point P. (The intersection may be a nonreduced scheme supported at P.)*

Proof. Let T_R and $A = \tau^{-1}(T_R)$ be as in Proposition 2.9. Recall that $C_i = \tau^{-1}(L)$ for some line L in T_R, so C_i is defined in A by a single quadric. The line M_j is contained in A by Proposition 2.9, but not in C_i by Lemma 2.8, so the dimension of the intersection is indeed 0. First suppose A is nonsingular. Then M_j is a line in one of the rulings on A. The curve C_i on A is of type $(2,2)$, so it intersects M_j twice, counted with multiplicity. The claim follows immediately. Now assume that A is singular. Without loss of generality we assume $V = V_0$. Suppose $P = (x_0 : y_0 : z_0 : w_0)$, so that A is given by $x_0^2x^2 + y_0^2y^2 + z_0^2z^2 + w_0^2w^2 = 0$. The facts that A is singular and that $P \in U$ imply that exactly one of the coordinates of P is zero. We deduce that A is the cone over a smooth conic, and that P is not contained in any of the lines on V, as their equations imply that if one coordinate is zero, then so is another. Therefore C_i is smooth, so, as C_i is defined in A by a single equation, it does not contain the vertex S of A. This means we can naively apply the usual intersection theory to study the intersections of C_i with other curves. The line M_j is a line on A through the vertex S and intersects any hyperplane section that does not go through S once. Since C_i is a quadratic hypersurface section, we find $M_j \cdot C_i = 2$. Again, the claim follows. □

Definition 2.12. Following Corollary 2.11, we define two morphisms $e_i, e_j : U \to V$. The morphism e_i sends any point R to the unique second intersection point between the fibre of f_i through R and the tangent at R to the fibre of f_j through R. The morphism e_j is defined by interchanging the roles of i and j.

By definition, e_i and e_j respect the fibrations f_i and f_j respectively.

Corollary 2.13. *Let T_P denote the tangent plane to V at P, and set $D = T_P \cap V$. If C_j is irreducible, then the line M_j intersects D in the divisor $3(P) + (e_i(P))$ of D. If C_i is irreducible, then T_P intersects C_i also in $3(P)+(e_i(P))$, but as a divisor on C_i.*

Proof. Let A be as in Proposition 2.9. Then we have $M_j \subset A$ and $A \cap V = C_i \cup C_j$. If C_j is irreducible, then M_j is not contained in C_j, and by Lemma 2.8 also not in C_i, so $M_j \not\subset V$, and $M_j \cap D$ is 0-dimensional. If C_i is irreducible, then it is not contained in T_P, so $T_P \cap C_i$ is 0-dimensional. Both intersections have degree 4 by Bézout's Theorem, applied in T_P and $\mathbb{P}^3$ respectively. By Proposition 2.9 the intersection at P has multiplicity at least 3, so it suffices to show that $e_i(P)$ is contained in both intersections. This follows from $e_i(P) \in C_i \subset V$ and $e_i(P) \in M_j \subset T_P$. □

Remark 2.14. Since the M_k intersect D at P with multiplicity at least 3, they are exactly the "tangent" lines to the node on D at P that we discussed in the introduction. By Corollary 2.13 this means that the $e_i(P)$ are the two points obtained from P as described there.

Remark 2.15. Let H be a hyperplane section of the generic fibre $\mathcal{V}_i/\overline{\mathbb{Q}}(t)$ in $\mathbb{P}^3$ of f_i. By Proposition 2.13, if we identify $\mathcal{V}_i$ with $\mathrm{Pic}^1(\mathcal{V}_i)$, then e_i is given by sending R to $H - 3R$ for any point R on $\mathcal{V}_i$.

Although a computer is useful, even by hand it is not impossible to check that, on an open subset, e_1 and e_2 are given by sending $(x_0 : y_0 : z_0 : w_0)$ to $(x_1 : y_1 : z_1 : w_1)$ with

$$\begin{aligned} x_1 &= x_0\big((3bcy_0^4z_0^4 + adx_0^4w_0^4)(ax_0^4 + dw_0^4) + 4Nx_0^2y_0^2z_0^2w_0^2(by_0^4 - cz_0^4)\big), \\ y_1 &= y_0\big((3acx_0^4z_0^4 + bdy_0^4w_0^4)(by_0^4 + dw_0^4) + 4Nx_0^2y_0^2z_0^2w_0^2(cz_0^4 - ax_0^4)\big), \\ z_1 &= z_0\big((3abx_0^4y_0^4 + cdz_0^4w_0^4)(cz_0^4 + dw_0^4) + 4Nx_0^2y_0^2z_0^2w_0^2(ax_0^4 - by_0^4)\big), \\ w_1 &= w_0\big(cdz_0^4w_0^4(cz_0^4 + dw_0^4) - abx_0^4y_0^4(9cz_0^4 + dw_0^4)\big), \end{aligned} \tag{1}$$

where N is one of the two square roots of $abcd$.

Definition 2.16. For each coordinate v of $\mathbb{P}^3$, let $\sigma_v \in \mu$ denote the automorphism of $\mathbb{P}^3$ that negates the v-coordinate. For a pair of coordinates u, v, let $\sigma_{uv} = \sigma_u \sigma_v$.

