1. Let $G = \mathbb{Z}_4$, the group of integers modulo 4, and let $H$ be the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let $f: G \to H$ be a homomorphism. Show that the kernel of $f$ must contain the element 2 of $G$.

**Solution:** Notice that for any element $h \in H$, we have $2h = 0$ (we write the group operation as addition, so the identity is 0). Therefore, in particular, $2f(1) = 0$. But $2f(1) = f(1 + 1) = f(2)$, so $f(2) = 0$, as desired. ♦

2. Let $G$ be a group, and let $\phi: G \to G$ be the function defined by $\phi(g) = g^{-1}$. If $\phi$ is a homomorphism, must $G$ be abelian? Prove your answer, or give a counterexample.

**Solution:** Yes, $G$ must be abelian. Assume that $\phi$ is a homomorphism, and let $x$ and $y$ be any elements of $G$. Then $\phi(xy) = \phi(x)\phi(y)$, so $(xy)^{-1} = x^{-1}y^{-1}$. If we multiply both sides by $xy$ on the left, we get $1 = xyx^{-1}y^{-1}$. If we then multiply both sides by $y$ on the right, we get $y = xyx^{-1}$, and if we finally multiply both sides by $x$ on the right, we get $yx = xy$. Thus, $G$ must be abelian, as advertised. ♦

3. Let $G$ be a group, and let $f: GL_2(\mathbb{R}) \to G$ be a homomorphism. Consider the following matrices:

$$A = \begin{pmatrix} 6 & 0 \\ 0 & 9 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 5 & -2 \\ 2 & 10 \end{pmatrix}$$

Prove that $f(A) = 1$ if and only if $f(B) = 1$.

**Solution:** If we can show that $A$ and $B$ are similar matrices, then we’ll know that there is an invertible matrix $P$ such that $A = P^{-1}BP$. If that’s true, then $f(A) = 1$ if and only if $f(P^{-1}BP) = 1$, which is true if and only if $f(P)^{-1}f(B)f(P) = 1$, which is true if and only if $f(B) = 1$.

Thus, this question reduces to the question of whether or not $A$ and $B$ are similar. This is a familiar question from linear algebra. Since $A$ is a diagonal matrix, our strategy will be to diagonalize $B$ and see if we get $A$, or something like it.
We all know how to diagonalize a matrix, so I’ll spare you the gory details. The short story is that the eigenvalues of $B$ are 6 and 9, so it follows that $A$ can be obtained by diagonalizing $B$. Thus, $A$ is in the kernel of $f$ if and only if $B$ is, and we’re done. ♣