1. Let $G$ be any group, and let $H$ be the subgroup generated by elements of the form $xyx^{-1}y^{-1}$. (This $H$ is called the commutator subgroup of $G$.)

(a) Prove that $H$ is a normal subgroup of $G$, and that $G/H$ is abelian.

Solution: First off, $H$ is a subgroup by definition, so we just have to show that it’s normal. This amounts to showing that for every $g \in G$, we have $g^{-1}Hg \subseteq H$. Let $h \in H$ be any element. Then we can write $h = (x_1y_1x_1^{-1}y_1^{-1}) \cdots (x_ny_nx_n^{-1}y_n^{-1})$ for some elements $x_i$ and $y_i$ of $G$. (Note that $(xyx^{-1}y^{-1})^{-1} = yxy^{-1}x^{-1}$, so we don’t have to worry about inverses in the description of an arbitrary element of $H$.)

Next, note that since conjugation by $g$ is a homomorphism, we get $g^{-1}hg = a_1b_1a_1^{-1}b_1^{-1} \cdots a_nb_na_n^{-1}b_n^{-1}$, where $a_i = g^{-1}x_ig$ and $b_i = g^{-1}y_ig$. Thus, $g^{-1}hg \in H$, and so $H$ is normal.

Now let’s consider $G/H$. Choose any elements $xH$ and $yH$ in $G/H$. We want to show that $(xH)(yH) = (yH)(xH)$. This amounts to showing that $xyH = yxH$, or that $xyx^{-1}y^{-1}H = H$. But this last is clear by the definition of $H$, so $G/H$ is abelian, as desired. ♣

(b) Let $\phi: G \to K$ be a homomorphism from $G$ to another group $K$. Prove that $\text{im} \phi$ is an abelian group if and only if $H \subseteq \ker \phi$.

Solution: Let’s do forwards first. If $\text{im} \phi$ is abelian, then for all $x$ and $y$ in $G$, we have $\phi(xy) = \phi(x)\phi(y) = \phi(y)\phi(x) = \phi(yx)$. Thus $\phi(xyx^{-1}y^{-1}) = \phi(xy)\phi(yx)^{-1} = 1$, so $H \subseteq \ker \phi$.

Now for backwards. Assume that $H \subseteq \ker \phi$. Then by the universal property of group quotients, we can find a homomorphism $\tilde{\phi}: G/H \to K$ such that $\tilde{\phi} \circ q = \phi$, where $q$ is the reduction-mod-$H$ map. In particular, we must have $\text{im} \tilde{\phi} = \text{im} \phi$. But $\text{im} \phi$ is clearly abelian, because $G/H$ is abelian, so therefore $\text{im} \phi$ is also abelian. ♣

2. Is there an onto homomorphism from $S_5$ to $S_4$? If so, write it down explicitly and prove that it works. If not, prove that there is no such homomorphism.

Solution: No, there is no onto homomorphism from $S_5$ to $S_4$. 

Let \( \phi : S_5 \to S_4 \) be a homomorphism, and assume that \( \phi \) is onto. By the UPQ (or the First Isomorphism Theorem), the image of \( \phi \) is isomorphic to \( S_5 / \ker \phi \). We know that \( S_5 \) has \( 5! = 120 \) elements, while \( S_4 \) has \( 4! = 24 \) elements. If the image of \( \phi \) is \( S_4 \), then it has 24 elements, and so the kernel of \( \phi \) must have \( 120/24 = 5 \) elements.

Let \( \sigma \in \ker \phi \) be any element. By Lagrange's Theorem, \( \sigma \) has order dividing 5, so its order is either 1 or 5. If \( \sigma \) has order 1, then it is the identity permutation, so there must be some other \( \sigma \in \ker \phi \) with order 5.

If \( \sigma \) has order 5, then it must be a 5-cycle. Since \( \ker \phi \) is a normal subgroup of \( S_5 \), it is closed under conjugation by elements of \( S_5 \), so \( \ker \phi \) must contain all the conjugates of \( \sigma \). But every 5-cycle is a conjugate of \( \sigma \), so \( \ker \phi \) must contain all 5-cycles, of which there are 24. This contradicts the fact that \( \ker \phi \) has only 5 elements, so we conclude that there is no onto homomorphism from \( S_5 \) to \( S_4 \), as advertised.

3. (a) Let \( G = \mathbb{Z}_{24} \), \( H = \langle 6 \rangle \). What is the order of 13 mod \( H \)? That is, what is the order of the element \( 13 + H \) in the quotient group \( G/H \)?

Solution: The order is 6. The question just asks: What is the smallest positive integer \( n \) such that \( 13n \) is a multiple of 6 modulo 24? Being a multiple of 6 modulo 24 is the same as just being a multiple of 6, so we’re looking for the smallest positive integer such that \( 13n \) is a multiple of 6. That integer is clearly 6. ⊲

(b) Let \( G = \mathbb{R} \) under addition, and let \( H \) be the subgroup \( \mathbb{Z} \). What is the order of \( 16/7 \mod H \)?

Solution: The order is 7. We’re looking for the smallest positive integer \( n \) such that \((16/7)n \in \mathbb{Z} \) — that integer is definitely 7. ⊲

(c) Let \( G = GL_2(\mathbb{Z}_5) \) be the group of invertible matrices with entries in \( \mathbb{Z}_5 \), and let \( H = SL_2(\mathbb{Z}_5) \) be the subgroup of such matrices with determinant 1. What is the order of \( 2I \mod H \), where \( I \) is the identity matrix?

Solution: The homomorphism \( \det : GL_2(\mathbb{Z}_5) \to \mathbb{Z}_5^* \) has kernel \( H \). Thus, the order of \( 2I \mod H \) is equal to the order of \( \det(2I) \) in \( \mathbb{Z}_5^* \). Since \( \det(2I) = 4 = -1 \), this order is two. ⊲