Proposition 2.17. *The automorphisms σ_u commute with the maps e_i on V and with all the maps $\iota_{a,b,c,d}$.*

Proof. For the e_i, this follows immediately from the equations in (1). For the $\iota_{a,b,c,d}$ it is obvious. □

Proposition 2.18. *Assume P is contained in a line that is an irreducible component of C_i. Then there are two different coordinates u, v of $\mathbb{P}^3$ such that $e_j(P) = \sigma_{uv}(P)$, while P and $e_i(P)$ lie on nonintersecting components of C_i and $e_i(P) \notin \Omega$. In particular, $e_i(P)$ lies on a line.*

Proof. By Proposition 2.17 we may assume $V = V_0'$. Let L be the line in C_i that contains P. Note that the group G acts by conjugation on the set of the three automorphisms of the form σ_{uv} with $u \neq v$. The group G also acts by conjugation on the pair of fibrations and acts accordingly on the set of the two sets Λ_i of lines. It follows that after acting with an appropriate element of G, we may assume that L is given by $x = z$ and $y = w$, so there are $s, t \in \overline{\mathbb{Q}}$ such that $P = (s : t : s : t)$. From $P \notin \Omega$ we get $st \neq 0$. Then f_i can be given by $(x : y : z : w) \mapsto (x^2 - z^2 : w^2 + y^2)$, while f_j can be given by $(x : y : z : w) \mapsto (x^2 + z^2 : w^2 + y^2)$. As e_i respects f_i, it is easy to check which equations in (1) give e_i. It turns out that e_i is given by (1) with $N = 1$, while e_j is given by $N = -1$. Substituting in (1), we find $e_j(P) = (-s : t : -s : t) = \sigma_{xz}(P)$ and $e_i(P) = (t^3 : -s^3 : -t^3 : s^3)$. It is clear that $e_i(P)$ is not contained in Ω and lies on the component of C_i given by $x + z = y + w = 0$, which is a line and does not intersect L. □

Proposition 2.19. *Let $\pi \in S_4$ be an automorphism of V_0 in $\mathbb{P}^3$ given by permutation of the coordinates, and let S_4 act on $V = V_{a,b,c,d}$ by conjugating the action on V_0 with $\iota_{a,b,c,d}$. Then $\pi e_i = e_k \pi$, where $k = i$ if the permutation underlying π is even and $k = j$ if it is odd.*

Proof. The statement is purely geometric, so we may assume $V = V_0$. Note that π switches the rulings on the quadric given by $x^2 + y^2 + z^2 + w^2 = 0$ if and only if π is odd, and it permutes the two elliptic fibrations on V accordingly. The statement therefore follows from Corollary 2.11. □

We can rephrase Proposition 2.19 by stating that inside the group $G = \mu \rtimes S_4$ we have $G_0 \cap S_4 = A_4$. We see that if we conjugate the equations for e_i in (1) by an element in A_4, then we obtain a new set of equations for e_i. In particular this will be true for the subgroup $V_4 \subset A_4$ of products of disjoint cycles. If instead we conjugate the equations for e_i by an odd element of S_4, we get a new set of equations for e_j.

Proposition 2.20. *The map e_i extends to an endomorphism of U. The sets of equations obtained by conjugating those in (1) with elements of V_4 are sufficient to define e_i on all of U.*

Proof. The set of all points R where none of these 4 sets of equations defines a regular map is determined on V by the vanishing of all 16 polynomials in the sets. One checks by computer that this set is supported on Ω. Now suppose that for some R we have $e_i(R) \in \Omega$, and let C_i be the fibre of f_i through R. Then $e_i(R)$ also lies on C_i, so C_i is a singular fibre, and Proposition 2.18 contradicts $e_i(R) \in \Omega$. We conclude that $e_i(U) \subset U$. □

Remark 2.21. In fact the e_i do not extend to any of the points in Ω. One way to prove this is to show that, for each point $R \in \Omega$ and each i, there are two lines that meet at R whose images under the e_i are disjoint. This reflects the fact that K3 surfaces do not have endomorphisms of degree greater than 1.

Proposition 2.22. *The following are equivalent: (a)* $e_1(P) = P$, *(b)* $e_2(P) = P$, *(c) exactly one of the coordinates of* P *is* 0.

Proof. Without loss of generality we suppose $V = V_0$. Let A be as in Proposition 2.9 and assume $P = (x_0 : y_0 : z_0 : w_0)$. Then A is given by $x_0^2x^2 + y_0^2y^2 + z_0^2z^2 + w_0^2w^2 = 0$. Suppose (c) holds. Then A is the cone over a nonsingular conic, and from $M_1, M_2 \subset A$ we conclude that $M_1 = M_2$ is the unique line on A through P and the vertex of A. This implies that the M_i are both tangent to both the C_i, so the second intersection point in $M_1 \cap C_2$ and $M_2 \cap C_1$ is again P, proving (a) and (b). Alternatively, we can verify the statement by direct calculation: we may assume $x_0 = 0$ by Proposition 2.19. Substituting into (1) and using $-y_0^4 = z_0^4 + w_0^4$, we simplify $e_i(P)$ to $(0 : -y_0^5z_0^4w_0^4 : -y_0^4z_0^5w_0^4 : -y_0^4z_0^4w_0^5) = (0 : y_0 : z_0 : w_0)$. Conversely, fix $k \in \{1, 2\}$ and assume $e_k(P) = P$. If C_k were singular, then P and $e_k(P)$ would lie on nonintersecting components of C_k by Proposition 2.18, so we conclude that C_k is nonsingular. By Corollary 2.13 the divisor $4(P) = 3(P) + (e_k(P))$ on C_k is linearly equivalent to a hyperplane section H. Since multiplication by 4 on the Jacobian of C_k has degree 16, there are exactly 16 points S on C_k for which $4S$ is linearly equivalent to H. By the implication (c) ⇒ (a) & (b), we can already account for all 16 of these points, namely the intersection points of the coordinate planes with C_k, as each of the 4 planes intersects C_k in 4 different points. This shows that P is one of these points, which proves (c). □

Proposition 2.23. *Suppose that* C_i *is irreducible. Then the divisor* $(e_i^2(P)) - (P)$ *on* C_i *is linearly equivalent to* $-2(e_i(P)) + 2(P)$.

Proof. Let H denote the hyperplane class on C_i. By Corollary 2.13 we have $3(S) + (e_i(S)) \sim H$ for any point S on C_i. Applying this to $S = P$ and $S = e_i(P)$, we obtain $3(P) + (e_i(P)) \sim 3(e_i(P)) + (e_i^2(P))$, from which the statement follows. □

Remark 2.24. Proposition 2.23 tells us that if we give C_i the structure of an elliptic curve with origin P, then we have $e_i^2(P) = -2e_i(P)$, and by induction we obtain $e_i^n(P) = a_n e_i(P)$ with $a_1 = 1$ and $a_{n+1} = -3a_n + 1$.

Corollary 2.25. *Assume that* C_i *is irreducible. Then the divisor* $(e_i(P)) - (P)$ *has order dividing* 3 *if and only if* $e_i^2(P) = e_i(P)$.

Proof. By Proposition 2.23, the order divides 3 if and only if $e_i^2(P)$ is linearly equivalent to $e_i(P)$. This is equivalent to $e_i^2(P) = e_i(P)$, because two different points cannot be linearly equivalent on a curve of positive genus. □

Our goal is to prove that the class of the divisor $(e_i(P)) - (P)$ on C_i is often of infinite order in the Jacobian of C_i. It turns out that this class is 2 times the class of a divisor that has a simple description.

Proposition 2.26. *Assume that C_i is nonsingular. Then the divisors $(e_i(P))-(P)$ and $2(\sigma_u P)-2(P)$ on C_i are linearly equivalent, where u is any of the coordinates on $\mathbb{P}^3$.*

Proof. As this statement is purely geometric, we may assume $V = V_0'$. By Proposition 2.19, we may assume that f_i is given by $(x:y:z:w) \mapsto (x^2-z^2 : y^2-w^2)$, and since A_4 acts transitively on the coordinates, also that $u = y$. Write $P = (x_0 : y_0 : z_0 : w_0)$. Adding $4(P)$ to both sides and using Corollary 2.13 to identify $(e_i(P)) + 3(P)$ with the hyperplane class on C_i, we see that it is enough to find a hyperplane whose intersection with C_i is $2(P) + 2(\sigma_y(P))$. A straightforward calculation shows that the plane $(s^2+t^2)(w_0x - x_0w) - (s^2-t^2)(w_0z - z_0w)$ has this property, where $(s:t) = (x_0^2 - z_0^2 : y_0^2 - w_0^2)$ is the image of P under f_i. □

Proposition 2.27. *Assume C_i to be irreducible. Then for any two coordinates u, v of $\mathbb{P}^3$, the divisor $2(\sigma_{uv}(P)) - 2(P)$ on C_i is principal.*

Proof. Twice applying Proposition 2.26 to $\sigma_u(P)$, once with the coordinate u and once with v, we see that the divisors $2(\sigma_{uv}(P)) - 2(\sigma_u(P))$ and $2(P) - 2(\sigma_u(P))$ are linearly equivalent. The result follows immediately. □

Proposition 2.28. *Let $P = (x_0 : y_0 : z_0 : w_0)$ and $V = V_0'$. Let C_1 be the fibre through P of the fibration given by $(x:y:z:w) \mapsto (x^2+z^2 : y^2+w^2)$, and let C_2 be the other fibre. For $k = 1, 2$, we let E_k be the elliptic curve C_k with P as origin, provided that C_k is nonsingular. Let i denote a fixed square root of -1. If C_1 is nonsingular, then the rational points of exact order 4 on E_1 are the $(\pm z_0 : \pm w_0 : \pm x_0 : \pm y_0)$ with three + signs and one − sign; the other points of exact order 4 are given by*

$$(w_0 : \pm i z_0 : \pm i y_0 : x_0), \quad (-w_0 : \mp i z_0 : \pm i y_0 : x_0),$$
$$(y_0 : \pm i x_0 : \pm i w_0 : -z_0), \quad (y_0 : \pm i x_0 : \mp i w_0 : z_0).$$

If C_2 is nonsingular, then on E_2, the $(\pm w_0, \pm z_0, \pm y_0, \pm x_0)$ with three + signs and one − sign are all the rational points of exact order 4; the others are given by

$$(\pm i z_0 : w_0 : \pm i x_0 : y_0), \quad (\pm i z_0 : -w_0 : \mp i x_0 : y_0),$$
$$(\mp i y_0 : x_0 : \pm i w_0 : z_0), \quad (\pm i y_0 : -x_0 : \pm i w_0 : z_0).$$

In particular, for each $k \in \{1, 2\}$ there is a set S_k of 12 automorphisms of V_0' defined over $\mathbb{Q}(i)$ such that, for R in an open subset of V_0', the class of the divisor $(e_k(R)) - (R)$ on the fibre of f_k through R is of exact order 4 if and only if $e_k(R) = \alpha(R)$ for some $\alpha \in S_k$.

Proof. One proof consists of finding a Weierstrass form for C_i, finding all 4-torsion points, and then pulling them back to our model. Instead, we show directly that for each S among the given points, the double $2S$ is one of the 2-torsion points $\sigma_{uv}(P)$ given in Proposition 2.27. Take for instance the point $S = (z_0 : w_0 : x_0 : -y_0)$ on E_1. The tangent plane to V at S is given by $l = 0$ with $l = z_0^3 x + w_0^3 y - x_0^3 z + y_0^3 w$. By Corollary 2.13 this plane intersects C_1 with multiplicity 3 at S and multiplicity 1 at $e_1(S)$. One checks, for instance using (1), that the plane given by $m = 0$ with $m = z_0 w_0 (w_0 x - z_0 y) - x_0 y_0 (y_0 z + x_0 w)$ contains S and $e_1(S)$. It follows that the function $g = m/l$ on C_1 is regular everywhere, except perhaps for a double pole at

S. Since C_1 is not contained in a plane, the function g is not constant. As C_1 has positive genus, the function g has more poles than just one simple pole, so g has exactly a double pole at S. Set $\lambda = g(P)$, then one easily checks that $g - \lambda$ also vanishes at $\sigma_{xz}(P)$, so we have $(g - \lambda) = P + \sigma_{xz}(P) - 2S$, which shows that on E_1 we have $2S = \sigma_{xz}(P)$, so by Proposition 2.27, the point S has order exactly 4. The other points can be handled similarly. □

The simplicity of the formulas in Proposition 2.28 reflects the well-known fact that the Mordell-Weil groups of the Jacobians of these fibrations over $\mathbb{Q}(i)(t)$ are both isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, cf. [13, paragraph 8].

Proposition 2.29. *Assume that P is defined over $\mathbb{Q}$ and that all its coordinates are nonzero. If $(e_i(P)) - (P)$ is a torsion divisor class on C_i, its order is at most 4.*

Proof. Since the coordinates of P are all nonzero, the 2-torsion subgroup of the Mordell-Weil group of the Jacobian of C_i over $\mathbb{Q}$ has order 4 by Proposition 2.27. By Mazur's theorem [10], the torsion subgroup of C_i is therefore $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2k\mathbb{Z}$ where $k \leq 4$. Accordingly, if $(e_i(P)) - (P)$ is torsion, then, since it is a multiple of 2 by Proposition 2.26, its order is at most 4. □

Of course the order of $(e_i(P)) - (P)$ is 1 if and only if $e_i(P) = P$; that is, by Proposition 2.22, if and only if one of the coordinates of P is 0. In this case, P is fixed by both endomorphisms e_k, and the construction of this paper does not allow us to exhibit infinitely many rational points on V.

Proposition 2.30. *Assume that C_i is nonsingular. Then the exact order of the divisor $(e_i(P)) - (P)$ is 2 if and only if P lies on a line.*

Proof. Suppose that P lies on a line L. Then none of its coordinates is 0, as the intersection of any line with any coordinate plane is contained in Ω. By Proposition 2.22 this implies $e_i(P) \neq P$, so $e_i(P)$ and P are not linearly equivalent, as no two different points are linearly equivalent to each other on a curve of positive genus. The line L is a component of C_j with $j \neq i$, so by Proposition 2.18 we have $e_i(P) = \sigma_{uv}(P)$ for some coordinates u, v of $\mathbb{P}^3$. By Proposition 2.27 we find immediately that $(e_i(P)) - (P)$ has order 2, thus proving one implication.

Note that each L of the 24 lines in the fibres of f_j intersect C_i in two points by Corollary 2.11, which are different by Lemma 2.8. This already gives 48 points S on C_i with $2(e_i(S)) \sim 2(S)$. By Proposition 2.22, the 16 intersection points of C_i with the coordinate planes also satisfy $2(e_i(S)) \sim 2(S)$, so that is 64 points total. Note that $3(S) + e_i(S)$ is linearly equivalent to a hyperplane section H on C_i for every point S on C_i by Corollary 2.13, so we have $2(e_i(S)) \sim 2(S)$ if and only if $8(S)$ is linearly equivalent to $2H$. Since multiplication by 8 on the Jacobian of C_i has degree 64, there are no points S with $8(S) \sim 2H$ other than the ones already described. This proves the converse. □

3. Proof of the main theorem

Definition 3.1. Let k be a positive integer and $i \in \{1, 2\}$. Define T_{ki} to be the closure of the locus of points P on V for which C_i, the fibre of f_i through P, is nonsingular and where $(e_i(P)) - (P)$ is a divisor of exact order k on C_i.

In these terms, Proposition 2.22 states that $T_{11} = T_{12}$ is the intersection of V with the union of the coordinate planes. Likewise, in Proposition 2.30 we showed that $T_{21} \cup T_{22}$ is contained in the union of the lines on V. Therefore, if a point P satisfies the hypotheses of our main theorem (Theorem 1.1, repeated below as Theorem 3.4), it does not lie in T_{ki} for any $k, i \in \{1, 2\}$.

By Corollary 2.25, we have that T_{3i} is the closure of $e_i^{-1}(T_{1i}) \setminus T_{1i}$. Now, T_{1i} consists of the intersections of the coordinate planes with V. Calling these X_1, X_2, X_3, X_4, a straightforward computer calculation shows that $e_i^{-1}(X_j) = X_j \cup Y_{ij}$, where the Y_{ij} are geometrically irreducible. In other words, each T_{3i} is the union of four irreducible components. Also, as in Proposition 2.28, the exact order of $e_i(P) - P$ is 4 if and only if $e_i(P) = \alpha(P)$, where α ranges over a certain set of 12 automorphisms. So T_{4i} is the union of 12 such curves, which are in fact geometrically irreducible.

Theorem 3.2. *The set T_{3i} is the union of geometrically irreducible curves of geometric genus* 41, *while T_{4i} is the union of geometrically irreducible curves of geometric genus* 13.

Proof. From the above, we can compute the T_{3i} and T_{4i} explicitly using any computer algebra package. Without loss of generality we assume that $V = V_0'$, given by $x^4 + y^4 = z^4 + w^4$, and that the fibration f_i is given by $(x : y : z : w) \mapsto (x^2 - z^2 : y^2 + w^2)$, so that in terms of Proposition 2.28 we have $i = 2$. Then e_i is given by (1) with $N = 1$.

We first consider T_{4i}. Consider the automorphism $\alpha\colon V \to V$ given by $(x : y : z : w) \mapsto (-w : z : y : x)$, so that α is one of the 12 automorphisms of Proposition 2.28 for which for any point $P \in U$ with $e_i(P) = \alpha(P)$, the divisor $(e_i(P)) - (P)$ has exact order 4 on the fiber of f_i through P. It is easily computed that the locus of such points is an irreducible curve D whose unon with the skew quadrilateral $Q \subset V$ given by $w^2 + x^2 = y^2 - z^2 = 0$ is the degree-4 hypersurface section given by $yz(y^2 - z^2) = xw(x^2 + w^2)$. Note that a skew quadrilateral of lines has self-intersection 0 and intersects a hyperplane section with multiplicity 4. We deduce that the self-intersection D^2 equals 32, so that by the adjunction formula the arithmetic genus of D equals $\frac{1}{2}(D^2 + 2) = 17$. One also checks that D has four ordinary double points, namely at the singular points of Q. It follows that D has geometric genus $17 - 4 = 13$. The other components can be dealt with similarly, or by symmetry under the action of the group G_0 of Definiton 2.6.

For T_{3i} we consider the locus of points P for which $e_i(P)$ is contained in the coordinate plane $x = 0$. One checks that this locus is the union of this coordinate plane itself and an irreducible curve E whose union with the two disjoint skew quadrilaterals given by $x^2 - iy^2 = z^2 + iw^2 = 0$ and $x^2 + iy^2 = z^2 - iw^2 = 0$ is the degree-8 hypersurface section given by $x^2y^2(z^4 - w^4) = w^2z^2(x^4 + 3y^4)$. We deduce that the self-intersection E^2 equals 128, so that by the adjunction formula the arithmetic genus of E equals $\frac{1}{2}(E^2 + 2) = 65$. One also checks that E has an ordinary double point at each of the 24 points of Ω of Definition 2.7. It follows that E has geometric genus $65 - 24 = 41$. For the remaining coordinates we get three more isomorphic curves. □

Proposition 3.3. *The intersection $(T_{31} \cup T_{41}) \cap (T_{32} \cup T_{42})$ does not contain any rational points outside the coordinate planes and the* 48 *lines on V.*

Proof. Suppose P is a rational point in the given intersection that is not on any of the coordinate planes, and set $Q = \iota_{a,b,c,d}(P) \in V_0$. Then the ratios of the fourth powers of the coordinates of Q are rational. By computer calculation, we will see that up to the action of G, there is only one such point on V_0 with this property. Working on V_0, we intersect every component of T_{31} and T_{41} with every component of T_{32} and T_{42}, and remove extraneous points with one of the coordinates 0 or that lie on a line. Then we resolve these schemes into primary components over $\mathbb{Q}$. Since all of the intersections have dimension 0, we may use Magma to find all the $\overline{\mathbb{Q}}$-points on these components. The only points for which the fourth powers of the coordinates have rational ratios are in the orbit under G (acting on V_0) of $(\eta : 1 : 1 : 1)$ with $\eta^4 = -3$, so Q is one of these points, defined over $\mathbb{Q}(i, \eta)$, where i denotes a square root of -1, and P is its inverse image under $\iota_{a,b,c,d}$. Let $\alpha, \beta, \gamma, \delta$ be the fourth roots of a, b, c, d respectively that determine $\iota_{a,b,c,d}$, viewed as elements of a field $K = \mathbb{Q}(i, \alpha, \beta, \gamma, \delta, \eta)$. Fix an extension v_3 of the 3-adic valuation of $\mathbb{Q}$ to K. Note that η is a uniformizer for v_3; we normalize so that $v_3(\eta) = 1$. Given a K-point $R = (r_0 : r_1 : r_2 : r_3)$ of V, let $\nu(R) = \sum_{i=0}^{3} v_3(r_i)$ viewed as an element of $\mathbb{Z}/4\mathbb{Z}$. It is clear that $\nu(R)$ is well-defined, G-invariant, and 0 if R is defined over $\mathbb{Q}$. However, $\nu(P) = v_3(\eta/(\alpha\beta\gamma\delta))$, and from the fact that $abcd$ is a square it follows that $v_3(\alpha\beta\gamma\delta)$ is even. This is a contradiction, because $v_3(\eta) = 1$ and $\nu(P) = 0$. We conclude that such a P does not exist. □

We are now ready to prove the main theorem, repeated here.

Theorem 3.4. *Let $a, b, c, d \in \mathbb{Q}^*$ be nonzero rational numbers with abcd square. Let $P = (x_0 : y_0 : z_0 : w_0)$ be a rational point on $V_{a,b,c,d}$, and suppose that $x_0y_0z_0w_0 \neq 0$ and that P does not lie on any of the 48 lines of the surface. Then the set of rational points of the surface is dense in both the Zariski and the real analytic topology.*

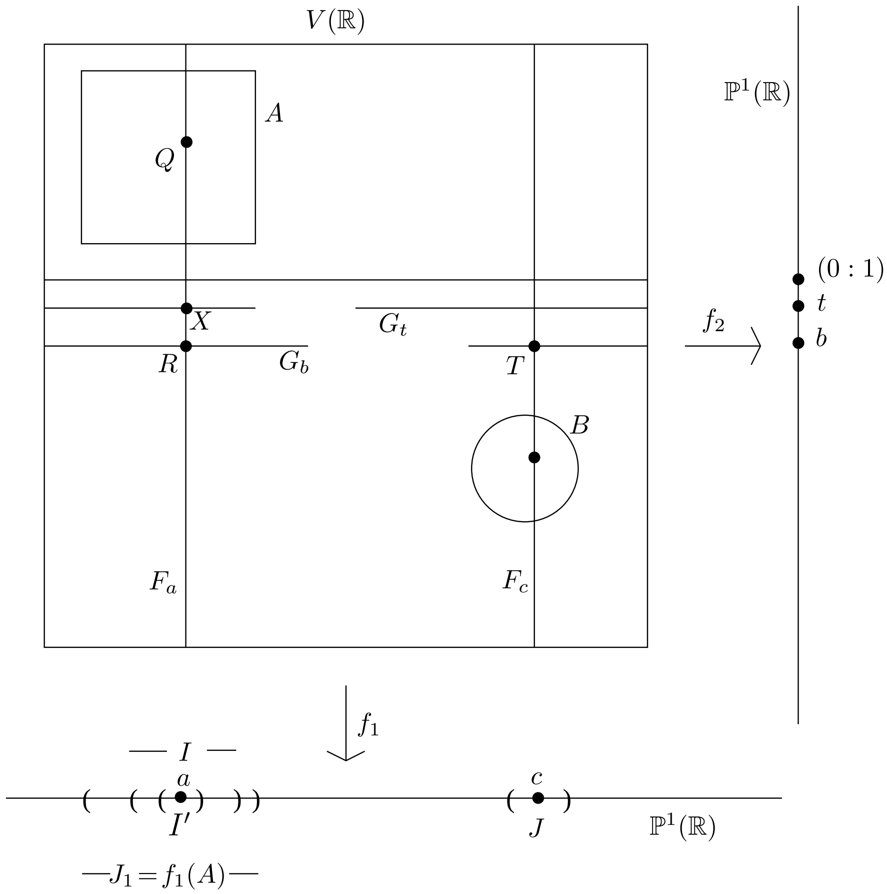
Proof. For $i = 1, 2$, let C_i denote the fibre of f_i through P, endowed with the structure of an elliptic curve with P as the origin. By assumption, $e_i(P)$ does not have order 1 or 2 on either C_i. That being so, Proposition 3.3 assures us that for some i the order of $e_i(P)$ is infinite. Say (without loss of generality) that this i is 1. Then the subgroup S of $C_1(\mathbb{R})$ generated by $e_1(P)$ and the 2-torsion points is infinite and, in fact, dense in the real analytic topology. For each point Q in S, consider the divisor class $(e_2(Q)) - (Q)$ on the fibre of f_2 passing through Q. Its order is 1 or 2 finitely often, by Propositions 2.22 and 2.30; by Theorem 3.2 it is 3 or 4 finitely often, because C_1 does not have genus 41 or 13 and so cannot be one of the curves on which the order of $(e_2(R)) - (R)$ is 3 or 4 for all R. In other words, there are only finitely many points Q in S for which the fibre of f_2 through Q contains only finitely many rational points. If $R \in S$ is not one of these finitely many points, then similarly the set of rational points on the fibre of f_2 through R is infinite and dense in the real analytic topology. Of course C_1 meets any fibre of f_2 in only finitely many points, so there are infinitely many distinct fibres of f_2 with infinitely many rational points. Zariski density follows.

Now we treat the real analytic topology. If a, b, c, d were all of the same sign, then $V = V_{a,b,c,d}$ would not have any real points, so we conclude that not all of them have the same sign. Since $abcd$ is a nonzero square, it is positive, so two of a, b, c, d are positive and two are negative. Without loss of generality, we assume that $a, d > 0$ and $b, c < 0$, and we choose real $\alpha, \beta, \gamma, \delta$ such that $\alpha^4 = a$, $\beta^4 = -b$, $\gamma^4 = -c$, and $\delta^4 = d$. Then over $\mathbb{R}$, one of the elliptic fibrations, say f_1, is given by $(x : y : z : w) \mapsto (\alpha^2x^2 - \beta^2y^2 : \gamma^2z^2 - \delta^2w^2) = (\gamma^2z^2 + \delta^2w^2 : \alpha^2x^2 + \beta^2y^2)$, up to a linear automorphism of $\mathbb{P}^1$. The fibration f_2 can be given by $(x : y : z : w) \mapsto (\alpha^2x^2 - \beta^2y^2 : \gamma^2z^2 + \delta^2w^2) = (\gamma^2z^2 - \delta^2w^2 : \alpha^2x^2 + \beta^2y^2)$. Let L be the line defined by $\alpha x = \beta y$ and $\gamma z = \delta w$. Then L is contained in the fibre of f_2 above $(0:1)$ and it is easy to check that $L(\mathbb{R})$ maps surjectively to $f_1(V(\mathbb{R})) \subset \mathbb{P}^1(\mathbb{R})$.

We now show that there exists a nonempty open subset A of $V(\mathbb{R})$ in which the subset of rational points is dense. The locus of points on V where f_1 and f_2 do not give local parameters is of codimension 1. Since the set of rational points is Zariski dense, we can choose a rational point Q not on a line or a coordinate plane such that f_1 and f_2 give local parameters at Q. We choose Q such that the set of rational points on the fibre F of f_1 through Q is dense in $F(\mathbb{R})$ as well. Let $A \subset V(\mathbb{R})$ be an open neighbourhood of Q and $J_1, J_2 \subset \mathbb{P}^1(\mathbb{R})$ connected open subsets such that the map $f = (f_1, f_2)\colon A \to J_1 \times J_2$ is a homeomorphism. It suffices to show that $f(A \cap V(\mathbb{Q}))$ is dense in $J_1 \times J_2$. Set $s_i = f_i(Q)$, so that $f(Q) = (s_1, s_2)$. Choose $(r_1, r_2) \in J_1 \times J_2$. Since the rational points on F are dense in $F(\mathbb{R})$, following the proof of the density of rational points in the Zariski topology, we can choose a rational $t_2 \in J_2$, arbitrarily close to r_2, such that $(s_1, t_2) = f(R)$ for some $R \in F(\mathbb{Q})$ for which the rational points in the fibre G of f_2 through R are dense. Therefore, there is a t_1, arbitrarily close to r_1, such that $(t_1, t_2) = f(T)$ for some $T \in G(\mathbb{Q})$. Since we can (t_1, t_2) arbitrarily close to (r_1, r_2), it follows that $V(\mathbb{Q}) \cap A$ is dense in A. The following diagram depicts the remainder of the argument.

Let $I \subset \mathbb{P}^1(\mathbb{R})$ be a nonempty connected open subset contained in $f_1(A) = J_1$. Suppose B is a nonempty open subset of $V(\mathbb{R})$ and let $J \subset \mathbb{P}^1(\mathbb{R})$ be a connected open subset contained in $f_1(B)$. Since $f_1(L) = f_1(f_2^{-1}((0:1)))$ contains I and J, for $t \in \mathbb{P}^1(\mathbb{R})$ close enough to $(0:1)$ the set $f_1(f_2^{-1}(t))$ intersects both I and J nontrivially. Choose such a t close to $(0:1)$, let G_t denote the fibre $f_2^{-1}(t)$, and choose a nonempty connected open subset $I' \subset \mathbb{P}^1(\mathbb{R})$ contained in $I \cap f_1(G_t(\mathbb{R}))$. Since $V(\mathbb{Q}) \cap A$ is dense in A, we may choose $Q \in V(\mathbb{Q}) \cap A$ such that $a = f_1(Q)$ is contained in I' and the set of rational points on the fibre $F_a = f_1^{-1}(a)$ is dense in $F_a(\mathbb{R})$; moreover, such that the set S of those rational points R on F_a for which the set of rational points on the fibre of f_2 through R is dense, is itself dense in $F_a(\mathbb{R})$. Since $a \in I'$ is contained in $f_1(G_t(\mathbb{R}))$, there is an $X \in G_t(\mathbb{R})$ with $f_1(X) = a$, so we can find $R \in S \subset F_a(\mathbb{Q})$ such that R is arbitrarily close to X and thus $b = f_2(R)$ is arbitrarily close to t. Since $f_1(G_t(\mathbb{R}))$ intersects J nontrivially, we may choose R so close to X that also $f_1(G_b(\mathbb{R}))$ intersects J nontrivially, with $G_b = f_2^{-1}(b)$. Since the set of rational points on G_b is dense in $G_b(\mathbb{R})$, we can find a point $T \in G_b(\mathbb{Q})$ such that $c = f_1(T)$ is contained in J; moreover, we can pick T so that the set of rational points on the fibre $F_c = f_1^{-1}(c)$ is dense in $F_c(\mathbb{R})$. Since $F_c(\mathbb{R})$ intersects B nontrivially and $F_c(\mathbb{Q})$ is dense in $F_c(\mathbb{R})$, we conclude that B contains at least one rational point. Thus, any nonempty open subset of $V(\mathbb{R})$ contains at least one rational point, and we conclude that $V(\mathbb{Q})$ is dense in $V(\mathbb{R})$. □

Remark 3.5. One might wonder about the possibility of proving that the rational points are dense in the p-adic topology as well as the real one. Sadly, the techniques of this paper are insufficient to prove this. To see this, note that our techniques start from a given rational point, and then move along fibres of the two fibrations to enter any given open set. In the p-adic topology, this is known to be impossible. For instance, there is an example of Swinnerton-Dyer (reference???), in which he shows that for the surface $x^4 + y^4 = 9z^4 + w^4$, every 3-adic point satisfies either $3|x/y$ or $3|y/x$, and all smooth fibres of each fibration contain only one of the two kinds of point.



It might be possible to prove something weaker using these techniques. For example, it might be possible to prove that there is some non-empty p-adic open set U on which the rational points are dense. We have not attempted to do this here.

The second theorem from the introduction, also repeated here, follows almost immediately.

Theorem 3.6. *Let $a, b, c, d \in \mathbb{Q}^*$ be nonzero rational numbers with $abcd$ square and $a + b + c + d = 0$. Assume that no two of a, b, c, d sum to 0. Then the set of rational points of the surface $V_{a,b,c,d}$ is dense in both the Zariski and the real analytic topology.*

Proof. The surface $V_{a,b,c,d}$ contains the point $P = (1 : 1 : 1 : 1)$, which does not lie on a coordinate plane. Each of the 48 lines on V is contained in one of the sets $ax^4 + by^4 = 0$, $ax^4 + cz^4 = 0$, or $ax^4 + dw^4 = 0$. Since no two of a, b, c, d sum to zero, the point P does not lie on any of the lines. By Theorem 1.1, the set of rational points of the surface is dense in both the Zariski and the real analytic topology. □

Remark 3.7. Theorem 1.2 is included to give a large family of surfaces for which we can prove unconditionally that the set of rational points is dense. Each surface

$V = V_{a,b,c,d}$ with $a+b+c+d=0$ contains the point $P = (1:1:1:1)$ and if $N^2 = abcd$, then V also contains the less trivial point $Q = (x:y:z:w)$ with

$$\begin{aligned} x &= (3bc+ad)(a+d) + 4N(b-c), \\ y &= (3ac+bd)(b+d) + 4N(c-a), \\ z &= (3ab+cd)(c+d) + 4N(a-b), \\ w &= -d(ab+ac+bc) - 9abc, \end{aligned} \tag{2}$$

which equals $e_i(P)$ for some $i \in \{1,2\}$ by (1). Theorem 1.2 appears weaker than Theorem 1.1 because of the condition $a+b+c+d=0$, but in fact Theorem 1.1 follows directly from Theorem 1.2. Indeed, given a point $P' = (x_0 : y_0 : z_0 : w_0)$ on $V' = V_{a',b',c',d'}$, the map $(x:y:z:w) \mapsto (x_0^{-1}x : y_0^{-1}y : z_0^{-1}z : w_0^{-1}w)$ sends P' to $P = (1:1:1:1)$ and induces an isomorphism $\tau_{P'}$ from V' to $V = V_{a,b,c,d}$ with $a = a'x_0^4$, $b = b'y_0^4$, $c = c'z_0^4$, and $d = d'w_0^4$, satisfying $a+b+c+d=0$. The point P' lies on a line in V' if and only if P lies on a line in V, which is the case if and only if two of a,b,c,d sum to 0. In a conversation, Andrew Granville reduced (1) to the equations in (2) and noted that the endomorphism e_i on V' can be recovered from these simpler formulas, as we have $e_i(P') = \tau_{P'}^{-1}(Q)$.

Remark 3.8. Without reference to the real analytic topology, Theorem 1.1 and its proof also apply to rational function fields over $\mathbb{Q}$. Take, for instance, the function field $K = \mathbb{Q}(a,b,c)$, set $d = -a-b-c$, and define $L = K[x]/(x^2 - abcd)$. Then, as in Theorem 1.2, we find that $V_{a,b,c,d}(L)$ is Zariski dense in $V_{a,b,c,d}$.

4. General number fields

Theorem 1.1 does not generalize immediately to number fields, as Mazur's theorem does not either. Samir Siksek pointed out to us that one can prove the following statement for general number fields. Note that Definition 2.3 applies to any number field.

Theorem 4.1. *There exists a Zariski open subset $U \subset V_{1,1,1,1}$, such that for each number field K there exists an integer n, such that for all $a,b,c,d \in K^*$ with $abcd \in K^{*2}$, if $\iota_{a,b,c,d}^{-1}(U) \subset V = V_{a,b,c,d}$ contains more than n points over K, then the set of K-rational points on V is Zariski dense.*

Proof. For each $P \in V$, let $o_i(P) \in \{1,2,3,\ldots\} \cup \{\infty\}$ denote the order of $e_i(P)$ on the fibre of f_i through P with P as origin. We refer to $o_i(P)$ as the order of $e_i(P)$.

Recall that for any positive integer N, the curve $X_1(N)$ parametrizes pairs (E,P), where E is an elliptic curve and P is a point of order N. The genus of $X_1(N)$ is at least 2 for $N = 13$ and $N \geq 16$ (see [12, p. 109]). For the remaining N, i.e., $N \in I := \{1,\ldots,12,14,15\}$, and $i \in \{1,2\}$, let $T_{i,N}$ be the closure of the locus of all points P on $V_{1,1,1,1}$ such that $o_i(P) = N$. Let $U \subset V_{1,1,1,1}$ be the complement of the $T_{i,N}$, so that for all $P \in U$ we have $o_i(P) \notin I$.

Suppose K is a number field. By Merel's Theorem [11, Corollaire], there is an integer B, depending in fact only on the degree of K, such that any K-rational point of finite order on an elliptic curve over K has order at most B. Set

$$s = \sum_{\substack{N \leq B \\ N \notin I}} \#X_1(N)(K).$$

Note that s is well defined because for each N in the sum, the genus of $X_1(N)$ is at least 2, so $\#X_1(N)(K)$ is finite by Faltings' Theorem [7]. We conclude that up to isomorphism over the algebraic closure of K, there are at most s elliptic curves over K containing a point of finite order $N \notin I$.

Take $a, b, c, d \in K^*$ with $abcd \in K^{*2}$, and let f_1, f_2 be the elliptic fibrations of $V = V_{a,b,c,d}$ over K as before. It is easy to check that the degree of the maps $j_i \colon \mathbb{P}^1 \to \mathbb{P}^1$ that send $t \in \mathbb{P}^1$ to the j-invariant of the fibre $f_i^{-1}(t)$ equals 24. Therefore there are at most $24s$ fibres of f_i, defined over K, of which the Jacobian contains a point over K of finite order $N \notin I$. Let $R = \sum_{k=1}^{B} k^2$, and take $n = 24sR$ and assume $U_{a,b,c,d} = \iota_{a,b,c,d}^{-1}(U)$ contains more than n points over K. Suppose that no fibre of f_1 contains more than R points of $U_{a,b,c,d}(K)$. Then there would at least be one point $P \in U_{a,b,c,d}(K)$ on a fibre of f_1, say C, such that all K-rational torsion points on the Jacobian of C have order in I. From $P \in U_{a,b,c,d}$ we derive $o_i(P) \notin I$, so $e_i(P)$ has infinite order and C has infinitely many rational points. We conclude that there is a fibre of f_1, say C_1, with more than R points of $U_{a,b,c,d}(K)$. By Merel's Theorem, at least one of these points has infinite order, so that there are infinitely many K-rational points on C_1.

As C_1 intersects $U_{a,b,c,d}$ nontrivially, infinitely many of these rational points Q lie in $U_{a,b,c,d}$, thus satisfying $o_2(Q) \notin I$. Since at most n points Q on V have finite order $o_2(Q) \notin I$ on the fiber of f_2 through Q, we get $o_2(Q) = \infty$ for infinitely many rational Q on C_1. It follows that infinitely many fibres of f_2 contain infinitely many rational points, so the set of rational points is Zariski dense. □

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